Differentiation and the Laplace Transform

In this chapter, we explore how the Laplace transform interacts with the basic operators of calculus: differentiation and integration. The greatest interest will be in the first identity that we will derive. This relates the transform of a derivative of a function to the transform of the original function, and will allow us to convert many initial-value problems to easily solved algebraic equations. But there are other useful relations involving the Laplace transform and either differentiation or integration. So we'll look at them, too.

25.1 Transforms of Derivatives The Main Identity

To see how the Laplace transform can convert a differential equation to a simple algebraic equation, let us examine how the transform of a function's derivative,

$$\mathcal{L}[f'(t)]|_{s} = \mathcal{L}\left[\frac{df}{dt}\right]|_{s} = \int_{0}^{\infty} \frac{df}{dt} e^{-st} dt = \int_{0}^{\infty} e^{-st} \frac{df}{dt} dt ,$$

is related to the corresponding transform of the original function,

$$F(s) = \mathcal{L}[f(t)]|_{s} = \int_{0}^{\infty} f(t)e^{-st} dt$$

The last formula above for $\mathcal{L}[f'(t)]$ clearly suggests using integration by parts, and to ensure that this integration by parts is valid, we need to assume f is continuous on $[0, \infty)$ and f' is at least piecewise continuous on $(0, \infty)$. Assuming this,

$$\mathcal{L}[f'(t)]|_{s} = \int_{0}^{\infty} \underbrace{e^{-st}}_{u} \underbrace{\frac{df}{dt}}_{dv} dt$$
$$= uv|_{t=0}^{\infty} - \int_{0}^{\infty} v \, du$$
$$= e^{-st} f(t)|_{t=0}^{\infty} - \int_{0}^{\infty} f(t)[-se^{-st}] \, dt$$

$$= \lim_{t \to \infty} e^{-st} f(t) - e^{-s \cdot 0} f(0) - \int_0^\infty \left[-s e^{-st} \right] f(t) dt$$
$$= \lim_{t \to \infty} e^{-st} f(t) - f(0) + s \int_0^\infty f(t) e^{-st} dt \quad .$$

Now, if f is of exponential order s_0 , then

$$\lim_{t \to \infty} e^{-st} f(t) = 0 \qquad \text{whenever} \quad s > s_0$$

and

$$F(s) = \mathcal{L}[f(t)]|_{s} = \int_{0}^{\infty} f(t)e^{-st} dt \quad \text{exists for} \quad s > s_{0} \quad .$$

Thus, continuing the above computations for $\mathcal{L}[f'(t)]$ with $s > s_0$, we find that

$$\mathcal{L}[f'(t)]|_{s} = \lim_{t \to \infty} e^{-st} f(t) - f(0) + s \int_{0}^{\infty} f(t) e^{-st} dt$$

= 0 - f(0) + s \mathcal{L}[f(t)]|_{s} ,

which is a little more conveniently written as

$$\mathcal{L}[f'(t)]|_{s} = s\mathcal{L}[f(t)]|_{s} - f(0)$$
(25.1a)

or even as

$$\mathcal{L}[f'(t)]|_{s} = sF(s) - f(0)$$
 (25.1b)

This will be a very useful result, well worth preserving in a theorem.

Theorem 25.1 (transform of a derivative)

Let $F = \mathcal{L}[f]$ where f is a continuous function of exponential order s_0 on $[0, \infty)$. If f' is at least piecewise continuous on $(0, \infty)$, then

$$\mathcal{L}[f'(t)]|_{s} = sF(s) - f(0) \quad \text{for } s > s_{0} \quad .$$

Extending these identities to formulas for the transforms of higher derivatives is easy. First, for convenience, rewrite equation (25.1a) as

$$\mathcal{L}[g'(t)]|_{s} = s\mathcal{L}[g(t)]|_{s} - g(0)$$

or, equivalently, as

$$\mathcal{L}\left[\frac{dg}{dt}\right]\Big|_{s} = s\mathcal{L}[g(t)]|_{s} - g(0) \quad .$$

(Keep in mind that this assumes g is a continuous function of exponential order, g' is piecewise continuous and s is larger than the order of g.) Now we simply apply this equation with g = f', g = f'', etc. Assuming all the functions are sufficiently continuous and are of exponential order, we see that

$$\mathcal{L}[f''(t)]|_{s} = \mathcal{L}\left[\frac{df'}{dt}\right]|_{s} = s\mathcal{L}[f'(t)]|_{s} - f'(0)$$
$$= s[sF(s) - f(0)] - f'(0)$$
$$= s^{2}F(s) - sf(0) - f'(0) - f'(0)$$

Using this, we then see that

$$\mathcal{L}[f'''(t)]|_{s} = \mathcal{L}\left[\frac{df''}{dt}\right]|_{s} = s\mathcal{L}[f''(t)]|_{s} - f''(0)$$

= $s[s^{2}F(s) - sf(0) - f'(0)] - f''(0)$
= $s^{3}F(s) - s^{2}f(0) - sf'(0) - f''(0)$.

Clearly, if we continue, we will end up with the following corollary to theorem 25.1:

Corollary 25.2 (transforms of derivatives)

Let $F = \mathcal{L}[f]$ where f is a continuous function of exponential order s_0 on $[0, \infty)$. If f' is at least piecewise continuous on $(0, \infty)$, then

$$\mathcal{L}[f'(t)]|_{s} = sF(s) - f(0) \quad \text{for} \quad s > s_0$$

If, in addition, f' is a continuous function of exponential order s_0 , and f'' is at least piecewise continuous, then

$$\mathcal{L}[f''(t)]|_{s} = s^{2}F(s) - sf(0) - f'(0) \quad \text{for } s > s_{0}$$

More generally, if f, f', f'', ..., and $f^{(n-1)}$ are all continuous functions of exponential order s_0 on $[0, \infty)$ for some positive integer n, and $f^{(n)}$ is at least piecewise continuous on $(0, \infty)$, then, for $s > s_0$,

$$\mathcal{L}[f^{(n)}(t)]|_{s} = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Using the Main Identity

Let us now see how these identities can be used in solving initial-value problems. We'll start with something simple:

!> Example 25.1: Consider the initial-value problem

$$\frac{dy}{dt} - 3y = 0 \qquad \text{with} \quad y(0) = 4$$

Observe what happens when we take the Laplace transform of the differential equation (i.e., we take the transform of both sides). Initially, we just have

$$\mathcal{L}\left[\frac{dy}{dt} - 3y\right]\Big|_{s} = \mathcal{L}[0]|_{s}$$

By the linearity of the transform and fact that $\mathcal{L}[0] = 0$, this is the same as

$$\mathcal{L}\left[\frac{dy}{dt}\right]\Big|_{s} - 3\mathcal{L}[y]\Big|_{s} = 0$$

.

Letting $Y = \mathcal{L}[y]$ and applying the "transform of the derivative identity" (theorem 25.1, above), our equation becomes

$$[sY(s) - y(0)] - 3Y(s) = 0 ,$$

which, since the initial condition is y(0) = 4, can be rewritten as

$$sY(s) - 4 - 3Y(s) = 0$$

This is a simple algebraic equation that we can easily solve for Y(s). First, gather the Y(s) terms together and add 4 to both sides,

$$[s-3]Y(s) = 4 \quad ,$$

and then divide through by s - 3,

$$Y(s) = \frac{4}{s-3} \quad .$$

Thus, we have the Laplace transform Y of the solution y to the original initial-value problem. Of course, it would be nice if we can recover the formula for y(t) from Y(s). And this is fairly easy done, provided we remember that

$$\frac{4}{s-3} = 4 \cdot \frac{1}{s-3} = 4 \mathcal{L}[e^{3t}]|_{s} = \mathcal{L}[4e^{3t}]|_{s}$$

Combining the last two equations with the definition of Y, we now have

$$\mathcal{L}[y(t)]|_{s} = Y(s) = \frac{4}{s-3} = \mathcal{L}[4e^{3t}]|_{s}$$

That is,

$$\mathcal{L}[y(t)] = \mathcal{L}[4e^{3t}] \quad ,$$

from which it seems reasonable to expect

$$y(t) = 4e^{3t}$$

We will confirm that this is valid reasoning when we discuss the "inverse Laplace transform" in the next chapter.

In general, it is fairly easy to find the Laplace transform of the solution to an initial-value problem involving a linear differential equation with constant coefficients and a 'reasonable' forcing function¹. Simply take the transform of both sides of the differential equation involved, apply the basic identities, avoid getting lost in the bookkeeping, and solve the resulting simple algebraic equation for the unknown function of s. But keep in mind that this is just the Laplace transform Y(s) of the solution y(t) to the original problem. Recovering y(t) from the Y(s) found will usually not be as simple as in the last example. We'll discuss this (the recovering of y(t) from Y(s)) in greater detail in the next chapter. For now, let us just practice finding the "Y(s)".

¹ i.e., a forcing function whose transform is easily computed

!> Example 25.2: Let's find the Laplace transform $Y(s) = \mathcal{L}[y(t)]|_s$ when y is the solution to the initial-value problem

$$y'' - 7y' + 12y = 16e^{2t}$$
 with $y(0) = 6$ and $y'(0) = 4$.

Taking the transform of the equation and proceeding as in the last example:

$$\mathcal{L}[y'' - 7y' + 12y]|_{s} = \mathcal{L}[16e^{2t}]|_{s}$$

$$\hookrightarrow \qquad \mathcal{L}[y'']|_{s} - 7\mathcal{L}[y']|_{s} + 12\mathcal{L}[y]|_{s} = 16\mathcal{L}[e^{2t}]|_{s}$$

$$\hookrightarrow \qquad [s^{2}Y(s) - sy(0) - y'(0)] \\ - 7[sY(s) - y(0)] + 12Y(s) = \frac{16}{s-2}$$

$$\hookrightarrow \qquad [s^{2}Y(s) - s6 - 4] \\ - 7[sY(s) - 6] + 12Y(s) = \frac{16}{s-2}$$

$$\hookrightarrow \qquad s^{2}Y(s) - 6s - 4 \\ - 7sY(s) + 7 \cdot 6 + 12Y(s) = \frac{16}{s-2}$$

$$\hookrightarrow \qquad [s^{2} - 7s + 12]Y(s) - 6s + 38 = \frac{16}{s-2}$$

$$\hookrightarrow \qquad [s^{2} - 7s + 12]Y(s) = \frac{16}{s-2} + 6s - 38 \quad .$$

Thus,

$$Y(s) = \frac{16}{(s-2)(s^2 - 7s + 12)} + \frac{6s - 38}{s^2 - 7s + 12} \quad . \tag{25.2}$$

If desired, we can obtain a slightly more concise expression for Y(s) by finding the common denominator and adding the two terms on the right,

$$Y(s) = \frac{16}{(s-2)(s^2-7s+12)} + \frac{(s-2)(6s-38)}{(s-2)(s^2-7s+12)}$$

obtaining

$$Y(s) = \frac{6s^2 - 50s + 92}{(s-2)(s^2 - 7s + 12)} \quad . \tag{25.3}$$

.

We will finish solving the above initial-value problem in example 26.6 on page 533. At that time, we will find the later expression for Y(s) to be more convenient. At this point, though, there is no significant advantage gained by reducing expression (25.2) to (25.3). When doing similar problems in the exercises, go ahead and "find the common denominator and add" if the algebra is relatively simple. Otherwise, leave your answers as the sum of two terms.

However, do observe that we did NOT multiply out the factors in the denominator, but left them as

$$(s-2)(s^2-7s+12)$$

Do the same in your own work. In the next chapter, we will see that leaving the denominator in factored form will simplify the task of recovering y(t) from Y(s).

25.2 Derivatives of Transforms

In addition to the "transforms of derivatives" identities just discussed, there are some "derivatives of transforms" identities worth discussing. To derive the basic identity, we start with a generic transform,

$$F(s) = \mathcal{L}[f(t)]|_s = \int_0^\infty f(t)e^{-st} dt \quad ,$$

and (naively) look at its derivative,

$$F'(s) = \frac{dF}{ds} = \frac{d}{ds} \int_0^\infty f(t)e^{-st} dt$$

=
$$\int_0^\infty \frac{\partial}{\partial s}e^{-st} f(t) dt$$

=
$$\int_0^\infty (-t)e^{-st} f(t) dt = -\underbrace{\int_0^\infty tf(t)e^{-st} dt}_{\mathcal{L}[tf(t)]|_s}$$

Cutting out the middle of the above set of equalities gives us the identity

$$\frac{dF}{ds} = -\mathcal{L}[tf(t)]|_s$$

Since we will often use this identity to compute transforms of functions multiplied by t, let's move the negative sign to the other side and rewrite this identity as

$$\mathcal{L}[tf(t)]|_{s} = -\frac{dF}{ds}$$

or, equivalently, as

$$\mathcal{L}[tf(t)]|_{s} = -\frac{d}{ds}\mathcal{L}[f(t)]|_{s}$$

The cautious reader may be concerned about the validity of

$$\frac{d}{ds}\int_0^\infty g(t,s)\,dt = \int_0^\infty \frac{\partial}{\partial s}[g(t,s)]\,dt$$

blithely used (with $g(t, s) = e^{-st} f(t)$) in the above derivation This is a legitimate concern, and is why we must consider the above a somewhat "naive" derivation, instead of a truly rigorous one. Fortunately, the above derivations can be rigorously verified whenever f is of exponential order s_0 and we restrict s to being greater than s_0 . This gives the following theorem:

Theorem 25.3 (derivatives of transforms)

Let $F = \mathcal{L}[f]$ where f is a piecewise continuous function of exponential order s_0 on $(0, \infty)$. Then F(s) is differentiable on $s > s_0$, and

$$\mathcal{L}[tf(t)]|_{s} = -\frac{dF}{ds} \quad \text{for} \quad s > s_0 \quad .$$
(25.5)

A rigorous proof of this theorem is not hard, but is a bit longer than our naive derivation. The interested reader can find it in the appendix starting on page 519

Now let's try using our new identity.

!> Example 25.3: Find the Laplace transform of $t \sin(3t)$. Here, we have

$$\mathcal{L}[t\sin(3t)]|_{s} = \mathcal{L}[tf(t)]|_{s} = -\frac{dF}{ds}$$

with $f(t) = \sin(3t)$. From the tables (or memory), we find that

$$F(s) = \mathcal{L}[f(t)]|_{s} = \mathcal{L}[\sin(3t)]|_{s} = \frac{3}{s^{2}+9}$$

Applying the identity just derived (identity (25.5)) yields

$$\mathcal{L}[t\sin(3t)]|_{s} = \mathcal{L}[tf(t)]|_{s} = -\frac{dF}{ds}$$
$$= -\frac{d}{ds} \left[\frac{3}{s^{2}+9}\right] = -\frac{-3\cdot 2s}{\left(s^{2}+9\right)^{2}} = \frac{6s}{\left(s^{2}+9\right)^{2}}$$

Deriving corresponding identities involving higher order derivatives and higher powers of t is straightforward. Simply use the identity in theorem 25.3 repeatedly, replacing f(t) with tf(t), $t^2 f(t)$, etc.:

$$\mathcal{L}[t^{2}f(t)]|_{s} = \mathcal{L}[t[tf(t)]]|_{s} = -\frac{d}{ds}\mathcal{L}[tf(t)]|_{s}$$

$$= -\frac{d}{ds}\left[-\frac{dF}{ds}\right] = (-1)^{2}\frac{d^{2}F}{ds^{2}} ,$$

$$\mathcal{L}[t^{3}f(t)]|_{s} = \mathcal{L}[t[t^{2}f(t)]]|_{s} = -\frac{d}{ds}\mathcal{L}[t^{2}f(t)]|_{s}$$

$$= -\frac{d}{ds}\left[(-1)^{2}\frac{d^{2}F}{ds^{2}}\right] = (-1)^{3}\frac{d^{3}F}{ds^{3}}$$

$$\mathcal{L}[t^{4}f(t)]|_{s} = \mathcal{L}[t[t^{3}f(t)]]|_{s} = -\frac{d}{ds}\mathcal{L}[t^{3}f(t)]|_{s}$$

$$\mathcal{L}[t^4 f(t)]|_s = \mathcal{L}[t[t^3 f(t)]]|_s = -\frac{d}{ds} \mathcal{L}[t^3 f(t)]|_s$$
$$= -\frac{d}{ds} \left[(-1)^3 \frac{d^3 F}{ds^3} \right] = (-1)^4 \frac{d^4 F}{ds^4}$$

and so on. Clearly, then, as a corollary to theorem 25.3, we have:

Corollary 25.4 (derivatives of transforms)

Let $F = \mathcal{L}[f]$ where f is a piecewise continuous function of exponential order s_0 . Then F(s) is infinitely differentiable for $s > s_0$, and

$$\mathcal{L}[t^n f(t)]|_s = (-1)^n \frac{d^n F}{ds^n}$$
 for $n = 1, 2, 3, ...$

For easy reference, all the Laplace transform identities we've derived so far are listed in table 25.1. Also in the table are two identities that will be derived in the next section.

,

 Table 25.1: Commonly Used Identities (Version 1)

In the following, $F(s) =$	$\mathcal{L}[f(t)] _{s}$.	
h(t)	$H(s) = \mathcal{L}[h(t)] _s$	Restrictions
f(t)	$\int_0^\infty f(t)e^{-st}dt$	
$e^{\alpha t}f(t)$	$F(s-\alpha)$	α is real
$\frac{df}{dt}$	sF(s) - f(0)	
$\frac{d^2f}{dt^2}$	$s^2 F(s) - sf(0) - f'(0)$	
$\frac{d^n f}{dt^n}$	$s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots - f^{(n-1)}(0)$	$n = 1, 2, 3, \ldots$
t f(t)	$-\frac{dF}{ds}$	
$t^n f(t)$	$(-1)^n \frac{d^n F}{ds^n}$	$n = 1, 2, 3, \ldots$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$	

25.3 Transforms of Integrals and Integrals of Transforms

Analogous to the differentiation identities

$$\mathcal{L}[f'(t)]|_{s} = sF(s) - f(0) \quad \text{and} \quad \mathcal{L}[tf(t)]|_{s} = -F'(s)$$

are a pair of identities concerning transforms of integrals and integrals of transforms. These identities will not be nearly as important to us as the differentiation identities, but they do have their uses and are considered to be part of the standard set of identities for the Laplace transform.

Before we start, however, take another look at the above differentiation identities. They show that, under the Laplace transform, the differentiation of one of the functions, f(t) or F(s), corresponds to the *multiplication* of the other by the appropriate variable. This may lead you to suspect that the analogous integration identities show that, under the Laplace transform, integration of one of the functions, f(t) or F(s), corresponds to the *division* of the other by the appropriate variable. Be suspicious. We will confirm (and use) this suspicion.

Transform of an Integral

Let

$$g(t) = \int_0^t f(\tau) d\tau \quad .$$

where f is piecewise continuous function on $(0, \infty)$ and of exponential order s_0 . From calculus, we (should) know the following:

- 1. g is continuous on $[0, \infty)$.²
- 2. g is differentiable at every point in $(0, \infty)$ at which f is continuous, and

$$\frac{dg}{dt} = \frac{d}{dt} \int_0^t f(\tau) d\tau = f(t)$$

3.
$$g(0) = \int_0^0 f(\tau) d\tau = 0$$

In addition, it is not that difficult to show (see the proof of lemma 25.7 on page 518) that g is also of exponential order s_1 with s_1 being any positive value greater than or equal to s_0 . So both f and g have Laplace transforms, which, as usual, will be denoted by F and G, respectively. Letting $s > s_1$, and using the second and third facts listed above, along with our first differentiation identity, we have

$$\frac{dg}{dt} = f(t)$$

$$\hookrightarrow \qquad \qquad \mathcal{L}\left[\frac{dg}{dt}\right]\Big|_{s} = \mathcal{L}[f(t)]|_{s}$$

$$\hookrightarrow \qquad \qquad sG(s) - \underbrace{g(0)}_{0} = F(s)$$

$$\hookrightarrow \qquad \qquad sG(s) = F(s)$$

Dividing through by s and recalling what G and g represent then gives us the following theorem:

Theorem 25.5 (transform of an integral)

Let $F = \mathcal{L}[f]$ where f is any piecewise continuous function on $(0, \infty)$ of exponential order s_0 , and let s_1 be any positive value greater than or equal to s_0 . Then

$$\int_0^t f(\tau) \, d\tau$$

is a continuous function of t on $[0, \infty)$ of exponential order s_1 , and, for each positive s greater than s_0 ,

$$\mathcal{L}\left[\int_0^t f(\tau) \, d\tau\right]\Big|_s = \frac{F(s)}{s} \quad .$$

² If the continuity of g is not obvious, take a look at the discussion of theorem 2.1 on page 35.



Figure 25.1: The graphs of (a) the ramp function at α from example 25.4 and (b) the sinc function from example 25.5.

!> Example 25.4: Let α be any nonnegative real number. The "ramp at α function" can be defined by

$$\operatorname{ramp}_{\alpha}(t) = \int_0^t \operatorname{step}_{\alpha}(\tau) d\tau$$

If $t \geq \alpha$,

$$\operatorname{ramp}_{\alpha}(t) = \int_{0}^{t} \operatorname{step}_{\alpha}(\tau) d\tau$$
$$= \int_{0}^{\alpha} \operatorname{step}_{\alpha}(\tau) d\tau + \int_{\alpha}^{t} \operatorname{step}_{\alpha}(\tau) d\tau$$
$$= \int_{0}^{\alpha} 0 d\tau + \int_{\alpha}^{t} 1 d\tau = 0 + t - \alpha$$

If $t < \alpha$,

$$\operatorname{ramp}_{\alpha}(t) = \int_0^t \operatorname{step}_{\alpha}(\tau) d\tau = \int_0^t 0 d\tau = 0$$

So

$$\operatorname{ramp}_{\alpha}(t) = \begin{cases} 0 & \text{if } t < \alpha \\ t - \alpha & \text{if } \alpha \le t \end{cases}$$

which is the function sketched in figure 25.1a.

By the integral formula we gave for $ramp_{\alpha}(t)$ and the identity in theorem 25.5,

$$\mathcal{L}\left[\operatorname{ramp}_{\alpha}(t)\right]\Big|_{s} = \mathcal{L}\left[\int_{0}^{t}\operatorname{step}_{\alpha}(\tau)\,d\tau\right]\Big|_{s} = \mathcal{L}\left[\int_{0}^{t}f(\tau)\,d\tau\right]\Big|_{s} = \frac{F(s)}{s}$$

where $f(t) = \operatorname{step}_{\alpha}(t)$, $s_0 = 0$, and

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}[\operatorname{step}_{\alpha}(t)]|_s = \frac{e^{-\alpha s}}{s} \quad \text{for } s > 0$$
.

So,

$$\mathcal{L}\left[\operatorname{ramp}_{\alpha}(t)\right]\Big|_{s} = \frac{F(s)}{s} = \frac{e^{-\alpha s}}{s \cdot s} = \frac{e^{-\alpha s}}{s^{2}} \quad \text{for } s > 0 \quad .$$

Integral of a Transform

The identity just derived should reinforce our suspicion that, under the Laplace transform, the *division* of f(t) by t should correspond to some integral of F. To confirm this suspicion and derive that integral, let's assume

$$g(t) = \frac{f(t)}{t}$$

where f is some piecewise continuous function on $(0, \infty)$ of exponential order s_0 . Let us further assume that

$$\lim_{t \to 0^+} \frac{f(t)}{t}$$

converges to some finite value. Clearly then, g is also piecewise continuous on $(0, \infty)$ and of exponential order s_0 .

Using an old trick along with the "derivative of a transform" identity (in theorem 25.3), we have

$$F(s) = \mathcal{L}[f(t)]|_{s} = \mathcal{L}\left[t \cdot \frac{f(t)}{t}\right]|_{s} = \mathcal{L}[tg(t)]|_{s} = -\frac{dG}{ds}$$

Cutting out the middle and renaming the variable as σ ,

$$F(\sigma) = -\frac{dG}{d\sigma}$$

allows us to use s as a limit when we integrate both sides,

$$\int_{a}^{s} F(\sigma) \, d\sigma = -\int_{a}^{s} \frac{dG}{d\sigma} \, d\sigma = -G(s) + G(a)$$

which we can then solve for G(s):

$$G(s) = G(a) - \int_a^s F(\sigma) d\sigma = G(a) + \int_s^a F(\sigma) d\sigma$$

All this, of course, assumes a and s are any two real numbers greater than s_0 . Since we are trying to get a formula for G, we don't know G(a). What we do know (from theorem 24.5 on page 495) is that $G(a) \rightarrow 0$ as $a \rightarrow \infty$. This, along with the fact that s and a are independent of each other, means that

$$G(s) = \lim_{a \to \infty} G(s) = \lim_{a \to \infty} \left[G(a) + \int_s^a F(\sigma) \, d\sigma \right] = 0 + \int_s^\infty F(\sigma) \, d\sigma \quad .$$

After recalling what G and g originally denoted, we discover that we have verified:

Theorem 25.6 (integral of a transform)

Let $F = \mathcal{L}[f]$ where f is piecewise continuous on $(0, \infty)$ and of exponential order s_0 . Assume further that

$$\lim_{t \to 0^+} \frac{f(t)}{t}$$

converges to some finite value. Then

$$\frac{f(t)}{t}$$

is also piecewise continuous on $(0, \infty)$ and of exponential order s_0 . Moreover,

$$\mathcal{L}\left[\frac{f(t)}{t}\right]\Big|_{s} = \int_{s}^{\infty} F(\sigma) \, d\sigma \qquad \text{for} \quad s > s_{0}$$

.

.

!► Example 25.5: The sinc function (pronounced "sink") is defined by

$$\operatorname{sinc}(t) = \frac{\sin(t)}{t} \quad \text{for } t \neq 0$$
.

It's limit as $t \to 0$ is easily computed using L'Hôpital's rule, and defines the value of sinc(t) when t = 0,

$$\operatorname{sinc}(0) = \lim_{t \to 0} \operatorname{sinc}(t) = \lim_{t \to 0} \frac{\sin(t)}{t} = \lim_{t \to 0} \frac{\frac{d}{dt} [\sin(t)]}{\frac{d}{dt} [t]} = \lim_{t \to 0} \frac{\cos(t)}{1} = 1 \quad .$$

The graph of the sinc function is sketched in figure 25.1b.

Now, by definition,

$$\operatorname{sinc}(t) = \frac{f(t)}{t}$$
 with $f(t) = \sin(t)$ for $t > 0$

Clearly, this f satisfies all the requirements for f given in theorem 25.6 (with $s_0 = 0$). Thus, for s > 0, we have

$$\mathcal{L}[\operatorname{sinc}(t)]|_{s} = \mathcal{L}\left[\frac{f(t)}{t}\right]|_{s} = \int_{s}^{\infty} F(\sigma) \, d\sigma$$

with

$$F(\sigma) = \mathcal{L}[f(t)]|_{\sigma} = \mathcal{L}[\sin(t)]|_{\sigma} = \frac{1}{\sigma^2 + 1}$$

So, for s > 0*,*

$$\mathcal{L}[\operatorname{sinc}(t)]|_{s} = \int_{s}^{\infty} \frac{1}{\sigma^{2} + 1} d\sigma$$

= $\operatorname{arctan}(\sigma)|_{s}^{\infty}$
= $\lim_{\sigma \to \infty} \arctan(\sigma) - \arctan(s) = \frac{\pi}{2} - \arctan(s)$

(By the way, you can derive the equivalent formula

$$\mathcal{L}[\operatorname{sinc}(t)]|_s = \arctan\left(\frac{1}{s}\right)$$

using either arctangent identities or the substitution $\sigma = \frac{1}{u}$ in the last integral above.)

Addendum

Here's a little fact used in deriving the "transform of an integral" identity in theorem 25.5. We prove it here because the proof could distract from the more important part of that derivation.

Lemma 25.7

Let f be any piecewise continuous function on $(0, \infty)$ of exponential order s_0 . Then the function g, given by

$$g(t) = \int_0^t f(\tau) d\tau \quad ,$$

is of exponential order s_1 where s_1 is any positive value greater than or equal to s_0 .

PROOF: Since f is piecewise continuous on $(0, \infty)$ and of exponential order s_0 , lemma 24.8 on page 500 assures us that there is a constant M such that

$$|f(t)| \leq Me^{s_0 t}$$
 whenever $0 < t$

Now let s_1 be any positive value greater than or equal to s_0 , and let t > 0. Using the above and the integral inequality from lemma 24.9 on page 500, we have

$$|g(t)| = \left| \int_0^t f(\tau) d\tau \right| \le \int_0^t |f(\tau)| d\tau$$

$$\le \int_0^t M e^{s_0 \tau} d\tau$$

$$\le \int_0^t M e^{s_1 \tau} d\tau = \frac{M}{s_1} \left[e^{s_1 t} - e^{s_1 \cdot 0} \right] \le \frac{M}{s_1} e^{s_1 t}$$

So, letting $M_1 = {}^{M_{/s_1}}$,

$$|g(t)| < M_1 e^{s_1 t}$$
 for $t > 0$

verifying that g is of exponential order s_1 .

25.4 Appendix: Differentiating the Transform The Main Issue

On page 512, we derived the "derivative of a transform" identity, $F' = -\mathcal{L}[tf(t)]$, naively using the "fact" that

$$\frac{d}{ds}\int_0^\infty g(t,s)\,dt = \int_0^\infty \frac{\partial}{\partial s} [g(t,s)]\,dt \quad . \tag{25.6}$$

The problem is that this "fact", while often true, is not always true.

!► Example 25.6: Let

$$g(t,s) = \frac{\sin(st)}{t}$$
 for $t > 0$ and $s > 0$

(Compare this function to the sinc function in example 25.5 on page 518.) It is easily verified that the graph of this function over the positive T-axis is as sketched in figure 25.2. Recalling the relation between integration and area, we also see that, for each s > 0,

$$\int_{0}^{\infty} g(t,s) dt = \lim_{T \to \infty} \int_{0}^{T} \frac{\sin(st)}{t} dt$$

= $A_1 - A_2 + A_3 - A_4 + A_5 - A_6 + \dots = \sum_{k=1}^{\infty} (-1)^{k+1} A_k$



Figure 25.2: The graph of $\frac{\sin(st)}{t}$ for some s > 0 with $t_k = \frac{k\pi}{s}$.

where each A_k is the area enclosed by the graph of g and the T-axis interval (t_{k-1}, t_k) described in figure 25.2. Notice that this last summation is an alternating series whose terms are steadily decreasing to zero. As you surely recall from your calculus course, any such summation is convergent. Hence, so is the above integral of g. That is, this integral is well defined and finite for each s > 0. Unfortunately, this integral cannot by evaluated by elementary means. Still, using the substitution $t = \frac{\tau}{s}$, we can reduce this integral to a slightly simpler form:

$$\int_0^\infty g(t,s) dt = \int_0^\infty \frac{\sin(st)}{t} dt = \int_0^\infty \frac{\sin(\tau)}{\tau/s} \frac{d\tau}{s} = \int_0^\infty \frac{\sin(\tau)}{\tau} d\tau$$

Thus, in fact, this integral does not depend on s. Consequently,

$$\frac{d}{ds}\int_0^\infty g(t,s)\,dt = \frac{d}{ds}\int_0^\infty \frac{\sin(\tau)}{\tau}\,d\tau = 0$$

On the other hand,

$$\int_0^\infty \frac{\partial}{\partial s} [g(t,s)] dt = \int_0^\infty \frac{\partial}{\partial s} \left[\frac{\sin(st)}{t} \right] dt$$
$$= \int_0^\infty \frac{\cos(st) \cdot t}{t} dt$$
$$= \int_0^\infty \cos(st) dt = \lim_{t \to \infty} \frac{\sin(st)}{s}$$

which is not 0 — it does not even converge! Thus, at least for this choice of g(t, s),

$$\frac{d}{ds}\int_0^\infty g(t,s)\,dt \neq \int_0^\infty \frac{\partial}{\partial s}[g(t,s)]\,dt$$

There are fairly reasonable conditions ensuring that equation (25.6) holds, and our use of it on page 512 in deriving the "derivative of the transform" identity can be justified once we know those "reasonable conditions". But instead, let's see if we can rigorously verify our identity just using basic facts from elementary calculus.

The Rigorous Derivation

Our goal is to prove theorem 25.3 on page 512. That is, we want to rigorously derive the identity

$$F'(s) = -\mathcal{L}[tf(t)]|_s \tag{25.7}$$

assuming $F = \mathcal{L}[f]$ with f being a piecewise continuous function of exponential order s_0 . We will also assume $s > s_0$.

First, you should verify that the results of exercise 24.18 on page 505 and lemma 24.8 on page 500 give us:

Lemma 25.8

If f(t) is of exponential order s_0 , and n is any positive integer, then $t^n f(t)$ is a piecewise continuous function on $(0, \infty)$ of exponential order $s_0 + \sigma$ for any $\sigma > 0$. Moreover, there is a constant M_{σ} such that

$$|t^n f(t)| \leq M_\sigma e^{(s_0+\sigma)t}$$
 for all $t > 0$.

Since we can always find a positive σ such that $s_0 < s_0 + \sigma < s$, this lemma assures us that $\mathcal{L}[t^n f(t)]|_s$ is well defined for $s > s_0$.

Now let's consider F'(s). By definition

$$F'(s) = \lim_{\Delta s \to 0} \frac{F(s + \Delta s) - F(s)}{\Delta s}$$

provided the limit exists. Taking $|\Delta s|$ small enough so that $s + \Delta s$ is also greater that s_0 (even if $\Delta s < 0$), we have

$$\frac{F(s+\Delta s)-F(s)}{\Delta s} = \frac{1}{\Delta s} \left[F(s+\Delta s) - F(s) \right]$$
$$= \frac{1}{\Delta s} \left[\int_0^\infty f(t) \underbrace{e^{-(s+\Delta s)t}}_{e^{-st}e^{-\Delta st}} dt - \int_0^\infty f(t) e^{-st} dt \right]$$

which simplifies to

$$\frac{F(s+\Delta s)-F(s)}{\Delta s} = \int_0^\infty f(t) \left(\frac{1}{\Delta s}\right) \left[e^{-(\Delta s)t}-1\right] e^{-st} dt \quad .$$
(25.8)

To deal with the integral in the last equation, we will use the fact that, for any value x, the exponential of x is given by its Taylor series,

$$e^{x} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k} = 1 + x + \frac{1}{2!} x^{2} + \frac{1}{3!} x^{3} + \frac{1}{4!} x^{4} + \cdots$$

So

$$e^{x} - 1 = x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \frac{1}{4!}x^{4} + \cdots$$

= $x + x^{2} E(x)$

where

$$E(x) = \frac{1}{2!} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \cdots$$

,

Consequently (using $x = -(\Delta s)t$),

$$\frac{1}{\Delta s} \left[e^{-(\Delta s)t} - 1 \right] = \frac{1}{\Delta s} \left[-(\Delta s)t + \left[-(\Delta s)t \right]^2 E(-(\Delta s)t) \right]$$
$$= -t + (\Delta s)t^2 E(-(\Delta s)t) \quad .$$

Combined with equation (25.8), this yields

$$\frac{F(s+\Delta s)-F(s)}{\Delta s} = \int_0^\infty f(t) \left[-t + (\Delta s)t^2 E(-(\Delta s)t)\right] e^{-st} dt$$
$$= \int_0^\infty f(t) \left[-t\right] e^{-st} dt + \int_0^\infty f(t) \left[(\Delta s)t^2 E(-(\Delta s)t)\right] e^{-st} dt$$

That is,

$$\frac{F(s+\Delta s)-F(s)}{\Delta s} = -\mathcal{L}[tf(t)]|_s + \Delta s \int_0^\infty t^2 f(t) E(-(\Delta s)t) e^{-st} dt \quad .$$
(25.9)

Obviously, the question now is *What happens to the second term on the right when* $\Delta s \rightarrow 0$? To help answer that, let us observe that, for all x,

$$|E(x)| = \left| \frac{1}{2!} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \frac{1}{5!}x^3 + \cdots \right|$$

$$\leq \frac{1}{2!} + \frac{1}{3!}|x| + \frac{1}{4!}|x|^2 + \frac{1}{5!}|x|^3 + \cdots$$

$$< 1 + |x| + \frac{1}{2!}|x|^2 + \frac{1}{3!}|x|^3 + \cdots = e^{|x|}$$

Moreover, as noted in lemma 25.8, for each $\sigma > 0$, there is a positive constant M_{σ} such that

$$\left|t^2 f(t)\right| \leq M_{\sigma} e^{(s_0 + \sigma)t}$$
 for all $t > 0$

Remember, $s > s_0$. And since σ can be chosen as close to 0 as desired, and we are taking the limit as $\Delta s \to 0$, we may assume that $s + [\sigma + |\Delta s|] > s_0$. Doing so and applying the above then yields

$$\begin{aligned} \left| \Delta s \int_0^\infty t^2 f(t) E(-(\Delta s)t) e^{-st} dt \right| &\leq \left| \Delta s \right| \int_0^\infty \left| t^2 f(t) \right| \left| E(-(\Delta s)t) \right| e^{-st} dt \\ &\leq \left| \Delta s \right| \int_0^\infty M_\sigma e^{(s_0 + \sigma)t} e^{\left| -(\Delta s)t \right|} e^{-st} dt \\ &= \left| \Delta s \right| M_\sigma \int_0^\infty e^{-(s-s_0 - \sigma - |\Delta s|)t} dt \\ &\leq \left| \Delta s \right| \frac{M_\sigma}{s-s_0 - \sigma - |\Delta s|} \end{aligned}$$

Thus,

$$\lim_{\Delta s \to 0} \left| \Delta s \int_0^\infty t^2 f(t) E(-(\Delta s)t) e^{-st} dt \right| = 0 \cdot \frac{M_\sigma}{s - s_0 - \sigma - 0} = 0$$

Combining this with equation (25.9), we finally obtain

$$\lim_{\Delta s \to 0} \frac{F(s + \Delta s) - F(s)}{\Delta s} = -\mathcal{L}[tf(t)]|_s + 0 \quad ,$$

verifying both the differentiability of F at s, and equation (25.7).

Additional Exercises

25.1. Find the Laplace transform Y(s) of the solution to each of the following initial-value problems. Just find Y(s) using the ideas illustrated in examples 25.1 and 25.2. Do NOT solve the problem using methods developed before we started discussing Laplace transforms and then computing the transform! Also, do not attempt to recover y(t) from each Y(s) you obtain.

a.
$$y' + 4y = 0$$
 with $y(0) = 3$
b. $y' - 2y = t^3$ with $y(0) = 4$
c. $y' + 3y = \text{step}_4(t)$ with $y(0) = 0$
d. $y'' - 4y = t^3$ with $y(0) = 1$ and $y'(0) = 3$
e. $y'' + 4y = 20e^{4t}$ with $y(0) = 3$ and $y'(0) = 12$
f. $y'' + 4y = \sin(2t)$ with $y(0) = 3$ and $y'(0) = 5$
g. $y'' + 4y = 3 \text{step}_2(t)$ with $y(0) = 0$ and $y'(0) = 5$
h. $y'' + 5y' + 6y = e^{4t}$ with $y(0) = 1$ and $y'(0) = 0$
i. $y'' - 5y' + 6y = t^2e^{4t}$ with $y(0) = 0$ and $y'(0) = 2$
j. $y'' - 5y' + 6y = 7$ with $y(0) = 2$ and $y'(0) = 4$
k. $y'' - 4y' + 13y = e^{2t}\sin(3t)$ with $y(0) = 4$ and $y'(0) = 3$
l. $y'' + 4y' + 13y = 4t + 2e^{2t}\sin(3t)$ with $y(0) = 4$ and $y'(0) = 3$
m. $y''' - 27y = e^{-3t}$ with $y(0) = 2$, $y'(0) = 3$ and $y''(0) = 4$

- **25.2.** Compute the Laplace transforms of the following functions using the given tables and the 'derivative of the transforms identities' from theorem 25.3 (and its corollary).
 - **a.** $t \cos(3t)$ **b.** $t^2 \sin(3t)$ **c.** te^{-7t} **d.** $t^3 e^{-7t}$ **e.** $t \operatorname{step}(t-3)$
- **25.3 a.** Verify the following identities using the 'derivative of the transforms identities' from theorem 25.3.
 - i. $\mathcal{L}[t\sin(\omega t)]|_s = \frac{2\omega s}{(s^2 + \omega^2)^2}$ ii. $\mathcal{L}[t\cos(\omega t)]|_s = \frac{s^2 \omega^2}{(s^2 + \omega^2)^2}$
 - **b.** Now, using the above results and linearity, verify that

$$\mathcal{L}[\sin(\omega t) - \omega t \cos(\omega t)]|_{s} = \frac{2\omega^{3}}{\left(s^{2} + \omega^{2}\right)^{2}}$$

25.4. For the following, y is the solution to

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0$$
 with $y(0) = 1$ and $y'(0) = 0$

The above differential equation, known as Bessel's equation of order zero, is important in many two-dimensional problems involving circular symmetry. The solution to this equation with the above initial values is also important in this problems. This solution is called Bessel's function (of the first kind) of order zero, and is universally denoted by J_0 . Thus, in the following,

 $y(t) = J_0(t)$ and $Y(s) = \mathcal{L}[y(t)]|_s = \mathcal{L}[J_0(t)]|_s$.

a. Using the differentiation identities from this chapter, show that

$$\left(s^2+1\right)\frac{dY}{ds}+sY = 0 \quad .$$

- **b.** The above differential equation for *Y* is a simple first-order differential equation. Find its general solution.
- **c.** It can be shown (trust me on this) that

$$\int_0^\infty J_0(t) = 1$$

What does this tell you about Y(0)?

- **d.** Using what you now know about Y(s), find $= \mathcal{L}[J_0(t)]|_s$.
- **25.5.** Compute the Laplace transforms using the tables provided. You will have to apply two different identities.

a.
$$te^{4t} \sin(3t)$$
 b. $te^{4t} \cos(3t)$ **c.** $te^{4t} \operatorname{step}(t-3)$

- **25.6.** The following concern the ramp at α function whose transform was computed in example 25.4 on page 516. Assume $\alpha > 0$.
 - **a.** Verify that the "ramp-squared at α " function,

$$\operatorname{ramp}_{\alpha}^{2}(t) = (\operatorname{ramp}_{\alpha}(t))^{2}$$

satisfies

$$\operatorname{ramp}_{\alpha}^{2}(t) = \int_{0}^{t} 2 \operatorname{ramp}_{\alpha}(\tau) d\tau$$

b. Using the above and the "transform of an integral" identity, find $\mathcal{L}[\operatorname{ramp}_{\alpha}^{2}(t)]|_{s}$.

25.7. The sine-integral function, Si, is given by

$$\operatorname{Si}(t) = \int_0^t \frac{\sin(\tau)}{\tau} d\tau$$

In example 25.5, it is shown that

$$\mathcal{L}\left[\frac{\sin(t)}{t}\right]\Big|_{s} = \arctan\left(\frac{1}{s}\right) \quad .$$

What is $\mathcal{L}[Si(t)]|_{s}$?

25.8. Verify that the limit of each of the following functions as $t \to 0$ is a finite number, and then find the Laplace transform of that function using the "integral of a transform" identity.

a.
$$\frac{1-e^{-t}}{t}$$
 b. $\frac{e^{2t}-1}{t}$ **c.** $\frac{e^{-2t}-e^{3t}}{t}$
d. $\frac{1-\cos(t)}{t}$ **e.** $\frac{1-\cosh(t)}{t}$ **f.** $\frac{\sin(3t)}{t}$