13

Reduction of Order

We shall take a brief break from developing the general theory for linear differential equations to discuss one method (the "reduction of order method") for finding the general solution to any linear differential equation. In some ways, this method may remind you of the material in chapter 11. Indeed, part of the method involves solving a higher-order equation via first-order methods as discussed in chapter 11. The general theory developed in chapter 12 will not, however, be used to any great extent. Instead, the material developed here will help us finish that general theory (at least partially confirming the suspicions raised at the end of the chapter), and will help lead us to the complete result on constructing general solutions from particular solutions.

But why worry about completing that general theory if any linear differential equation can be completely solved by this "reduction of order method"? Because this method requires that one solution to the differential equation already be known. This limits the method's applicability. Also, serious practical difficulties arise when the differential equation to be solved is of order three or more. Still, there are situations where the method is of practical value, and it will help us confirm suspicions we already have about general solutions.

Oh yes, there is another reason to develop this method: A rather powerful method for solving nonhomogeneous equations, the "variation of parameters" method described in chapter 23, is simply a clever refinement of the reduction of order method.

13.1 The General Idea

The "reduction of order method" is a method for converting any linear differential equation to another linear differential equation of lower order, and then constructing the general solution to the original differential equation using the general solution to the lower-order equation. In general, to use this method with an N^{th}-order linear differential equation

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = g \quad ,$$

we need one known nontrivial solution $y_1 = y_1(x)$ to the corresponding homogeneous differential equation. We then try a substitution of the form

$$y = y_1 u$$

where $u = u(x)$ is a yet unknown function (and $y_1 = y_1(x)$ is the aforementioned known solution). Plugging this substitution into the differential equation then leads to a linear differential equation for u. As we will see, because y_1 satisfies the homogeneous equation, the differential equation for u ends up being of the form

$$A_0 u^{(N)} + A_1 u^{(N-1)} + \cdots + A_{N-2} u'' + A_{N-1} u' = g$$

— remarkably, there is no " $A_N u$ " term. This means we can use the substitution

$$v = u' \quad ,$$

as discussed in chapter 11, to rewrite the differential equation for u as a $(N-1)^{\text{th}}$-order differential equation for v,

$$A_0 v^{(N-1)} + A_1 v^{(N-2)} + \cdots + A_{N-2} v' + A_{N-1} v = g \quad .$$

So we have reduced the order of the equation to be solved. If a general solution $v = v(x)$ for this equation can be found, then the most general formula for u can be obtained from v by integration (since $u' = v$). Finally then, by going back to the original substitution formula $y = y_1 u$, we can obtain a general solution to the original differential equation.

This method is especially useful for solving second-order homogeneous linear differential equations since (as we will see) it reduces the problem to one of solving relatively simple first-order differential equations. Accordingly, we will first concentrate on its use in finding general solutions to second-order, homogeneous linear differential equations. Then we will briefly discuss using reduction of order with linear homogeneous equations of higher order, and with nonhomogeneous linear equations.

13.2 Reduction of Order for Homogeneous Linear Second-Order Equations

The Method

Here we lay out the details of the "reduction of order method" for second-order homogeneous linear differential equations. To illustrate the method, we'll use the differential equation

$$x^2 y'' - 3xy' + 4y = 0 \quad .$$

Note that the first coefficient, x^2, vanishes when $x = 0$. From comments made in chapter 12 (see page 261), we should suspect that $x = 0$ ought not be in any interval of interest for this equation. So we will be solving over the intervals $(0, \infty)$ and $(-\infty, 0)$.

Before starting the reduction of order method, we need one nontrivial solution y_1 to our differential equation. Ways for finding that first solution will be discussed in later chapters. For now let us just observe that if

$$y_1(x) = x^2 \quad ,$$

then

$$\begin{aligned} x^2 y_1'' - 3x y_1' + 4y_1 &= x^2 \frac{d^2}{dx^2}\left[x^2\right] - 3x \frac{d}{dx}\left[x^2\right] + 4\left[x^2\right] \\ &= x^2[2 \cdot 1] - 3x[2x] + 4x^2 \\ &= x^2 \underbrace{[2 - (3 \cdot 2) + 4]}_{0\,!} = 0 \quad . \end{aligned}$$

Thus, one solution to the above differential equation is $y_1(x) = x^2$.

As already stated, this method is for finding a general solution to some homogeneous linear second-order differential equation

$$ay'' + by' + cy = 0$$

(where a, b, and c are 'known functions' with $a(x)$ never being zero on the interval of interest). We will assume that we already have one nontrivial particular solution $y_1(x)$ to this generic differential equation.

For our example (as already noted), we will seek a general solution to

$$x^2y'' - 3xy' + 4y = 0 \quad . \tag{13.1}$$

The one (nontrivial) solution we know is $y_1(x) = x^2$.

Here, now, are the details in using the reduction of order method to solve the above:

1. Let

$$y = y_1 u$$

where $u = u(x)$ is a function yet to be determined. To simplify the "plugging into the differential equation", go ahead and compute the corresponding formulas for the derivatives y' and y'' using the product rule:

$$y' = (y_1u)' = y_1{}'u + y_1u'$$

and

$$\begin{aligned} y'' = \left(y'\right)' &= \left(y_1{}'u + y_1u'\right)' \\ &= \left(y_1{}'u\right)' + \left(y_1u'\right)' \\ &= \left(y_1{}''u + y_1{}'u'\right) + \left(y_1{}'u' + y_1u''\right) \\ &= y_1{}''u + 2y_1{}'u' + y_1u'' \quad . \end{aligned}$$

For our example,

$$y = y_1 u = x^2u$$

where $u = u(x)$ *is the function yet to be determined. The derivatives of* y *are*

$$y' = \left(x^2u\right)' = 2xu + x^2u'$$

and

$$\begin{aligned} y'' = (y')' &= \left(2xu + x^2u'\right)' \\ &= (2xu)' + \left(x^2u'\right)' \\ &= \left(2u + 2xu'\right) + \left(2xu' + x^2u''\right) \\ &= 2u + 4xu' + x^2u'' \quad . \end{aligned}$$

2. Plug the formulas just computed for y, y' and y'' into the differential equation, group together the coefficients for u and each of its derivatives, and simplify as far as possible. (We'll do this with the example first and then look at the general case.)

Plugging the formulas just computed above for y, y' and y'' into equation (13.1), we get

$$\begin{aligned} 0 &= x^2 y'' - 3xy' + 4y \\ &= x^2\left[2u + 4xu' + x^2u''\right] - 3x\left[2xu + x^2u'\right] + 4\left[x^2u\right] \\ &= 2x^2u + 4x^3u' + x^4u'' - 6x^2u - 3x^3u' + 4x^2u \\ &= x^4u'' + \left[4x^3 - 3x^3\right]u' + \left[2x^2 - 6x^2 + 4x^2\right]u \\ &= x^4u'' + x^3u' + 0 \cdot u \quad . \end{aligned}$$

Notice that the u term drops out! So the resulting differential equation for u is simply

$$x^4u'' + x^3u' = 0 \quad ,$$

which we can further simplify by dividing out x^4,

$$u'' + \frac{1}{x}u' = 0$$

In general, plugging in the formulas for y and its derivatives into the given differential equation yields

$$\begin{aligned} 0 &= ay'' + by' + cy \\ &= a\left[y_1''u + 2y_1'u' + y_1u''\right] + b\left[y_1'u + y_1u'\right] + c\left[y_1u\right] \\ &= ay_1''u + 2ay_1'u' + ay_1u'' + by_1'u + by_1u' + cy_1u \\ &= ay_1u'' + \left[2ay_1' + by_1\right]u' + \left[ay_1'' + by_1' + cy_1\right]u \quad . \end{aligned}$$

That is, the differential equation becomes

$$Au'' + Bu' + Cu = 0$$

where

$$A = ay_1 \quad , \quad B = 2ay_1' + by_1 \quad \text{and} \quad C = ay_1'' + by_1' + cy_1 \quad .$$

But remember, y_1 is a solution to the homogeneous equation

$$ay'' + by' + cy = 0 \quad .$$

Consequently,

$$C = ay_1'' + by_1' + cy_1 = 0 \quad ,$$

and the differential equation for u automatically reduces to

$$Au'' + Bu' = 0 \quad .$$

The u term always drops out!

3. Now find the general solution to the second-order differential equation just obtained for u,

$$Au'' + Bu' = 0 \quad ,$$

via the substitution method discussed in section 11.1:

(a) Let $u' = v$ (and, thus, $u'' = v' = {}^{dv}/_{dx}$) to convert the second-order differential equation for u to the first-order differential equation for v ,

$$A\frac{dv}{dx} + Bv = 0 \quad .$$

(It is worth noting that this first-order differential equation will be both linear and separable.)

(b) Find the general solution $v(x)$ to this first-order equation. (Since it is both linear and separable, you can solve it using either the procedure developed for first-order linear equations or the approach developed for first-order separable equations.)

(c) Using the formula just found for v , integrate the substitution formula $u' = v$ to obtain the formula for u ,

$$u(x) = \int v(x)\,dx \quad .$$

Don't forget all the arbitrary constants.

In our example, we obtained

$$u'' + \frac{1}{x}u' = 0 \quad .$$

Letting $v = u'$ *and, thus,* $v' = u''$ *this becomes*

$$\frac{dv}{dx} + \frac{1}{x}v = 0 \quad .$$

Equivalently,

$$\frac{dv}{dx} = -\frac{1}{x}v \quad .$$

This is a relatively simple separable first-order equation. It has one constant solution, $v = 0$ *. To find the others, we divide through by* v *and proceed as usual with such equations:*

$$\frac{1}{v}\frac{dv}{dx} = -\frac{1}{x}$$

$$\hookrightarrow \qquad \int \frac{1}{v}\frac{dv}{dx}\,dx = -\int \frac{1}{x}\,dx$$

$$\hookrightarrow \qquad \ln|v| = -\ln|x| + c_0$$

$$\hookrightarrow \qquad v = \pm e^{-\ln|x| + c_0}$$

$$\hookrightarrow \qquad v = \pm e^{c_0}x^{-1} \quad .$$

Letting $A = \pm e^{c_0}$ *, this simplifies to*

$$v = \frac{A}{x} \quad ,$$

which also accounts for the constant solution (when $A = 0$ *).*

Since $u' = v$ *, it then follows that*

$$u(x) = \int v(x)\,dx = \int \frac{A}{x}\,dx = A\ln|x| + B \quad .$$

4. Finally, plug the formula just obtained for $u(x)$ into the first substitution,

$$y = y_1 u \quad ,$$

used to convert the original differential equation for y to a differential equation for u. The resulting formula for $y(x)$ will be a general solution for that original differential equation. (Sometimes that formula can be simplified a little. Feel free to do so.)

In our example, the solution we started with was $y_1(x) = x^2$. *Combined with the* $u(x)$ *just found, we have*

$$y = y_1 u = x^2[A \ln|x| + B] \quad .$$

That is,

$$y(x) = Ax^2 \ln|x| + Bx^2$$

is the general solution to equation (13.1).

An Observation About the Solution

Let us observe that, in the example, the general solution obtained was

$$y(x) = Ax^2 \ln|x| + Bx^2 \quad ,$$

which can be viewed as a linear combination of the two functions

$$y_1(x) = x^2 \qquad \text{and} \qquad y_2(x) = x^2 \ln|x| \quad .$$

Since the A and B in the above formula for $y(x)$ are arbitrary constants, and y_2 is given by that formula for y with $A = 1$ and $B = 0$, it must be that this y_2 is another particular solution to our original homogeneous linear differential equation. What's more, it is clearly not a constant multiple of y_1. This should strengthen an earlier suspicion that the general solution to a homogeneous linear second-order differential equation can be written as just such a linear combination.

We will later examine more closely the general form for general solution to any homogeneous linear differential equation. In the meantime, while practicing this method, do observe that the general solution you obtain for each second-order homogeneous linear differential equation can, invariably, be written as

$$y(x) = Ay_2(x) + By_1(x)$$

where y_1 and y_2 are two solutions that are not constant multiples of each other. Keep in mind that this form is the same as the form earlier anticipated, namely,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad .$$

We've just renamed the arbitrary constants from c_1 and c_2 to B and A, respectively.

13.3 Reduction of Order for Nonhomogeneous Linear Second-Order Equations

If you look back over our discussion in section 13.2, you will see that the reduction of order method applies almost as well in solving a nonhomogeneous equation

$$ay'' + by' + cy = g \quad ,$$

provided that "one solution y_1" is a solution to the corresponding *homogeneous* equation

$$ay'' + by' + cy = 0 \quad .$$

Then, letting $y = y_1 u$ in the nonhomogeneous equation and then replacing u' with v leads to an equation of the form

$$Av' + Bv = g$$

instead of

$$Av' + Bv = 0 \quad .$$

So we don't end up with a first-order equation for v which is both separable and linear; it is just linear. Still, we know how to solve such equations. Solving that first-order linear differential equation for v and continuing with the method already described finally yields the general solution to the desired nonhomogeneous differential equation.

We will do one example. Then I'll tell you why the method is rarely used in practice.

!►Example 13.1: *Let us try to solve the second-order nonhomogeneous linear differential equation*

$$x^2y'' - 3xy' + 4y = \sqrt{x} \tag{13.2}$$

over the interval $(0, \infty)$.

As we saw in our main example in the section 13.2, the corresponding homogeneous equation

$$x^2y'' - 3xy' + 4y = 0.$$

has $y_1(x) = x^2$ *as one solution (in fact, from that example, we know the entire general solution to this homogeneous equation, but we only need this one particular solution for the method). Let*

$$y = y_1 u = x^2 u$$

where $u = u(x)$ *is the function yet to be determined. The derivatives of* y *are*

$$y' = \left(x^2u\right)' = 2xu + x^2u'$$

and

$$\begin{aligned} y'' = (y')' &= \left(2xu + x^2u'\right)' \\ &= 2u + 2xu' + 2xu' + x^2u'' \\ &= 2u + 4xu' + x^2u'' \quad . \end{aligned}$$

Plugging these into equation (13.2) yields

$$\begin{aligned}\sqrt{x} &= x^2y'' - 3xy' + 4y \\ &= x^2\big[2u + 4xu' + x^2u''\big] - 3x\big[2xu + x^2u'\big] + 4\big[x^2u\big] \\ &= 2x^2u + 4x^3u' + x^4u'' - 6x^2u - 3x^3u' + 4x^2u \\ &= x^4u'' + \big[4x^3 - 3x^3\big]u' + \big[2x^2 - 6x^2 + 4x^2\big]u \\ &= x^4u'' + x^3u' + 0\cdot u \quad .\end{aligned}$$

As before, the u *term drops out. In this case, we are left with*

$$x^4u'' + x^3u' = \sqrt{x} \quad .$$

That is,

$$x^4v' + x^3v = x^{1/2} \qquad \text{with} \quad v = u' \quad .$$

This is a relatively simple first-order linear equation. To help find the integrating factor, we now divide through by x^4 *, obtaining*

$$\frac{dv}{dx} + \frac{1}{x}v = x^{-7/2} \quad .$$

Thus, the integrating factor is

$$\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln|x|} = |x| \quad .$$

Since we are just attempting to solve over the interval $(0, \infty)$ *, we really just have*

$$\mu = x \quad .$$

Multiplying the last differential equation for v *and proceeding as usual when solving first-order linear differential equations:*

$$x\left[\frac{dv}{dx} + \frac{1}{x}v\right] = x\left[x^{-7/2}\right]$$

$$\hookrightarrow \qquad x\frac{dv}{dx} + v = x^{-5/2}$$

$$\hookrightarrow \qquad \frac{d}{dx}\big[xv\big] = x^{-5/2}$$

$$\hookrightarrow \qquad \int \frac{d}{dx}\big[xv\big]\,dx = \int x^{-5/2}\,dx$$

$$\hookrightarrow \qquad xv = -\frac{2}{3}x^{-3/2} + c_1$$

$$\hookrightarrow \qquad v = -\frac{2}{3}x^{-5/2} + \frac{c_1}{x}$$

Recalling that $v = u'$ *, we can rewrite the last line as*

$$\frac{du}{dx} = -\frac{2}{3}x^{-5/2} + \frac{c_1}{x} \quad .$$

Thus,

$$u = \int \frac{du}{dx}\,dx = \int \left[-\frac{2}{3}x^{-5/2} + \frac{c_1}{x} \right] dx$$
$$= \left(\frac{2}{3}\right)^2 x^{-3/2} + c_1 \ln(x) + c_2$$
$$= \frac{4}{9}x^{-3/2} + c_1 \ln(x) + c_2 \quad ,$$

and the general solution to our nonhomogeneous equation is

$$y(x) = x^2 u(x) = x^2 \left[\frac{4}{9}x^{-3/2} + c_1 \ln(x) + c_2 \right]$$
$$= \frac{4}{9}x^{1/2} + c_1 x^2 \ln(x) + c_2 x^2 \quad .$$

For no obvious reason at this point, let's observe that we can write this solution as

$$y(x) = c_1 x^2 \ln(x) + c_2 x^2 + \frac{4}{9}\sqrt{x} \quad . \tag{13.3}$$

It should be observed that, in the above example, we only used one particular solution, $y_1(x) = x^2$, to the homogeneous differential equation

$$x^2 y'' - 3xy' + 4y = 0$$

even though we had already found the general solution

$$Ax^2 \ln|x| + Bx^2 \quad .$$

Later, in chapter 23, we will develop a refinement of the reduction of order method for solving second-order nonhomogeneous linear differential equations which makes use of the entire general solution to the corresponding homogeneous equation. This refinement (the "variation of parameters" method) has two distinct advantages over the reduction of order method when solving *non*homogeneous differential equations:

1. The computations required for the refined procedure tend to be simpler and more easily carried out.
2. With a few straightforward modifications, the refined procedure readily extends to being a useful method for dealing with nonhomogeneous linear differential equations of any order. For the reasons discussed in the next section, the same cannot be said about the basic reduction of order method.

That is why, in practice, the basic reduction of order method is rarely used with nonhomogeneous equations.

13.4 Reduction of Order in General

In theory, reduction of order can be applied to any linear equation of any order, homogeneous or not. Whether it's application is useful is another issue.

!► ***Example 13.2:*** *Consider the third-order homogeneous linear differential equation*

$$y''' - 8y = 0 \quad . \tag{13.4}$$

If you rewrite this equation as

$$y''' = 8y \quad ,$$

and think about what happens when you differentiate exponentials, you will realize that

$$y_1(x) = e^{2x}$$

is 'obviously' a solution to our differential equation (verify it yourself). Letting

$$y = y_1 u = e^{2x} u$$

and repeatedly using the product rule, we get

$$y' = \left(e^{2x}u\right)' = 2e^{2x}u + e^{2x}u' \quad ,$$

$$\begin{aligned} y'' = \left(e^{2x}u\right)'' &= \left(2e^{2x}u + e^{2x}u'\right)' \\ &= 4e^{2x}u + 2e^{2x}u' + 2e^{2x}u' + e^{2x}u'' \\ &= 4e^{2x}u + 4e^{2x}u' + e^{2x}u'' \quad , \end{aligned}$$

and

$$\begin{aligned} y''' = \left(e^{2x}u\right)''' &= \left(4e^{2x}u + 4e^{2x}u' + e^{2x}u''\right)' \\ &= 8e^{2x}u + 4e^{2x}u' + 8e^{2x}u' + 4e^{2x}u'' + 2e^{2x}u'' + e^{2x}u''' \\ &= 8e^{2x}u + 12e^{2x}u' + 6e^{2x}u'' + e^{2x}u''' \quad . \end{aligned}$$

So, using $y = e^{2x}u$ *,*

$$\begin{aligned} y''' - 8y &= 0 \\ \hookrightarrow \quad \left[8e^{2x}u + 12e^{2x}u' + 6e^{2x}u'' + e^{2x}u'''\right] - 8\left[e^{2x}u\right] &= 0 \\ \hookrightarrow \quad e^{2x}u''' + 6e^{2x}u'' + 12e^{2x}u' + \left[8e^{2x} - 8e^{2x}\right]u &= 0 \quad . \end{aligned}$$

Again, the u *term cancel out, leaving us with*

$$e^{2x}u''' + 6e^{2x}u'' + 12e^{2x}u' = 0 \quad .$$

Letting $v = u'$ *and dividing out the exponential, this becomes the second-order differential equation*

$$v'' + 6v' + 12v = 0 \quad . \tag{13.5}$$

Thus we have changed the problem of solving the third-order differential equation to one of solving a second-order differential equation. If we can now correctly guess a particular solution v_1 *to that second-order differential equation, we could again use reduction of order to get the general solution* $v(x)$ *to that second-order equation, and then use that and the fact that* $y = e^{2x}u$ *with* $v = u'$ *to obtain the general solution to our original differential equation. Unfortunately, even though the order is less, "guessing" a solution to equation (13.5) is a good deal more difficult than was guessing a particular solution to the original differential equation, equation (13.4).*

As the example illustrates, even if we can, somehow, obtain one particular solution to a given N^{th}-order linear homogeneous linear differential equation, and then use it to reduce the problem to solving an $(N-1)^{\text{th}}$-order differential equation, that lower order differential equation may be just as hard to solve as the original differential equation (unless $N = 2$). In fact, we will learn how to solve differential equations such as equation (13.5), but those methods can also be used to find the general solution to the original differential equation, equation (13.4), as well.

Still it does no harm to know that the problem of solving an N^{th}-order linear homogeneous linear differential equation can reduced to that of solving an $(N-1)^{\text{th}}$-order differential equation, especially since we may refer to this fact in the next chapter. For the record, here is a theorem to that effect:

Theorem 13.1 (reduction of order in homogeneous equations)
Let y *be any solution to some* N^{th}*-order homogeneous differential equation*

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = g \tag{13.6}$$

where g *and the* a_k*'s are known functions on some interval of interest* $\mathcal{I}$*, and let* y_1 *be a nontrivial particular solution to the corresponding homogeneous equation*

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0 \quad .$$

Set

$$u = \frac{y}{y_1} \qquad (\text{so that } y = y_1 u) \quad .$$

Then $v = u'$ *satisfies an* $(N-1)^{th}$*-order differential equation*

$$A_0 v^{(N-1)} + A_1 v^{(N-2)} + \cdots + A_{N-2} v' + A_{N-1} v = g \quad .$$

where the A_k*'s are functions on the interval* $\mathcal{I}$ *that can be determined from the* a_k*'s along with* y_1 *and its derivatives.*

The proof is relatively straightforward: You see what happens when you repeatedly use the product rule with $y = y_1 u$, and plug the results into the equation (13.6). I'll leave the details to you (see exercise 13.3).

Additional Exercises

13.1. *For each of the following, first verify that the given y_1 is a solution to the given differential equation, and then find the general solution to the differential equation using the given y_1 with the method of reduction of order.*

a. $y'' - 5y' + 6y = 0 \quad , \quad y_1(x) = e^{2x}$

b. $y'' - 10y' + 25y = 0 \quad , \quad y_1(x) = e^{5x}$

c. $x^2y'' - 6xy' + 12y = 0 \quad , \quad y_1(x) = x^3$

d. $4x^2y'' + y = 0 \quad$ *on* $\quad x > 0 \quad , \quad y_1(x) = \sqrt{x}$

e. $y'' - \left(4 + \frac{2}{x}\right)y' + \left(4 + \frac{4}{x}\right)y = 0 \quad , \quad y_1(x) = e^{2x}$

f. $y'' - \frac{1}{x}y' - 4x^2y = 0 \quad , \quad y_1(x) = e^{-x^2}$

g. $y'' + y = 0 \quad , \quad y_1(x) = \sin(x)$

h. $xy'' + (2 + 2x)y' + 2y = 0 \quad , \quad y_1(x) = x^{-1}$

i. $\sin^2(x)y'' - 2\cos(x)\sin(x)y' + \left(1 + \cos^2(x)\right)y = 0 \quad , \quad y_1(x) = \sin(x)$

j. $x^2y'' - 2xy' + \left(x^2 + 2\right)y = 0 \quad , \quad y_1(x) = x\sin(x)$

k. $x^2y'' + xy' + y = 0 \quad , \quad y_1(x) = \sin(\ln|x|)$

l. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \quad , \quad y_1(x) = x^{-1/2}\cos(x)$

13.2. *Several nonhomogeneous differential equations are given below. For each, first verify that the given y_1 is a solution to the corresponding homogeneous differential equation, and then find the general solution to the given nonhomogeneous differential equation using reduction of order with the given y_1.*

a. $y'' - 4y' + 3y = 9e^{2x} \quad , \quad y_1(x) = e^{3x}$

b. $y'' - 6y' + 8y = e^{4x} \quad , \quad y_1(x) = e^{2x}$

c. $x^2y'' + xy' - y = \sqrt{x} \quad , \quad y_1(x) = x$

d. $x^2y'' - 20y = 27x^5 \quad , \quad y_1(x) = x^5$

e. $xy'' + (2 + 2x)y' + 2y = 8e^{2x} \quad , \quad y_1 = x^{-1}$

f. $(x + 1)y'' + xy' - y = (x + 1)^2 \quad , \quad y_1 = e^{-x}$

13.3. *Prove the claims in theorem 13.1 assuming:*

a. $N = 3$ **b.** $N = 4$ **c.** N *is any positive integer*

13.4. *Each of the following is one of the relatively few third- and fourth-order differential equations that can be easily solved via reduction of order. For each, first verify that the* *given y_1 is a solution to the given differential equation or to the corresponding homogeneous equation (as appropriate), and then find the general solution to the differential equation using the given y_1 with the method of reduction of order.*

a. $y''' - 9y'' + 27y' - 27y = 0 \quad , \quad y_1 = e^{3x}$

b. $y''' - 9y'' + 27y' - 27y = e^{3x}\sin(x) \quad , \quad y_1 = e^{3x}$

c. $y^{(4)} - 8y''' + 24y'' - 32y' + 16y = 0 \quad , \quad y_1 = e^{2x}$

d. $x^3y''' - 4y'' + 10y' - 12y = 0 \quad , \quad y_1 = x^2$