2

Integration and Differential Equations

Often, when attempting to solve a differential equation, we are naturally led to computing one or more integrals — after all, integration is the inverse of differentiation. Indeed, we have already solved one simple second-order differential equation by repeated integration (the one arising in the simplest falling object model, starting on page 10). Let us now briefly consider the general case where integration is immediately applicable, and also consider some practical aspects of using both the indefinite integral and the definite integral.

2.1 Directly-Integrable Equations

We will say that a given first-order differential equation is *directly integrable* if (and only if) it can be (re)written as

$$\frac{dy}{dx} = f(x) \tag{2.1}$$

where $f(x)$ is some known function of just x (no y's). More generally, any N^{th}-order differential equation will be said to be *directly integrable* if and only if it can be (re)written as

$$\frac{d^N y}{dx^N} = f(x) \tag{2.1$'$}$$

where, again, $f(x)$ is some known function of just x (no y's or derivatives of y).

!▶Example 2.1: *Consider the equation*

$$x^2 \frac{dy}{dx} - 4x = 6 \quad . \tag{2.2}$$

Solving this equation for the derivative:

$$x^2 \frac{dy}{dx} = 4x + 6$$

$$\hookrightarrow \qquad \frac{dy}{dx} = \frac{4x + 6}{x^2} \quad .$$

Since the right-hand side of the last equation depends only on x, we do have

$$\frac{dy}{dx} = f(x) \qquad \left(\textit{with } f(x) = \frac{4x + 6}{x^2} \right) \quad .$$

So equation (2.2) is directly integrable.

!►Example 2.2: *Consider the equation*

$$x^2 \frac{dy}{dx} - 4xy = 6 \quad . \tag{2.3}$$

Solving this equation for the derivative:

$$x^2 \frac{dy}{dx} = 4xy + 6$$

$$\hookrightarrow \qquad \frac{dy}{dx} = \frac{4xy + 6}{x^2} \quad .$$

Here, the right-hand side of the last equation depends on both x *and* y *, not just* x *. So equation (2.3) is* not *directly integrable.*

Solving a directly-integrable equation is easy: First solve for the derivative to get the equation into form (2.1) or (2.1′), then integrate both sides as many times as needed to eliminate the derivatives, and, finally, do whatever simplification seems appropriate.

!►Example 2.3: *Again, consider*

$$x^2 \frac{dy}{dx} - 4x = 6 \quad . \tag{2.4}$$

In example 2.1, we saw that it is directly integrable and can be rewritten as

$$\frac{dy}{dx} = \frac{4x + 6}{x^2} \quad .$$

Integrating both sides of this equation with respect to x *(and doing a little algebra):*

$$\int \frac{dy}{dx}\,dx = \int \frac{4x+6}{x^2}\,dx \tag{2.5}$$

$$\begin{aligned} \hookrightarrow \qquad y(x) + c_1 &= \int \left[\frac{4}{x} + \frac{6}{x^2} \right] dx \\ &= 4 \int x^{-1}\,dx + 6 \int x^{-2}\,dx \\ &= 4 \ln|x| + c_2 - 6x^{-1} + c_3 \end{aligned}$$

where c_1 *,* c_2 *, and* c_3 *are arbitrary constants. Rearranging things slightly and letting* $c = c_2 + c_3 - c_1$ *, this last equation simplifies to*

$$y(x) = 4 \ln|x| - 6x^{-1} + c \quad . \tag{2.6}$$

This is our general solution to differential equation (2.4). Since both $\ln|x|$ *and* x^{-1} *are discontinuous at* $x = 0$ *, the solution can be valid over any interval not containing* $x = 0$ *.*

?►Exercise 2.1: *Consider the differential equation in example 2.2 and explain why the* y *, which is an unknown function of* x *, makes it impossible to completely integrate both sides of*

$$\frac{dy}{dx} = \frac{4xy + 6}{x^2}$$

with respect to x *.*

2.2 On Using Indefinite Integrals

This is a good point to observe that, whenever we take the indefinite integrals of both sides of an equation, we obtain a bunch of arbitrary constants — c_1, c_2, ... (one constant for each integral) — that can be combined into a single arbitrary constant c. In the future, rather than note all the arbitrary constants that arise and how they combine into a single arbitrary constant c that is added to the right-hand side in the end, let us agree to simply add that c at the end. Let's not explicitly note all the intermediate arbitrary constants. If, for example, we had agreed to this before doing the last example, then we could have replaced all that material from equation (2.5) to equation (2.6) with

$$\begin{aligned} \int \frac{dy}{dx}\,dx &= \int \frac{4x+6}{x^2}\,dx \\ \hookrightarrow \qquad y(x) &= \int \left[\frac{4}{x} + \frac{6}{x^2}\right] dx \\ &= 4\int x^{-1}\,dx + 6\int x^{-2}\,dx \\ &= 4\ln|x| - 6x^{-1} + c \quad . \end{aligned}$$

This should simplify our computations a little.

This convention of "implicitly combining all the arbitrary constants" also allows us to write

$$y(x) = \int \frac{dy}{dx}\,dx \tag{2.7}$$

instead of

$$y(x) + \text{some arbitrary constant} = \int \frac{dy}{dx}\,dx \quad .$$

By our new convention, that "some arbitrary constant" is still in equation (2.7) — it's just been moved to the right-hand side of the equation and combined with the constants arising from the integral there.

Finally, like you, this author will get tired of repeatedly saying "where c is an arbitrary constant" when it is obvious that the c (or the c_1 or the A or ...) that just appeared in the previous line is, indeed, some arbitrary constant. So let us not feel compelled to constantly repeat the obvious, and agree that, when a new symbol suddenly appears in the computation of an indefinite integral, then, yes, that is an arbitrary constant. Remember, though, to use different symbols for the different constants that arise when integrating a function already involving an arbitrary constant.

!►*Example 2.4:* *Consider solving*

$$\frac{d^2y}{dx^2} = 18x^2 \quad . \tag{2.8}$$

Clearly, this is directly integrable and will require two integrations. The first integration yields

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx = \int 18x^2\,dx = \frac{18}{3}x^3 + c_1 \quad .$$

Cutting out the middle leaves

$$\frac{dy}{dx} = 6x^3 + c_1 \quad .$$

Integrating this, we have

$$y(x) = \int \frac{dy}{dx}\,dx = \int \left[6x^3 + c_1\right] dx = \frac{6}{4}x^4 + c_1 x + c_2 \quad .$$

So the general solution to equation (2.8) is

$$y(x) = \frac{3}{2}x^4 + c_1 x + c_2 \quad .$$

In practice, rather than use the same letter with different subscripts for different arbitrary constants (as we did in the above example), you might just want to use different letters, say, writing

$$y(x) = \frac{3}{2}x^4 + ax + b$$

instead of

$$y(x) = \frac{3}{2}x^4 + c_1 x + c_2 \quad .$$

This sometimes prevents dumb mistakes due to bad handwriting.

2.3 On Using Definite Integrals

Basic Ideas

We have been using the *indefinite* integral to recover $y(x)$ from ${}^{dy}/_{dx}$ via the relation

$$\int \frac{dy}{dx}\,dx = y(x) + c \quad .$$

Here, c is some constant (which we've agreed to automatically combine with other constants from other integrals).

We could just about as easily have used the corresponding *definite* integral relation

$$\int_a^x \frac{dy}{ds}\,ds = y(x) - y(a) \tag{2.9}$$

to recover $y(x)$ from its derivative. Note that, here, we've used s instead of x to denote the variable of integration. This prevents the confusion that can arise when using the same symbol for both the variable of integration *and* the upper limit in the integral. The lower limit, a, can be chosen to be any convenient value. In particular, if we are also dealing with initial values, then it makes sense to set a equal to the point at which the initial values are given. That way (as we will soon see) we will obtain a general solution in which the undetermined constant is simply the initial value.

Aside from getting it into the form

$$\frac{dy}{dx} = f(x) \quad ,$$

there are two simple steps that should be taken before using the definite integral to solve a first-order, directly-integrable differential equation:

1. Pick a convenient value for the lower limit of integration a. In particular, if the value of $y(x_0)$ is given for some point x_0, set $a = x_0$.

2. Rewrite the differential equation with s denoting the variable instead of x (i.e., replace x with s),
$$\frac{dy}{ds} = f(s) \quad . \tag{2.10}$$

After that, simply integrate both sides of equation (2.10) with respect to s from a to x:

$$\int_a^x \frac{dy}{ds}\,ds = \int_a^x f(s)\,ds$$

$$\hookrightarrow \qquad y(x) - y(a) = \int_a^x f(s)\,ds \quad .$$

Then solve for $y(x)$ by adding $y(a)$ to both sides,

$$y(x) = \int_a^x f(s)\,ds + y(a) \quad . \tag{2.11}$$

This is a general solution to the given differential equation. It should be noted that the integral here is a definite integral. Its evaluation does not lead to any arbitrary constants. However, the value of $y(a)$, until specified, can be anything; so $y(a)$ is the "arbitrary constant" in this general solution.

!▶ ***Example 2.5:*** *Consider solving the initial-value problem*

$$\frac{dy}{dx} = 3x^2 \qquad \text{with} \qquad y(2) = 12 \quad .$$

Since we know the value of $y(2)$, we will use 2 as the lower limit for our integrals. Rewriting the differential equation with s replacing x gives

$$\frac{dy}{ds} = 3s^2 \quad .$$

Integrating this with respect to s from 2 to x:

$$\int_2^x \frac{dy}{ds}\,ds = \int_2^x 3s^2\,ds$$

$$\hookrightarrow \qquad y(x) - y(2) = s^3\Big|_2^x = x^3 - 2^3 \quad .$$

Solving for $y(x)$ (and computing 2^3) then gives us

$$y(x) = x^3 - 8 + y(2) \quad .$$

This is a general solution to our differential equation. To find the particular solution that also satisfies $y(2) = 12$, as desired, we simply replace the $y(2)$ in the general solution with its given value,

$$\begin{aligned} y(x) &= x^3 - 8 + y(2) \\ &= x^3 - 8 + 12 = x^3 + 4 \quad . \end{aligned}$$

Of course, rather than go through the procedure just outlined to solve

$$\frac{dy}{dx} = f(x) \quad ,$$

we could, after determining a and $f(s)$, just plug these into equation (2.11),

$$y(x) = \int_a^x f(s)\,ds + y(a) \quad ,$$

and compute the integral. That is, after all, what we derived for any choice of f.

Advantages of Using Definite Integrals

By using definite integrals instead of indefinite integrals we avoid dealing with arbitrary constants and end up with expressions explicitly involving initial values. This is sometimes convenient.

A much more important advantage of using definite integrals is that they result in concrete, computable formulas even when the corresponding indefinite integrals can*not* be evaluated. Let us look at a classic example.

!►*Example 2.6:* *Consider solving the initial-value problem*

$$\frac{dy}{dx} = e^{-x^2} \qquad \text{with} \qquad y(0) = 0 \quad .$$

In particular, determine the value of $y(x)$ when $x = 10$.

Using indefinite integrals yields

$$y(x) = \int \frac{dy}{dx}\,dx = \int e^{-x^2}\,dx \quad .$$

Unfortunately, this integral was not one you learned to evaluate in calculus.[1] *And if you check the tables, you will discover that no one else has discovered a usable formula for this integral. Consequently, the above formula for $y(x)$ is not very usable. Heck, we can't even isolate an arbitrary constant or see how the solution depends on the initial value.*

On the other hand, using definite integrals, we get

$$\int_0^x \frac{dy}{ds}\,ds = \int_0^x e^{-s^2}\,ds$$

$$\hookrightarrow \qquad y(x) - y(0) = \int_0^x e^{-s^2}\,ds$$

$$\hookrightarrow \qquad y(x) = \int_0^x e^{-s^2}\,ds + y(0) \quad .$$

This last formula explicitly describes how $y(x)$ depends on the initial value $y(0)$. Since we are assuming $y(0) = 0$, this reduces to

$$y(x) = \int_0^x e^{-s^2}\,ds \quad .$$

[1] Well, you could expand e^{-x^2} in a Taylor series and integrate the series.

We still cannot find a computable formula for this integral, but, if we choose a specific value for x *, say,* $x = 10$ *, this expression becomes*

$$y(10) = \int_0^{10} e^{-s^2}\,ds \quad .$$

The value of this integral can be very accurately approximated using any of a number of numerical integration methods such as the trapezoidal rule or Simpson's rule. In practice, of course, we'll just use the numerical integration command in our favorite computer math package (Maple, Mathematica, etc.). Using any such package, you will find that

$$y(10) = \int_0^{10} e^{-s^2}\,ds \approx 0.886 \quad .$$

In one sense,

$$y(x) = \int f(x)\,dx \tag{2.12}$$

and

$$y(x) = \int_a^x f(s)\,ds + y(a) \tag{2.13}$$

are completely equivalent mathematical expressions. In practice, either can be used just about as easily *provided* a reasonable formula for the indefinite integral in (2.12) can be found. If no such formula can be found, however, then expression (2.13) is much more useful because it can still be used, along with a numerical integration routine, to evaluate $y(x)$ for specific values of x . Indeed, one can compute $y(x)$ for a large number of values of x , plot each of these values of $y(x)$ against x , and thereby construct a very accurate approximation of the graph of y .

There are other ways to approximate solutions to differential equations, and we will discuss some of them. However, if you can express your solution in terms of definite integrals — even if the integral must be computed approximately — then it is usually best to do so. The other approximation methods for differential equations are typically more difficult to implement, and more likely to result in poor approximations.

Important "Named" Definite Integrals with Variable Limits

You should be familiar with a number of "named" functions (such as the natural logarithm and the arctangent) that can be given by definite integrals. For the two examples just cited,

$$\ln(x) = \int_1^x \frac{1}{s}\,ds \qquad \text{for} \quad x > 0$$

and

$$\arctan(x) = \int_0^x \frac{1}{1+s^2}\,ds \quad .$$

While $\ln(x)$ and $\arctan(x)$ can be defined independently of these integrals, their alternative definitions do not provide us with particularly useful ways to compute these functions by hand (unless x is something special, such as 1). Indeed, if you need the value of $\ln(x)$ or $\arctan(x)$ for, say, $x = 18$, then you are most likely to "compute" these values by having your calculator or computer or published tables[2] tell you the (approximate) value of $\ln(18)$ or $\arctan(18)$. Thus,

[2] if you are an old-timer

for computational purposes, we might as well just view $\ln(x)$ and $\arctan(x)$ as names for the above integrals, and be glad that their values can be easily looked up electronically or in published tables.

It turns out that other integrals arise often enough in applications that workers dealing with these applications have decided to "name" these integrals, and to have their values tabulated. Two noteworthy "named integrals" are:

- The *error function*, denoted by erf and given by

$$\operatorname{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-s^2}\, ds \quad .$$

- The *sine-integral function*, denoted by Si and given by[3]

$$\operatorname{Si}(x) = \int_0^x \frac{\sin(s)}{s}\, ds \quad .$$

Both of these are considered to be well-known functions, at least among certain groups of mathematicians, scientists and engineers. They (the functions, not the people) can be found in published tables and standard mathematical software (e.g., Maple, Mathematica, and MathCad) alongside such better-known functions as the natural logarithm and the trigonometric functions. Moreover, using tables or software, the value of $\operatorname{erf}(x)$ and $\operatorname{Si}(x)$ for any real value of x can be accurately computed just as easily as can the value of $\arctan(x)$. For these reasons, and because "$\operatorname{erf}(x)$" and "$\operatorname{Si}(x)$" take up less space than the integrals they represent, we will often follow the lead of others and use these function names instead of writing out the integrals.

!►Example 2.7: *In example 2.6, above, we saw that the solution to*

$$\frac{dy}{dx} = e^{-x^2} \qquad \text{with} \qquad y(0) = 0 \quad .$$

is

$$y(x) = \int_0^x e^{-s^2}\, ds \quad .$$

Since this integral is the same as the integral for the error function with $2/\sqrt{\pi}$ *divided out, we can also express our answer as*

$$y(x) = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) \quad .$$

[3] This integral is clearly mis-named since it is not the integral of the sine. In fact, the function being integrated, $\sin(x)/x$, is often called the "sinc" function (pronounced "sink"), so Si should really be called the "sinc-integral function". But nobody does.

2.4 Integrals of Piecewise-Defined Functions

Computing the Integrals

Be aware that the functions appearing in differential equations can be piecewise defined, as in

$$\frac{dy}{dx} = f(x) \qquad \text{where} \quad f(x) = \begin{cases} x^2 & \text{if} \quad x < 2 \\ 1 & \text{if} \quad 2 \leq x \end{cases} \quad .$$

Indeed, two such functions occur often enough that they have their own names: the *step function*, given by

$$\text{step}(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad 0 \leq x \end{cases} \quad ,$$

and the *ramp function*, given by

$$\text{ramp}(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad 0 \leq x \end{cases} \quad .$$

The reasons for these names should be obvious from their graphs (see figure 2.1)

Such functions regularly arise when we attempt to model things reacting to discontinuous influences. For example, if $y(t)$ is the amount of energy produced up to time t by some light-sensitive device, and the rate at which this energy is produced depends proportionally on the intensity of the light received by the device, then

$$\frac{dy}{dt} = \text{step}(t)$$

models the energy production of this device when it's kept in the dark until a light bulb (of unit intensity) is suddenly switched on at $t = 0$.

Computing the integrals of such functions is simply a matter of computing the integrals of the various "pieces", and then putting the integrated pieces together appropriately. Precisely how you do that depends on whether you are using indefinite integrals or definite integrals. Either can be used, but there is a good reason to prefer definite integrals: They automatically yield continuous solutions (if such solutions exist). With indefinite integrals you must do extra work to ensure the necessary continuity. To illustrate the basic ideas, let us solve the differential equation given at the start of this section both ways: first using definite integrals, then using indefinite integrals.

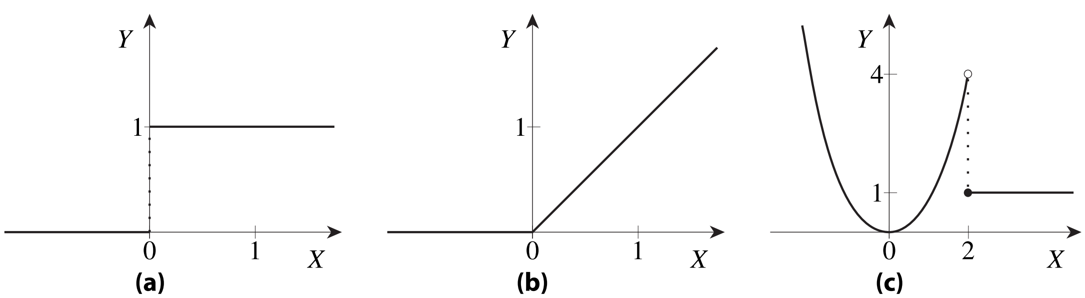

Figure 2.1: Three piecewise defined functions: **(a)** the step function, **(b)** the ramp function, **(c)** $f(x)$ from example 2.8.

!►*Example 2.8 (using definite integrals):* *We seek a general solution to*

$$\frac{dy}{dx} = f(x) \qquad \text{where} \quad f(x) = \begin{cases} x^2 & \text{if} \quad x < 2 \\ 1 & \text{if} \quad 2 \le x \end{cases} \quad .$$

Taking the definite integral (starting, for no good reason, at 0 *), we have*

$$y(x) = \int_0^x f(s)\,ds + y(0) \qquad \text{where} \quad f(s) = \begin{cases} s^2 & \text{if} \quad s < 2 \\ 1 & \text{if} \quad 2 \le s \end{cases} \quad .$$

Now, if $x \le 2$ *, then* $f(s) = s^2$ *for every value of* s *in the interval* $(0, x)$ *. So, when* $x \le 2$ *,*

$$\int_0^x f(s)\,ds = \int_0^x s^2\,ds = \left.\frac{1}{3}s^3\right|_{s=0}^{x} = \frac{1}{3}x^3 \quad .$$

(Notice that this integral is valid for $x = 2$ *even though the formula used for* $f(s)$ *,* s^2 *, was only valid for* $s < 2$ *.)*

On the other hand, if $2 < x$ *, we must break the integral into two pieces, the one over* $(0, 2)$ *and the one over* $(2, x)$ *:*

$$\begin{aligned} \int_0^x f(s)\,ds &= \int_0^2 f(s)\,ds + \int_2^x f(s)\,ds \\ &= \int_0^2 s^2\,ds + \int_2^x 1\,ds \\ &= \left.\frac{1}{3}s^3\right|_{s=0}^{2} + \left. s\right|_{s=2}^{x} \\ &= \left[\frac{1}{3}\cdot 2^3 - 0\right] + [x - 2] = x + \frac{2}{3} \quad . \end{aligned}$$

Thus, our general solution is

$$y(x) = \int_0^x f(s)\,ds + y(0) = \begin{cases} \frac{1}{3}x^3 + y(0) & \text{if} \quad x \le 2 \\ x + \frac{2}{3} + y(0) & \text{if} \quad 2 < x \end{cases} \quad .$$

Keep in mind that solutions to differential equations are required to be continuous. After checking the above formulas, it should be obvious that the $y(x)$ obtained in the last example is continuous everywhere except, possibly, at $x = 2$. With a little work we could also verify that, in fact, we also have continuity at $x = 2$. We simply have to recall the limit definition of continuity, and verify that the appropriate requirements are satisfied. But we won't bother because, in a little bit, it will seen that solutions so obtained via definite integration are guaranteed to be continuous, provided the discontinuities in the function being integrated are not too bad.

On the other hand, as the next example illustrates, the continuity of the solution is an issue when we use indefinite integrals.

!►*Example 2.9 (using indefinite integrals):* *Again, our differential equation is*

$$\frac{dy}{dx} = f(x) \qquad \text{where} \quad f(x) = \begin{cases} x^2 & \text{if} \quad x < 2 \\ 1 & \text{if} \quad 2 \le x \end{cases} \quad .$$

The indefinite integral of $f(x)$ is computed by simply finding the indefinite integral of each "piece", noting the values of the variable for which the integration is valid. Thus,

$$y(x) = \int f(x)\,dx = \begin{Bmatrix} \int x^2\,dx & \text{if} \quad x < 2 \\ \int 1\,dx & \text{if} \quad 2 \le x \end{Bmatrix} = \begin{cases} \frac{1}{3}x^3 + c_1 & \text{if} \quad x < 2 \\ x + c_2 & \text{if} \quad 2 \le x \end{cases} .$$

Again, I remind you that solutions to differential equations are required to be continuous. And, again, it should be obvious that the $y(x)$ just obtained is continuous everywhere except, possibly, at $x = 2$. *Now, recall what is means to say "$y(x)$ is continuous at $x = 2$" — it means*

$$\lim_{x \to 2} y(x) = y(2) \quad .$$

Here, however, $y(x)$ is given by different formulas on either side of $x = 2$. So we will have to consider both the left- and the right-hand limits, and require that

$$\lim_{x \to 2^-} y(x) = y(2) = \lim_{x \to 2^+} y(x) \quad .$$

Using the above set of formulas for $y(x)$, we see that

$$y(2) = 2 + c_2 \quad ,$$

$$\lim_{x \to 2^-} y(x) = \lim_{x \to 2} \left[\frac{1}{3}x^3 + c_1\right] = \frac{1}{3} \cdot 2^3 + c_1 = \frac{8}{3} + c_1$$

and

$$\lim_{x \to 2^+} y(x) = \lim_{x \to 2} [x + c_2] = 2 + c_2 \quad .$$

So our requirement for continuity at $x = 2$,

$$\lim_{x \to 2^-} y(x) = y(2) = \lim_{x \to 2^+} y(x) \quad ,$$

becomes

$$\frac{8}{3} + c_1 = 2 + c_2 \quad .$$

This, in turn, means that the "arbitrary constants" c_1 and c_2 are not completely arbitrary; they must be related by

$$\frac{8}{3} + c_1 = 2 + c_2 \quad .$$

Consequently, we must insist that

$$c_1 = c_2 + 2 - \frac{8}{3} = c_2 - \frac{2}{3}$$

or, equivalently, that

$$c_2 = c_1 + \frac{2}{3} \quad .$$

Choosing the later, we finally get a valid general solution to our differential equation, namely,

$$y(x) = \begin{Bmatrix} \frac{1}{3}x^3 + c_1 & \text{if} \quad x < 2 \\ x + c_2 & \text{if} \quad 2 \le x \end{Bmatrix} = \begin{cases} \frac{1}{3}x^3 + c_1 & \text{if} \quad x < 2 \\ x + \frac{2}{3} + c_1 & \text{if} \quad 2 \le x \end{cases} .$$

In practice, a given piecewise defined function may have more than two "pieces", and the differential equation may have order higher than one. For example, you may be called upon to solve

$$\frac{d^2y}{dx^2} = f(x) \qquad \text{where} \quad f(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 1 & \text{if} \quad 1 \le x < 2 \\ 0 & \text{if} \quad 2 \le x \end{cases}$$

or even something involving infinitely many pieces, such as

$$\frac{d^4y}{dx^4} = \text{stair}(x) \qquad \text{where} \quad \text{stair}(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad 0 \le x < 1 \\ 2 & \text{if} \quad 2 \le x < 3 \\ 3 & \text{if} \quad 3 \le x < 4 \\ 4 & \text{if} \quad 4 \le x < 5 \\ \vdots & \end{cases} . \tag{2.14}$$

The methods illustrated in the two examples can still be applied; you just have more integrals to keep track of, and the accompanying bookkeeping becomes more involved. If you use indefinite integrals, make sure to relate all the "arbitrary" constants to each other so that your solution is continuous. If you use definite integrals, then any concerns about the continuity of your solutions can probably be aleviated by the discussion in the next subsection.

Continuity of the Integrals

The continuity of g, where

$$g(x) = \int_a^x f(s)\,ds \quad ,$$

follows from the fact that this integral is closely related to the area of the region enclosed by the graph of f and the S–axis over the interval between a and x. In particular, if $a < x$ and f is a positive function on (a, x), then this integral is the area of the region bounded above by the graph of f, below by the S–axis, and on the sides by the lines $s = a$ and $s = x$. Changing x by just a little changes the base of this region by just a little, and this changes $g(x)$, the area given by the above integral, by only a little. That, essentially, is what continuity is all about — small changes in x can only result in small changes in $g(x)$.

More generally, f might not be a positive function, and we may be concerned with the above integral when $x < a$. Still, as long as f does not behave too badly, "area arguments" can assure us that the above g is a continuous function of x. What would be bad and would make these arguments impossible would be for the area to become infinite, as happens when we (foolishly) try to integrate x^{-1} across $x = 0$. This, of course, cannot happen unless the function being integrated "blows up" at some point (as x^{-1} does at 0). In practice, this means that such a point cannot be in the interval over which the differential equation can be solved. At best, it is an endpoint of our interval of interest.

All this leads to the following theorem, which gives an easily applied condition under which the continuity of

$$g(x) = \int_a^x f(s)\,ds$$

is guaranteed. It does not give the most general conditions, but it should cover all cases you are likely to encounter in the foreseeable future.

Theorem 2.1

Let f be a function on an interval (α, β) and let a be a point in that interval. Suppose, further, that f is continuous at all but, at most, a finite number of points in (α, β), and that, at each such point x_0 of discontinuity, the left- and right-hand limits

$$\lim_{x \to x_0^-} f(x) \qquad \text{and} \qquad \lim_{x \to x_0^+} f(x)$$

exist (and are finite).[4] *Then the function given by*

$$g(x) = \int_a^x f(s)\, ds$$

is continuous on (α, β).

We will prove this theorem in a little bit. First, let's apply it to verify that the solution obtained in example 2.8 is continuous.

!►*Example 2.10:* *In example 2.8, we integrated to get*

$$y(x) = \int_0^x f(s)\, ds + y(0) = \begin{cases} \frac{1}{3}x^3 + y(0) & \text{if } x \leq 2 \\ x + \frac{2}{3} + y(0) & \text{if } 2 < x \end{cases} \tag{2.15}$$

as a general solution to

$$\frac{dy}{dx} = f(x) \qquad \text{where} \qquad f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 1 & \text{if } 2 \leq x \end{cases} .$$

In this case, $f(x)$ is continuous everywhere except at $x = 2$. Attempting to compute the left- and right-hand limits of $f(x)$ at that point yields

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} x^2 = 2^2 = 4$$

and

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} 1 = 1 .$$

So the one-sided limits exist (as finite values). Theorem 2.1 then assures us that

$$g(x) = \int_0^x f(s)\, ds$$

is a continuous function on $(-\infty, \infty)$. Hence, so is $y(x)$ in formula (2.15), above.

[4] Such such discontinuities are said to be *finite-jump* discontinuities.

Theorem 2.1 can still be applied in those rare instances where the function being integrated has infinitely many discontinuities (as with the "stair function", defined above in line (2.14)), provided the function only has finitely many discontinuities on each finite subinterval, and each of these discontinuities is only a finite-jump discontinuity. For example, the stair function has "jumps" at

$$x = 0, 1, 2, 3, 4, 5, \ldots \quad .$$

However, on the finite interval $(-N, N)$, where N is any positive integer, the only discontinuities of $\text{stair}(x)$ are at

$$x = 0, 1, 2, 3, 4, \ldots, N-1 \quad .$$

So $\text{stair}(x)$ has only a finite number of discontinuities on $(-N, N)$, and each of these is a finite-jump discontinuity. The theorem then tells us that

$$g(x) = \int_0^x \text{stair}(s)\, ds$$

is continuous at each x in $(-N, N)$. Since N can be made as large as we wish, we can conclude that, in fact, $g(x)$ is continuous at every x in $(-\infty, \infty)$.

Now let's prove our theorem. For the proof, we will use facts based on "area arguments" that you should recall from your elementary calculus course.

PROOF (of theorem 2.1): First of all, note that the two requirements placed on f ensure

$$g(x) = \int_a^x f(s)\, ds$$

is well defined for any x in (α, β) using any of the definitions for the integral found in most calculus texts (check this out yourself, using the definition in your calculus text). They also prevent $f(x)$ from "blowing up" on any closed subinterval $[\alpha', \beta']$ of (α, β). Thus, for each such closed subinterval $[\alpha', \beta']$, there is a corresponding finite constant M such that[5]

$$|f(s)| \le M \qquad \text{whenever} \quad \alpha' \le s \le \beta' \quad .$$

Now, to verify the claimed continuity of g, we must show that

$$\lim_{x \to x_0} g(x) = g(x_0) \tag{2.16}$$

for any x_0 in (α, β). But by the definition of g and well-known properties of integration,

$$\begin{aligned} \lim_{x \to x_0} g(x) &= \lim_{x \to x_0} \int_a^x f(s)\, ds \\ &= \lim_{x \to x_0} \left[\int_a^{x_0} f(s)\, ds + \int_{x_0}^x f(s)\, ds \right] \\ &= \lim_{x \to x_0} \left[g(x_0) + \int_{x_0}^x f(s)\, ds \right] \\ &= g(x_0) + \lim_{x \to x_0} \int_{x_0}^x f(s)\, ds \quad . \end{aligned}$$

[5] The constant M can be the maximum value of $|f(s)|$ on $[\alpha', \beta']$, provided that maximum exists. It may change if either endpoint α' or β' is changed.

So, to show equation (2.16) holds, it suffices to confirm that

$$\lim_{x \to x_0} \int_{x_0}^{x} f(s)\,ds = 0 \quad ,$$

which, in turn, is equivalent to confirming that

$$\lim_{x \to x_0{}^+} \left| \int_{x_0}^{x} f(s)\,ds \right| = 0 \qquad \text{and} \qquad \lim_{x \to x_0{}^-} \left| \int_{x_0}^{x} f(s)\,ds \right| = 0 \quad . \tag{2.17}$$

That is what we will show.

To do this, pick any two finite values α' and β' satisfying $\alpha < \alpha' < x_0 < \beta' < \beta$. As noted, there is some finite constant M bigger than $|f(s)|$ on $[\alpha', \beta']$. So, if $x_0 \le x \le \beta'$,

$$0 \le \left| \int_{x_0}^{x} f(s)\,ds \right| \le \int_{x_0}^{x} |f(s)|\,ds \le \int_{x_0}^{x} M\,ds = M[x - x_0] \quad .$$

Similarly, if $\alpha' < x < x_0$, then

$$\begin{aligned} 0 \le \left| \int_{x_0}^{x} f(s)\,ds \right| &= \left| -\int_{x}^{x_0} f(s)\,ds \right| = \left| \int_{x}^{x_0} f(s)\,ds \right| \\ &\le \int_{x}^{x_0} |f(s)|\,ds \le \int_{x}^{x_0} M\,ds = M[x_0 - x] \quad . \end{aligned}$$

Hence,

$$0 \le \lim_{x \to x_0{}^+} \left| \int_{x_0}^{x} f(s)\,ds \right| \le \lim_{x \to x_0{}^+} M[x - x_0] = M[x_0 - x_0] = 0$$

and

$$0 \le \lim_{x \to x_0{}^-} \left| \int_{x_0}^{x} f(s)\,ds \right| \le \lim_{x \to x_0{}^-} M[x_0 - x] = M[x_0 - x_0] = 0 \quad , \tag{2.18}$$

which, of course, means that equation set (2.17) holds.

Additional Exercises

2.2. *Determine whether each of the following differential equations is or is not directly integrable:*

a. $\dfrac{dy}{dx} = 3 - \sin(x)$

b. $\dfrac{dy}{dx} = 3 - \sin(y)$

c. $\dfrac{dy}{dx} + 4y = e^{2x}$

d. $x\dfrac{dy}{dx} = \arcsin(x^2)$

e. $y\dfrac{dy}{dx} = 2x$

f. $\dfrac{d^2y}{dx^2} = \dfrac{x+1}{x-1}$

g. $x^2\dfrac{d^2y}{dx^2} = 1$

h. $y^2\dfrac{d^2y}{dx^2} = 8x^2$

i. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 8y = e^{-x^2}$

2.3. *Find a general solution for each of the following directly integrable equations. (Use indefinite integrals on these.)*

a. $\dfrac{dy}{dx} = 4x^3$

b. $\dfrac{dy}{dx} = 20e^{-4x}$

c. $x\dfrac{dy}{dx} + \sqrt{x} = 2$

d. $\sqrt{x+4}\,\dfrac{dy}{dx} = 1$

e. $\dfrac{dy}{dx} = x\cos\left(x^2\right)$

f. $\dfrac{dy}{dx} = x\cos(x)$

g. $x = \left(x^2 - 9\right)\dfrac{dy}{dx}$

h. $1 = \left(x^2 - 9\right)\dfrac{dy}{dx}$

i. $1 = x^2 - 9\dfrac{dy}{dx}$

j. $\dfrac{d^2y}{dx^2} = \sin(2x)$

k. $\dfrac{d^2y}{dx^2} - 3 = x$

l. $\dfrac{d^4y}{dx^4} = 1$

2.4. *Solve each of the following initial-value problems (using the indefinite integral). Also, state the largest interval over which the solution is valid (i.e., the maximal possible interval of interest).*

a. $\dfrac{dy}{dx} = 4x + 10e^{2x}$ *with* $y(0) = 4$

b. $\sqrt[3]{x+6}\,\dfrac{dy}{dx} = 1$ *with* $y(2) = 10$

c. $\dfrac{dy}{dx} = \dfrac{x-1}{x+1}$ *with* $y(0) = 8$

d. $x\dfrac{dy}{dx} + 2 = \sqrt{x}$ *with* $y(1) = 6$

e. $\cos(x)\dfrac{dy}{dx} - \sin(x) = 0$ *with* $y(0) = 3$

f. $\left(x^2 + 1\right)\dfrac{dy}{dx} = 1$ *with* $y(0) = 3$

g. $x\dfrac{d^2y}{dx^2} + 2 = \sqrt{x}$ *with* $y(1) = 8$ *and* $y'(1) = 6$

2.5 a. *Using definite integrals (as in example 2.5 on page 27), find the general solution to*

$$\frac{dy}{dx} = \sin\left(\frac{x}{2}\right)$$

with $y(0)$ *acting as the arbitray constant.*

b. *Using the formula just found for* $y(x)$ *:*

i. *Find* $y(\pi)$ *when* $y(0) = 0$.

ii. *Find* $y(\pi)$ *when* $y(0) = 3$.

iii. *Find* $y(2\pi)$ *when* $y(0) = 3$.

2.6 a. *Using definite integrals (as in example 2.5 on page 27), find the general solution to*

$$\frac{dy}{dx} = 3\sqrt{x+3}$$

with $y(1)$ *acting as the arbitray constant.*

b. *Using the formula just found for* $y(x)$ *:*

i. *Find* $y(6)$ *when* $y(1) = 16$. **ii.** *Find* $y(6)$ *when* $y(1) = 20$.

iii. *Find* $y(-2)$ *when* $y(1) = 0$.

2.7. *Using definite integrals (as in example 2.5 on page 27), find the solution to each of the following initial-value problems. (In some cases, you may want to use the error function or the sine-integral function.)*

a. $\frac{dy}{dx} = x\,e^{-x^2}$ *with* $y(0) = 3$

b. $\frac{dy}{dx} = \frac{x}{\sqrt{x^2+5}}$ *with* $y(2) = 7$

c. $\frac{dy}{dx} = \frac{1}{x^2+1}$ *with* $y(1) = 0$

d. $\frac{dy}{dx} = e^{-9x^2}$ *with* $y(0) = 1$

e. $x\frac{dy}{dx} = \sin(x)$ *with* $y(0) = 4$

f. $x\frac{dy}{dx} = \sin\left(x^2\right)$ *with* $y(0) = 0$

2.8. *Using an appropriate computer math package (such as Maple, Mathematica or Mathcad), graph each of the following over the interval* $0 \le x \le 10$ *:*

a. *the error function,* $\mathrm{erf}(x)$. **b.** *the sine integral function,* $\mathrm{Si}(x)$.

c. *the solution to*

$$\frac{dy}{dx} = \ln\left|2 + x^2\sin(x)\right| \qquad \text{with} \quad y(0) = 0 \quad .$$

2.9. *Each of the following differential equations involves a function that is (or can be) piecewise defined. Sketch the graph of each of these piecewise defined functions, and find the general solution of each differential equation. If an initial value is also given, then also solve the given initial-value problem:*

a. $\frac{dy}{dx} = \mathrm{step}(x)$ *with* $y(0) = 0$ *and* $\mathrm{step}(x)$ *as defined on page 31*

b. $\frac{dy}{dx} = f(x)$ *with* $y(0) = 2$ *and* $f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } 1 \le x \end{cases}$

c. $\frac{dy}{dx} = f(x)$ *with* $y(0) = 0$ *and* $f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ 0 & \text{if } 2 \le x \end{cases}$

d. $\frac{dy}{dx} = |x - 2|$

e. $\frac{dy}{dx} = \text{stair}(x)$ *with* $y(0) = 0$ *and* $\text{stair}(x)$ *as defined on page 34*