Chapter 30: Power Series Solutions I: Basic Computational Methods

30.3 a. Set

$$y = y(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{and} \qquad y' = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^k\right] = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad .$$

Then

$$\begin{aligned} 0 = y' - 2y &= \sum_{k=1}^{\infty} k a_k x^{k-1} - 2\sum_{k=0}^{\infty} a_k x^k \\ &= \underbrace{\sum_{k=1}^{\infty} k a_k x^{k-1}}_{n=k-1} + \underbrace{\sum_{k=0}^{\infty} (-2) a_k x^k}_{n=k} \\ &= \sum_{n+1=1}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} (-2) a_n x^n \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} (-2) a_n x^n \\ &= \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - 2a_n\right] x^n \quad . \end{aligned}$$

Thus,

$$(n+1)a_{n+1} - 2a_n = 0 \qquad \text{for} \quad n = 0, 1, 2, 3, \dots \quad .$$

Solving for a_{n+1} yields

$$a_{n+1} = \frac{2}{n+1} a_n \qquad \text{for} \quad n = 0, 1, 2, 3, \dots \quad .$$

Reindexing by letting $k = n + 1$ then yields the recursion formula

$$a_k = \frac{2}{k} a_{k-1} \qquad \text{for} \quad k = 1, 2, 3, 4, \dots \quad .$$

Using this,

$$\begin{aligned} a_1 &= \frac{2}{1} a_{1-1} = \frac{2}{1} a_0 \quad , \\ a_2 &= \frac{2}{2} a_{2-1} = \frac{2}{2} a_1 = \frac{2}{2} \cdot \frac{2}{1} a_0 = \frac{2^2}{2 \cdot 1} a_0 \quad , \\ a_3 &= \frac{2}{3} a_{3-1} = \frac{2}{3} a_2 = \frac{2}{3} \cdot \frac{2^2}{2 \cdot 1} a_0 = \frac{2^3}{3 \cdot 2 \cdot 1} a_0 \quad , \\ a_4 &= \frac{2}{4} a_{4-1} = \frac{2}{4} a_3 = \frac{2}{4} \cdot \frac{2^3}{3 \cdot 2 \cdot 1} a_0 = \frac{2^4}{4!} a_0 \quad , \\ &\vdots \end{aligned}$$

Clearly,

$$a_k = \frac{2^k}{k!} a_0 \qquad \text{for} \quad k = 1, 2, 3, 4, \dots \quad .$$

But also observe that, with $k = 0$, the right side of the last equation gives

$$\frac{2^0}{0!}a_0 = \frac{1}{1}a_0 = a_0 \quad .$$

So, in fact,

$$a_k = \frac{2^k}{k!}a_0 \qquad \text{for} \quad k = 0,\ 1,\ 2,\ 3,\ 4,\ \ldots \quad .$$

Hence,

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{2^k}{k!}a_0 x^k = a_0 \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$$

(which happens to also equal $a_0 e^{2x}$).

30.3 c. First, multiply the equation by $2x - 1$ to get it into the preferred form,

$$(2x-1)y' + 2y = 0 \quad .$$

Set

$$y = y(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{and} \qquad y' = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^k\right] = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad .$$

Then

$$\begin{aligned}
0 &= (2x-1)y' + 2y \\
&= (2x-1)\sum_{k=1}^{\infty} k a_k x^{k-1} + 2\sum_{k=0}^{\infty} a_k x^k \\
&= 2x\sum_{k=1}^{\infty} k a_k x^{k-1} - 1\sum_{k=1}^{\infty} k a_k x^{k-1} + 2\sum_{k=0}^{\infty} a_k x^k \\
&= \underbrace{\sum_{k=1}^{\infty} 2k a_k x^k}_{n=k} + \underbrace{\sum_{k=1}^{\infty}(-1)k a_k x^{k-1}}_{n=k-1} + \underbrace{\sum_{k=0}^{\infty} 2a_k x^k}_{n=k} \\
&= \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty}(-1)(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} 2a_n x^n \\
&= \sum_{n=1}^{\infty} 2n a_n x^n + \left[-(0+1)a_{0+1}x^0 + \sum_{n=1}^{\infty}(-1)(n+1)a_{n+1}x^n\right] \\
&\qquad + \left[2a_0 x^0 + \sum_{n=1}^{\infty} 2a_n x^n\right] \\
&= [-a_1 + 2a_0]x^0 + \sum_{n=1}^{\infty}\left[2n a_n - (n+1)a_{n+1} + 2a_n\right]x^n \\
&= [-a_1 + 2a_0]x^0 + \sum_{n=1}^{\infty}\left[2(n+1)a_n - (n+1)a_{n+1}\right]x^n \quad .
\end{aligned}$$

Thus,

$$-a_1 + 2a_0 = 0 \quad \rightarrowtail \quad a_1 = 2a_0 \quad , \qquad (\star)$$

and

$$2(n+1)a_n - (n+1)a_{n+1} = 0 \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

$$\hookrightarrow \qquad a_{n+1} = 2a_n \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

$$\hookrightarrow \qquad a_k = 2a_{k-1} \qquad \text{for} \quad k = 2, 3, 4, \ldots$$

The last equation is the recursion formula. Note that it also holds for $k = 1$ (see line $(\star)$). Using it, we get

$$a_1 = 2a_0 \quad ,$$

$$a_2 = 2a_{2-1} = 2a_1 = 2 \cdot 2a_0 = 2^2 a_0 \quad ,$$

$$a_3 = 2a_{3-1} = 2a_2 = 2 \cdot 2^2 a_0 = 2^3 a_0 \quad ,$$

$$a_4 = 2a_{4-1} = 2a_3 = 2 \cdot 2^3 a_0 = 2^4 a_0 \quad ,$$

$$\vdots$$

Clearly,

$$a_k = 2^k a_0 \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

The last equation also holds trivially for $k = 0$. So.

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} 2^k a_0 x^k = a_0 \sum_{k=0}^{\infty} 2^k x^k \quad ,$$

which is a geometric series that reduces to $\quad y(x) = \dfrac{a_0}{1-2x} \quad .$

30.3 e. The equation is already in preferred form. Plugging the series for y and y' into the differential equation, we get

$$\begin{aligned}
0 &= \left(1 + x^2\right) y' - 2xy \\
&= \left(1 + x^2\right) \sum_{k=1}^{\infty} k a_k x^{k-1} - 2x \sum_{k=0}^{\infty} a_k x^k \\
&= 1 \sum_{k=1}^{\infty} k a_k x^{k-1} + x^2 \sum_{k=1}^{\infty} k a_k x^{k-1} - 2x \sum_{k=0}^{\infty} a_k x^k \\
&= \underbrace{\sum_{k=1}^{\infty} k a_k x^{k-1}}_{n=k-1} + \underbrace{\sum_{k=1}^{\infty} k a_k x^{k+1}}_{n=k+1} + \underbrace{\sum_{k=0}^{\infty} (-2) a_k x^{k+1}}_{n=k+1} \\
&= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + \sum_{n=1}^{\infty} (-2) a_{n-1} x^n
\end{aligned}$$

$$= \left[(0+1)a_{0+1}x^0 + (1+1)a_{1+1}x^1 + \sum_{n=2}^{\infty}(n+1)a_{n+1}x^n\right]$$

$$+ \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + \left[-2a_{1-1}x^1 + \sum_{n=2}^{\infty}(-2)a_{n-1}x^n\right]$$

$$= a_1x^0 + [2a_2 - 2a_0]\,x^1 + \sum_{n=2}^{\infty}\big[(n+1)a_{n+1} + (n-3)a_{n-1}\big]\,x^n \quad .$$

So,

$$a_1 = 0 \quad ,$$

$$2a_2 - 2a_0 = 0 \quad \rightarrowtail \quad a_2 = a_0$$

and

$$(n+1)a_{n+1} + (n-3)a_{n-1} \qquad \text{for} \quad n \geq 2$$

$$\hookrightarrow \qquad a_{n+1} = -\frac{n-3}{n+1}a_{n-1} \qquad \text{for} \quad n \geq 2$$

$$\hookrightarrow \qquad a_k = -\frac{k-1-3}{k}a_{k-1-1} = \frac{4-k}{k}a_{k-2} \qquad \text{for} \quad k \geq 3 \quad .$$

The last is the recursion formula. Using the above:

$$a_1 = 0 \quad ,$$

$$a_2 = \frac{4-2}{2}a_{2-2} = \frac{2}{2}a_0 \quad ,$$

$$a_3 = \frac{4-3}{3}a_{3-2} = \frac{1}{3}a_1 = \frac{1}{3}\cdot 0 = 0 \quad ,$$

$$a_4 = \frac{4-4}{4}a_{4-2} = 0 \quad ,$$

$$a_5 = \frac{4-5}{5}a_{5-2} = \frac{-1}{5}a_3 = \frac{-1}{5}\cdot 0 = 0 \quad ,$$

$$a_6 = \frac{4-6}{6}a_{6-2} = \frac{-2}{6}a_4 = \frac{-2}{6}\cdot 0 = 0 \quad ,$$

$$\vdots$$

Clearly, $a_k = 0$ if $k \geq 3$. Thus,

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2$$

$$= a_0 + 0x + \frac{2}{2}a_0 x^2 = a_0\left[1 + x^2\right] \quad .$$

30.3 g. We get the differential equation into preferred form by multiplying through by $x - 1$:

$$(x-1)y' + y = 0 \quad .$$

Since $x_0 = 3$, we set $Y(X) = y(x)$ with $X = x - 3$. Hence $x = X + 3$,

$$x - 1 = [X + 3] - 1 = X + 2$$

and, in terms of X and Y, the last differential equation is

$$(X+2)Y' + Y = 0 \quad .$$

Plugging in the power series for $Y(X)$ about $X = 0$:

$$\begin{aligned}
0 &= (X+2)Y' + Y \\
&= (X+2)\sum_{k=1}^{\infty} k a_k X^{k-1} + \sum_{k=0}^{\infty} a_k X^k \\
&= X\sum_{k=1}^{\infty} k a_k X^{k-1} + 2\sum_{k=1}^{\infty} k a_k X^{k-1} + \sum_{k=0}^{\infty} a_k X^k \\
&= \underbrace{\sum_{k=1}^{\infty} k a_k X^k}_{n=k} + \underbrace{\sum_{k=1}^{\infty} 2k a_k X^{k-1}}_{n=k-1} + \underbrace{\sum_{k=0}^{\infty} a_k X^k}_{n=k} \\
&= \sum_{n=1}^{\infty} n a_n X^n + \sum_{n=0}^{\infty} 2(n+1)a_{n+1} X^n + \sum_{n=0}^{\infty} a_n X^n \\
&= \sum_{n=1}^{\infty} n a_n X^n + \left[2(0+1)a_{0+1}X^0 + \sum_{n=1}^{\infty} 2(n+1)a_{n+1}X^n\right] \\
&\qquad + \left[a_0 X^0 + \sum_{n=1}^{\infty} a_n X^n\right] \\
&= [2a_1 + a_0]\,X^0 + \sum_{n=1}^{\infty}\big[(n+1)a_n + 2(n+1)a_{n+1}\big]\,X^N \quad .
\end{aligned}$$

Thus,

$$2a_1 + a_0 = 0 \quad \rightarrowtail \quad a_1 = -\frac{1}{2}a_0$$

and

$$(n+1)a_n + 2(n+1)a_{n+1} = 0 \qquad \text{for} \quad n \geq 1$$

$$\hookrightarrow \qquad a_{n+1} = -\frac{1}{2}a_n \qquad \text{for} \quad n \geq 1$$

$$\hookrightarrow \qquad a_k = -\frac{1}{2}a_{k-1} \qquad \text{for} \quad k \geq 2 \quad .$$

The last is the recursion formula. Note that, because of the formula for a_1, it actually holds for $k \geq 1$. Applying this formula:

$$\begin{aligned}
a_1 &= -\frac{1}{2}a_0 \quad , \\
a_2 &= -\frac{1}{2}a_1 = -\frac{1}{2}\left[-\frac{1}{2}a_0\right] = (-1)^2\frac{1}{2^2}a_0 \quad , \\
a_3 &= -\frac{1}{2}a_2 = -\frac{1}{2}\left[(-1)^2\frac{1}{2^2}a_0\right] = (-1)^3\frac{1}{2^3}a_0 \quad , \\
a_4 &= -\frac{1}{2}a_3 = -\frac{1}{2}\left[(-1)^3\frac{1}{2^3}a_0\right] = (-1)^4\frac{1}{2^4}a_0 \quad , \\
&\ \vdots
\end{aligned}$$

Clearly, for $k \geq 1$,

$$a_k = (-1)^k \frac{1}{2^k} a_0 \quad .$$

This equation also holds trivially for $k = 0$. So

$$Y(X) = \sum_{k=0}^{\infty} a_k X^k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k} a_0 X^k = a_0 \sum_{k=0}^{\infty} \left(\frac{-1}{2}\right)^k X^k \quad ,$$

and, since $X = x - 3$,

$$y(x) = a_0 \sum_{k=0}^{\infty} \left(\frac{-1}{2}\right)^k (x-3)^k \quad ,$$

which happens to be a geometric series that reduces to $\quad y(x) = \dfrac{2a_0}{x-1} \quad .$

30.3 i. The equation is already in preferred form. Plugging in the series for y and y':

$$\begin{aligned}
0 &= (2 - x^3)y' - 3x^2 y \\
&= 2\sum_{k=1}^{\infty} k a_k x^{k-1} - x^3 \sum_{k=1}^{\infty} k a_k x^{k-1} - 3x^2 \sum_{k=0}^{\infty} a_k x^k \\
&= \underbrace{\sum_{k=1}^{\infty} 2k a_k x^{k-1}}_{n=k-1} + \underbrace{\sum_{k=1}^{\infty} (-1)k a_k x^{k+2}}_{n=k+2} + \underbrace{\sum_{k=0}^{\infty} (-3) a_k x^{k+2}}_{n=k+2} \\
&= \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n + \sum_{n=3}^{\infty} (-1)(n-2)a_{n-2}x^n + \sum_{n=2}^{\infty} (-3)a_{n-2}x^n \\
&= \left[2(0+1)a_{0+1}x^0 + 2(1+1)a_{1+1}x^1 + 2(2+1)a_{2+1}x^2 + \sum_{n=3}^{\infty} 2(n+1)a_{n+1}x^n\right] \\
&\qquad + \sum_{n=3}^{\infty} (-1)(n-2)a_{n-2}x^n + \left[-3a_{2-2}x^2 + \sum_{k=3}^{\infty} (-3)a_{n-2}x^n\right] \\
&= 2a_1 x^0 + 4a_2 x^2 + [2 \cdot 3a_3 - 3a_0] + \sum_{n=3}^{\infty} \left[2(n+1)a_{n+1} - (n+1)a_{n-2}\right] x^n \quad .
\end{aligned}$$

Hence, we must have

$$a_1 = 0 \quad , \quad a_2 = 0 \quad , \quad a_3 = \frac{1}{2} a_0$$

and

$$a_{n+1} = \frac{1}{2} a_{n-2} \quad \text{for} \quad n \geq 3 \quad .$$

Letting $k = n + 1$ and noting the above formula for a_3, we see that we have the recursion formula

$$a_k = \frac{1}{2} a_{k-3} \quad \text{for} \quad k \geq 3 \quad .$$

Using the recursion formula and fact that $a_1 = 0 = a_2$, we get that all the terms are 0 except a_0 and

$$\begin{aligned}
a_3 &= \frac{1}{2}a_0 \quad , \\
a_6 &= \frac{1}{2}a_3 = \frac{1}{2}\cdot\frac{1}{2}a_0 = \frac{1}{2^2}a_0 \quad , \\
a_9 &= \frac{1}{2}a_6 = \frac{1}{2}\cdot\frac{1}{2^2}a_0 = \frac{1}{2^3}a_0 \quad , \\
a_{12} &= \frac{1}{2}a_9 = \frac{1}{2}\cdot\frac{1}{2^3}a_0 = \frac{1}{2^4}a_0 \quad , \\
&\vdots
\end{aligned}$$

So

$$\begin{aligned}
y(x) &= a_0x^0 + a_3x^3 + a_6x^6 + a_9x^9 + a_{12}x^{12} + \dots \\
&= a_0\left[x^0 + \frac{1}{2}x^3 + \frac{1}{2^2}x^6 + \frac{1}{2^3}x^9 + \frac{1}{2^4}x^{12} + \dots\right] \\
&= a_0\left[x^{3\cdot 0} + \frac{1}{2}x^{3\cdot 1} + \frac{1}{2^2}x^{3\cdot 2} + \frac{1}{2^3}x^{3\cdot 3} + \frac{1}{2^4}x^{3\cdot 4} + \dots\right] \\
&= a_0\sum_{m=0}^{\infty}\frac{1}{2^m}x^{3m} \quad .
\end{aligned}$$

30.3 k.

$$\begin{aligned}
0 &= (1+x)\,y' - xy \\
&= 1\sum_{k=1}^{\infty}ka_kx^{k-1} + x\sum_{k=1}^{\infty}ka_kx^{k-1} - x\sum_{k=0}^{\infty}a_kx^k \\
&= \underbrace{\sum_{k=1}^{\infty}ka_kx^{k-1}}_{n=k-1} + \underbrace{\sum_{k=1}^{\infty}ka_kx^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty}(-1)a_kx^{k+1}}_{n=k+1} \\
&= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty}na_nx^n + \sum_{n=1}^{\infty}(-1)a_{n-1}x^n \\
&= \left[(0+1)a_{0+1}x^0 + \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n\right] + \sum_{n=1}^{\infty}na_nx^n + \sum_{n=1}^{\infty}(-1)a_{n-1}x^n \\
&= a_1x^0 + \sum_{n=1}^{\infty}\left[(n+1)a_{n+1} + na_n - a_{n-1}\right]x^n \quad .
\end{aligned}$$

So, $a_1 = 0$ and

$$(n+1)a_{n+1} + na_n - a_{n-1} = 0 \qquad \text{for} \quad n \ge 1$$

$$\hookrightarrow \qquad a_{n+1} = \frac{1}{n+1}a_{n-1} - \frac{n}{n+1}a_n \qquad \text{for} \quad n \ge 1$$

$$\hookrightarrow \qquad a_k = \frac{1}{k}a_{k-2} - \frac{k-1}{k}a_{k-1} \qquad \text{for} \quad k \ge 2 \quad .$$

This last is the recursion formula. Using it and the fact that $a_1 = 0$ (and not attempting to find any patterns), we get

$$\begin{aligned}
a_1 &= 0 \quad , \\
a_2 &= \frac{1}{2}a_{2-2} - \frac{2-1}{2}a_{2-1} = \frac{1}{2}a_0 - \frac{1}{2}a_1 = \frac{1}{2}a_0 \quad , \\
a_3 &= \frac{1}{3}a_{3-2} - \frac{3-1}{3}a_{3-1} = \frac{1}{3}a_1 - \frac{2}{3}a_2 = -\frac{2}{3}\cdot\frac{1}{2}a_0 = -\frac{1}{3}a_0 \quad , \\
a_4 &= \frac{1}{4}a_{4-2} - \frac{4-1}{4}a_{4-1} = \frac{1}{4}a_2 - \frac{3}{4}a_3 = \frac{1}{4}\cdot\frac{1}{2}a_0 - \frac{3}{4}\left[-\frac{1}{3}a_0\right] = \frac{3}{8}a_0 \quad , \\
a_5 &= \frac{1}{5}a_{5-2} - \frac{5-1}{5}a_{5-1} = \frac{1}{5}a_3 - \frac{4}{5}a_4 = \frac{1}{5}\left[-\frac{1}{3}a_0\right] - \frac{4}{5}\cdot\frac{3}{8}a_0 = -\frac{11}{30}a_0 \quad , \\
&\ \vdots
\end{aligned}$$

So,

$$\begin{aligned}
y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^5 + a_5x^5 + \cdots \\
&= a_0\left[1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{8}x^4 - \frac{11}{30}x^5 + \cdots\right] \quad .
\end{aligned}$$

30.4 a. Since the coefficients of the differential equation are nonzero constants, there are no singular points. Hence the radius of analyticity, R, is infinite and the power series solution is valid on $I = (-\infty, \infty)$.

30.4 c. Writing the differential equation in preferred form,

$$(2x - 1)y' + y = 0 \quad ,$$

we see that the first coefficient, $2x - 1$, is 0 if and only if $x = \frac{1}{2}$. So this differential equation has one singular point, $z_s = \frac{1}{2}$.

The radius of analyticity about $x_0 = 0$ is then

$$R = |z_s - x_0| = \left|\frac{1}{2} - 0\right| = \frac{1}{2} \quad ,$$

and the interval over which the power series solution is valid is

$$I = (x_0 - R, x_0 + R) = \left(0 - \frac{1}{2}, 0 + \frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right) \quad .$$

30.4 e. The equation,

$$\left(1 + x^2\right)y' - 2xy = 0 \quad ,$$

is in preferred form, and the singular points are where the first coefficient is 0:

$$1 + {z_s}^2 = 0 \quad \rightarrowtail \quad z_s = \pm i \quad .$$

Both are equally close to $x_0 = 0$. So

$$R = |z_s - x_0| = |i - 0| = |i| = 1 \quad ,$$

and

$$I = (x_0 - R, x_0 + R) = (0 - 1, 0 + 1) = (-1, 1) \quad .$$

30.4 g. Writing the differential equation in preferred form,

$$(x-1)y' + y = 0 \quad ,$$

we see that the first coefficient, $x-1$, is 0 if and only if $x=1$. So this differential equation has one singular point, $z_s = 1$.

The radius of analyticity about $x_0 = 3$ is then

$$R = |z_s - x_0| = |1-3| = 2 \quad ,$$

and the interval over which the power series solution is valid is

$$I = (x_0 - R, x_0 + R) = (3-2, 3+2) = (1,5) \quad .$$

30.4 i. The equation,

$$(2-x^3)y' - 3x^2 y = 0 \quad ,$$

is in preferred form, and the singular points are where the first coefficient is 0. Using basic algebra (hint: factor out $z - \sqrt[3]{2}$):

$$2 - {z_s}^3 = 0 \quad \rightarrowtail \quad z_s = \sqrt[3]{2}, \ \frac{-1 \pm i\sqrt{3}}{\sqrt[3]{4}} \quad .$$

Now,

$$\left|\sqrt[3]{2} - 0\right| = \sqrt[3]{2}$$

and

$$\left|\frac{-1 \pm i\sqrt{3}}{\sqrt[3]{4}} - 0\right| = \frac{\sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}}{\sqrt[3]{4}} = \cdots = \sqrt[3]{2} \quad .$$

So all the singular points are the same distance from $x_0 = 0$,

$$R = |z_s - x_0| = \left|\sqrt[3]{2} - 0\right| = \sqrt[3]{2} \quad ,$$

and

$$I = (x_0 - R, x_0 + R) = \left(-\sqrt[3]{2}, \sqrt[3]{2}\right) \quad .$$

30.4 k. The equation,

$$(1+x)\,y' - xy = 0 \quad ,$$

is in preferred form, and the only singular point is where the first coefficient is 0, $z_s = -1$. So

$$R = |z_s - x_0| = |-1-0| = 1 \quad ,$$

and

$$I = (x_0 - R, x_0 + R) = (-1, 1) \quad .$$

30.5 a. Set

$$y = y(x) = \sum_{k=0}^{\infty} a_k x^k \quad , \quad y' = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^k\right] = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y'' = \sum_{k=0}^{\infty} \frac{d^2}{dx^2}\left[a_k x^k\right] = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} \quad .$$

Then:

$$\begin{aligned}
0 &= \left(1+x^2\right)y'' - 2y \\
&= \left(1+x^2\right)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2\sum_{k=0}^{\infty} a_k x^k \\
&= 1\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + x^2\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2\sum_{k=0}^{\infty} a_k x^k \\
&= \underbrace{\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}}_{n=k-2} + \underbrace{\sum_{k=2}^{\infty} k(k-1)a_k x^{k}}_{n=k} + \underbrace{\sum_{k=0}^{\infty} (-2)a_k x^k}_{n=k} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (-2)a_n x^n \\
&= \left[(0+2)(0+1)a_{0+2}x^0 + (1+2)(1+1)a_{1+2}x^1 + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n\right] \\
&\quad + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \left[-2a_0x^0 - 2a_1x^1 + \sum_{n=2}^{\infty}(-2)a_n x^n\right] \\
&= [2a_2 - 2a_0]\,x^0 + [3\cdot 2a_3 - 2a_1]\,x^1 \\
&\quad + \sum_{n=2}^{\infty}\big[(n+2)(n+1)a_{n+2} + [n(n-1)-2]a_n\big]\,x^n
\end{aligned}$$

So,

$$2a_2 - 2a_0 = 0 \quad \rightarrowtail \quad a_2 = a_0 \quad ,$$

$$3\cdot 2a_3 - 2a_1 = 0 \quad \rightarrowtail \quad a_3 = \frac{1}{3}a_1 \quad ,$$

and, for $n \geq 2$,

$$(n+2)(n+1)a_{n+2} + [n(n-1)-2]a_n = 0$$

$$\hookrightarrow \qquad (n+2)(n+1)a_{n+2} + [n^2-n-2]a_n = 0$$

$$\hookrightarrow \qquad (n+2)(n+1)a_{n+2} + (n-2)(n+1)a_n = 0$$

$$\hookrightarrow \qquad a_{n+2} = -\frac{n-2}{n+2}a_n = 0 \quad .$$

Letting $k = n+2$ in the last equation gives the recursion formula

$$a_k = -\frac{k-4}{k}a_{k-2} \qquad \text{for} \quad k \geq 4 \quad .$$

And if you check the formulas for a_2 and a_3, you see that this recursion formula actually holds for $k \geq 2$.

Computing the coefficients from the above:

$$
\begin{aligned}
a_2 &= a_0 \quad , \\
a_3 &= \frac{1}{3}a_1 \quad , \\
a_4 &= -\frac{4-4}{4}a_{4-2} = 0 \quad , \\
a_5 &= -\frac{5-4}{5}a_{5-2} = -\frac{1}{5}a_3 = -\frac{1}{5}\cdot\frac{1}{3}a_1 = -\frac{1}{5\cdot 3}a_1 \quad , \\
a_6 &= -\frac{6-4}{6}a_{6-2} = -\frac{2}{6}a_4 = -\frac{2}{6}\cdot 0 = 0 \quad , \\
a_7 &= -\frac{7-4}{7}a_{7-2} = -\frac{3}{7}a_5 = -\frac{3}{7}\cdot\frac{-1}{5\cdot 3}a_1 = \frac{1}{7\cdot 5}a_1 \quad , \\
a_8 &= -\frac{8-4}{8}a_{8-2} = -\frac{4}{8}a_6 = -\frac{4}{8}\cdot 0 = 0 \quad , \\
a_9 &= -\frac{9-4}{9}a_{9-2} = -\frac{5}{9}a_7 = -\frac{5}{9}\cdot\frac{1}{7\cdot 5}a_1 = -\frac{1}{9\cdot 7}a_1 \quad , \\
a_{10} &= -\frac{10-4}{10}a_{10-2} = -\frac{6}{10}a_8 = -\frac{6}{10}\cdot 0 = 0 \quad , \\
a_{11} &= -\frac{11-4}{11}a_{11-2} = -\frac{7}{11}a_9 = -\frac{7}{11}\cdot\frac{-1}{9\cdot 7}a_1 = \frac{1}{11\cdot 9}a_1 \quad , \\
&\vdots
\end{aligned}
$$

There are two patterns, with the even indexed coefficients being 0 if $k > 2$. Because of the two different patterns, we will split our summation into two summations, one with even terms and one with odd terms:

$$
\begin{aligned}
y(x) &= \sum_{k}^{\infty} a_k x^k \\
&= \left[a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10}\cdots\right] \\
&\quad + \left[a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + a_{11}x^{11} + \cdots\right] \\
&= a_0\left[1 + 1x^2 + 0x^4 + 0x^6 + 0x^8 + 0\right] \\
&\quad + a_1\left[x + \frac{1}{3}x^3 - \frac{1}{5\cdot 3}x^5 + \frac{1}{7\cdot 5}x^7 - \frac{1}{9\cdot 7}x^9 + \frac{1}{11\cdot 9}x^{11} + \cdots\right] \\
&= a_0y_1(x) + a_1y_2(x)
\end{aligned}
$$

with

$$y_1(x) = 1 + x^2$$

and

$$y_2(x) = x + \frac{1}{3}x^3 - \frac{1}{5\cdot 3}x^5 + \frac{1}{7\cdot 5}x^7 - \frac{1}{9\cdot 7}x^9 + \frac{1}{11\cdot 9}x^{11} + \cdots \quad .$$

To get a 'simple' pattern for the coefficients in y_2, recall that $k = 2m + 1$ generates the nonnegative odd integers for $m = 0, 1, 2, 3, \ldots$, and observe that, using $k = 2m + 1$ for $m = 2, 3, 4, 5$, we get

$$a_{2\cdot 2+1} = a_5 = -\frac{1}{5\cdot 3}a_1 = (-1)^{2+1}\frac{1}{(2\cdot 2+1)(2\cdot 2-1)}a_1 \quad ,$$

$$a_{2\cdot 3+1} = a_7 = \frac{1}{7\cdot 5}a_1 = (-1)^{3+1}\frac{1}{(2\cdot 3+1)(2\cdot 3-1)}a_1 \quad ,$$

$$a_{2\cdot 4+1} = a_9 = -\frac{1}{9\cdot 7}a_1 = (-1)^{4+1}\frac{1}{(2\cdot 4+1)(2\cdot 4-1)}a_1 \quad ,$$

and

$$a_{2\cdot 5+1} = a_{11} = \frac{1}{11\cdot 9}a_1 = (-1)^{5+1}\frac{1}{(2\cdot 5+1)(2\cdot 5-1)}a_1 \quad .$$

indicating that we have a pattern

$$a_{2m+1} = (-1)^{m+1}\frac{1}{(2m+1)(2m-1)}a_1$$

at least for $m \geq 2$. Checking this pattern with $m = 0$ and $m = 1$ and the formulas for a_1 and a_3, we have

$$(-1)^{0+1}\frac{1}{(2\cdot 0+1)(2\cdot 0-1)}a_1 = -\frac{1}{-1}a_1 = a_1$$

and

$$(-1)^{1+1}\frac{1}{(2\cdot 1+1)(2\cdot 1-1)}a_1 = \frac{1}{3\cdot 1}a_1 = \frac{1}{3}a_1 = a_3 \quad .$$

So, we will use

$$a_{2m+1} = (-1)^{m+1}\frac{1}{(2m+1)(2m-1)}a_1 \qquad \text{for} \quad m \geq 0 \quad .$$

That is,

$$\begin{aligned} y_2(x) &= a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + a_{11}x^{11} + \cdots \\ &= \sum_{m=0}^{\infty} a_{2m+1}x^{2m+1} \\ &= a_1\sum_{m=0}^{\infty}(-1)^{m+1}\frac{1}{(2m+1)(2m-1)}x^{2m+1} \quad . \end{aligned}$$

30.5 c. Set

$$y = y(x) = \sum_{k=0}^{\infty} a_k x^k \qquad , \qquad y' = \sum_{k=0}^{\infty}\frac{d}{dx}\left[a_k x^k\right] = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y'' = \sum_{k=0}^{\infty}\frac{d^2}{dx^2}\left[a_k x^k\right] = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} \quad .$$

Then:

$$\begin{aligned} 0 &= \left(4+x^2\right)y'' + 2xy' \\ &= \left(4+x^2\right)\sum_{k=2}^{\infty}k(k-1)a_kx^{k-2} + 2x\sum_{k=1}^{\infty}ka_kx^{k-1} \end{aligned}$$

$$= 4\sum_{k=2}^{\infty}k(k-1)a_kx^{k-2} + x^2\sum_{k=2}^{\infty}k(k-1)a_kx^{k-2} + 2x\sum_{k=1}^{\infty}ka_kx^{k-1}$$

$$= \underbrace{\sum_{k=2}^{\infty}4k(k-1)a_kx^{k-2}}_{n=k-2} + \underbrace{\sum_{k=2}^{\infty}k(k-1)a_kx^{k}}_{n=k} + \underbrace{\sum_{k=1}^{\infty}2ka_kx^{k}}_{n=k}$$

$$= \sum_{n=0}^{\infty}4(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}n(n-1)a_nx^n + \sum_{n=1}^{\infty}2na_nx^n$$

$$= \left[4(0+2)(0+1)a_{0+2}x^0 + 4(1+2)(1+1)a_{1+2}x^1 + \sum_{n=2}^{\infty}4(n+2)(n+1)a_{n+2}x^n\right]$$

$$+ \sum_{n=2}^{\infty}n(n-1)a_nx^n + \left[2\cdot 1a_1x^1 + \sum_{n=2}^{\infty}2na_nx^n\right]$$

$$= 4\cdot 2a_2x^0 + [4\cdot 3\cdot 2a_3 + 2a_1]\,x^1$$

$$+ \sum_{n=2}^{\infty}\big[4(n+2)(n+1)a_{n+2} + [n(n-1)+2n]a_n\big]\,x^n$$

$$= 4\cdot 2a_2x^0 + 2\,[4\cdot 3a_3 + a_1]\,x^1 + \sum_{n=2}^{\infty}\big[4(n+2)(n+1)a_{n+2} + n(n+1)a_n\big]\,x^n \quad .$$

So,

$$4\cdot 2a_2 = 0 \quad \rightarrowtail \quad a_2 = 0 \quad ,$$

$$2\,[4\cdot 3a_3 + a_1] = 0 \quad \rightarrowtail \quad a_3 = -\frac{1}{4\cdot 3}a_1$$

and

$$4(n+2)(n+1)a_{n+2} + n(n+1)a_n = 0 \qquad \text{for} \quad n \geq 2$$

$$\hookrightarrow \qquad a_{n+2} = -\frac{n}{4(n+2)}a_n \qquad \text{for} \quad n \geq 2$$

$$\hookrightarrow \qquad a_k = -\frac{k-2}{4k}a_{k-2} \qquad \text{for} \quad k \geq 4 \quad .$$

The last is the recursion formula. Checking back at the formulas for a_2 and $a+3$, we see that it actually holds for $k \geq 2$.

Using the above to compute the a_k's:

$$a_2 = 0 \quad ,$$

$$a_3 = \frac{-1}{4\cdot 3}a_1 \quad ,$$

$$a_4 = -\frac{4-2}{4\cdot 4}a_{4-2} = -\frac{2}{4\cdot 4}a_2 = -\frac{2}{4\cdot 4}\,[0] = 0 \quad ,$$

$$a_5 = -\frac{5-2}{4\cdot 5}a_{5-2} = -\frac{3}{4\cdot 5}a_3 = -\frac{3}{4\cdot 5}\left[\frac{-1}{4\cdot 3}a_1\right] = \frac{(-1)^2}{4^2\cdot 5}a_1 \quad ,$$

$$a_6 = -\frac{6-2}{4\cdot 6}a_{6-2} = -\frac{4}{4\cdot 6}a_4 = -\frac{4}{4\cdot 6}\,[0] = 0 \quad ,$$

$$
\begin{aligned}
a_7 &= -\frac{7-2}{4\cdot 7}a_{k-2} = -\frac{5}{4\cdot 7}a_5 = -\frac{5}{4\cdot 7}\left[\frac{(-1)^2}{4^2\cdot 5}a_1\right] = \frac{(-1)^3}{4^3\cdot 7} \quad , \\
a_8 &= -\frac{8-2}{4\cdot 8}a_{8-2} = -\frac{6}{4\cdot 8}a_6 = -\frac{6}{4\cdot 8}[0] = 0 \quad , \\
a_9 &= -\frac{9-2}{4\cdot 9}a_{9-2} = -\frac{7}{4\cdot 9}a_7 = -\frac{7}{4\cdot 9}\left[\frac{(-1)^3}{4^3\cdot 7}\right] = \frac{(-1)^4}{4^4\cdot 9} \quad , \\
a_{10} &= -\frac{10-2}{4\cdot 10}a_{10-2} = -\frac{8}{4\cdot 10}a_8 = -\frac{8}{4\cdot 10}[0] = 0 \quad , \\
a_{11} &= -\frac{11-2}{4\cdot 11}a_{11-2} = -\frac{9}{4\cdot 11}a_9 = -\frac{9}{4\cdot k}\left[\frac{(-1)^4}{4^4\cdot 9}\right] = \frac{(-1)^5}{4^5\cdot 11} \quad , \\
&\vdots
\end{aligned}
$$

There are two patterns, with the even indexed coefficients being 0 if $k > 0$. Because of the two different patterns, we will split our summation into two summations, one with even terms and one with odd terms:

$$
\begin{aligned}
y(x) &= \sum_{k}^{\infty} a_k x^k \\
&= \left[a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + a_{10}x^{10}\cdots\right] \\
&\quad + \left[a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + a_{11}x^{11} + \cdots\right] \\
&= a_0\left[1 + 0x^2 + 0x^4 + 0x^6 + 0x^8 + 0\right] \\
&\quad + a_1\left[x + \frac{-1}{4\cdot 3}x^3 + \frac{(-1)^2}{4^2\cdot 5}x^5 + \frac{(-1)^3}{4^3\cdot 7}x^7 + \frac{(-1)^4}{4^4\cdot 9}x^9 + \cdots\right] \\
&= a_0y_1(x) + a_1y_2(x)
\end{aligned}
$$

with

$$
y_1(x) = 1
$$

and

$$
\begin{aligned}
y_2(x) &= x + \frac{-1}{4\cdot 3}x^3 + \frac{(-1)^2}{4^2\cdot 5}x^5 + \frac{(-1)^3}{4^3\cdot 7}x^7 + \frac{(-1)^4}{4^4\cdot 9}x^9 + \cdots \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m(2m+1)} \quad .
\end{aligned}
$$

30.5 e. Using the power series about $x_0 = 0$ for y, y' and y'':

$$
\begin{aligned}
0 &= \left(4 - x^2\right)y'' - 5xy' - 3y \\
&= \left(4 - x^2\right)\sum_{k=2}^{\infty} k(k-1)a_kx^{k-2} - 5x\sum_{k=1}^{\infty} ka_kx^{k-1} - 3\sum_{k=0}^{\infty} a_kx^k
\end{aligned}
$$

$$
\begin{aligned}
&= 4\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} \;-\; x^2\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} \\
&\quad -\; 5x\sum_{k=1}^{\infty} k a_k x^{k-1} \;-\; 3\sum_{k=0}^{\infty} a_k x^k \\
&= \underbrace{\sum_{k=2}^{\infty} 4k(k-1)a_k x^{k-2}}_{n=k-2} \;+\; \underbrace{\sum_{k=2}^{\infty}(-1)k(k-1)a_k x^k}_{n=k} \\
&\quad +\; \underbrace{\sum_{k=1}^{\infty}(-5)k a_k x^k}_{n=k} \;+\; \underbrace{\sum_{k=0}^{\infty}(-3)a_k x^k}_{n=k} \\
&= \sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2}x^n \;+\; \sum_{n=2}^{\infty}(-1)n(n-1)a_n x^n \\
&\quad +\; \sum_{n=1}^{\infty}(-5)n a_n x^n \;+\; \sum_{n=0}^{\infty}(-3)a_n x^n \\
&= \Bigg[4(0+2)(0+1)a_{0+2}x^0 + 4(1+2)(1+1)a_{1+2}x^1 \\
&\quad + \sum_{n=2}^{\infty} 4(n+2)(n+1)a_{n+2}x^n\Bigg] \;+\; \sum_{n=2}^{\infty}(-1)n(n-1)a_n x^n \\
&\quad +\; \Bigg[-5a_1x^1 + \sum_{n=2}^{\infty}(-5)n a_n x^n\Bigg] \;+\; \Bigg[-3a_0x^0 - 3a_1x^1 + \sum_{n=2}^{\infty}(-3)a_n x^n\Bigg] \\
&= [4\cdot 2a_2 - 3a_0]\,x^0 \;+\; [4\cdot 3\cdot 2a_3 - 8a_1]\,x^1 \\
&\quad +\; \sum_{n=2}^{\infty}\big[4(n+2)(n+1)a_{n+2} + [-n(n-1) - 5n - 3]a_n\big]\,x^n \\
&= [4\cdot 2a_2 - 3a_0]\,x^0 \;+\; 4\cdot 2\,[3a_3 - a_1]\,x^1 \\
&\quad +\; \sum_{n=2}^{\infty}\big[4(n+2)(n+1)a_{n+2} - [(n+3)(n+1)]a_n\big]\,x^n \quad .
\end{aligned}
$$

So

$$a_2 \;=\; \frac{3}{4\cdot 2}a_0 \qquad , \qquad a_3 \;=\; \frac{1}{3}a_1$$

and

$$a_{n+2} \;=\; \frac{n+3}{4(n+2)}a_n \qquad \text{for} \quad n \geq 2$$

$$\hookrightarrow \qquad a_k \;=\; \frac{k+1}{4k}a_{k-2} \qquad \text{for} \quad k \geq 4 \quad .$$

The last is the recursion formula. Checking the above formulas for a_2 and a_3 it is seen that, in fact, the recursion formula holds for $k \geq 2$.

Using the recursion formula:

$$
\begin{aligned}
a_2 &= \frac{2+1}{4\cdot 2}a_{2-2} = \frac{3}{4\cdot 2}a_0 \quad , \\
a_3 &= \frac{3+1}{4\cdot 3}a_{3-2} = \frac{4}{4\cdot 3}a_1 \quad , \\
a_4 &= \frac{4+1}{4\cdot 4}a_{4-2} = \frac{5}{4\cdot 4}a_2 = \frac{5}{4\cdot 4}\left[\frac{3}{4\cdot 2}a_0\right] = \frac{5\cdot 3}{4^2(4\cdot 2)}a_0 \quad , \\
a_5 &= \frac{5+1}{4\cdot 5}a_{5-2} = \frac{6}{4\cdot 5}a_3 = \frac{6}{4\cdot 5}\left[\frac{4}{4\cdot 3}a_1\right] = \frac{6\cdot 4}{4^2(5\cdot 3)}a_1 \quad , \\
a_6 &= \frac{6+1}{4\cdot 6}a_{6-2} = \frac{7}{4\cdot 6}a_4 = \frac{7}{4\cdot 6}\left[\frac{5\cdot 3}{4^2(4\cdot 2)}a_0\right] = \frac{7\cdot 5\cdot 3}{4^3(6\cdot 4\cdot 2)}a_0 \quad , \\
a_7 &= \frac{7+1}{4\cdot 7}a_{7-2} = \frac{8}{4\cdot 7}a_5 = \frac{8}{4\cdot 7}\left[\frac{6\cdot 4}{4^2(5\cdot 3)}a_1\right] = \frac{8\cdot 6\cdot 4}{4^3(7\cdot 5\cdot 3)}a_1 \quad , a_8 = \frac{8+1}{4\cdot 8}a_{8-2} = \frac{9}{4\cdot 8}a_6 = \frac{9}{4\cdot 8}\left[\frac{7}{4^3(\cdots\right. \\
&\vdots
\end{aligned}
$$

So,

$$
\begin{aligned}
y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\
&= \cdots = a_0y_1(x) + a_1y_2(x)
\end{aligned}
$$

where

$$
y_1(x) = 1 + \frac{3}{4\cdot 2}x^2 + \frac{5\cdot 3}{4^2(4\cdot 2)}x^4 + \frac{7\cdot 5\cdot 3}{4^3(6\cdot 4\cdot 2)}x^6 + \cdots
$$

and

$$
y_2(x) = x + \frac{4}{4\cdot 3}x^3 + \frac{6\cdot 4}{4^2(5\cdot 3)}x^5 + \frac{8\cdot 6\cdot 4}{4^3(7\cdot 5\cdot 3)}x^7 + \cdots \quad .
$$

By the way, it can be shown that

$$
y_1(x) = \sum_{m=0}^{\infty}\frac{(2m+1)!}{4^{2m}(m!)^2}x^{2m} \qquad \text{and} \qquad y_2(x) = \sum_{m=0}^{\infty}\frac{(m+1)!m!}{(2m+1)!}x^{2m+1} \quad .
$$

30.5 g. Using the power series about $x_0 = 0$ for y, y' and y'':

$$
\begin{aligned}
0 &= y'' - 2xy' + 6y \\
&= \sum_{k=2}^{\infty}k(k-1)a_kx^{k-2} - 2x\sum_{k=1}^{\infty}ka_kx^{k-1} + 6\sum_{k=0}^{\infty}a_kx^k \\
&= \underbrace{\sum_{k=2}^{\infty}k(k-1)a_kx^{k-2}}_{n=k-2} + \underbrace{\sum_{k=1}^{\infty}(-2k)a_kx^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty}6a_kx^k}_{n=k} \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}(-2n)a_nx^n + \sum_{n=0}^{\infty}6a_nx^n
\end{aligned}
$$

$$= \left[(0+2)(0+1)a_{0+2}x^0 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n\right] + \sum_{n=1}^{\infty}(-2n)a_n x^n$$

$$+ \left[6a_0x^0 + \sum_{n=1}^{\infty}6a_n x^n\right]$$

$$= 2\,[a_2 + 3a_0]\,x^0 + \sum_{n=1}^{\infty}\big[(n+2)(n+1)a_{n+2} - 2(n-3)a_n\big]\,x^n \quad .$$

So,

$$a_2 = -3a_0$$

and

$$a_{n+2} = \frac{2(n-3)}{(n+2)(n+1)}a_n \qquad \text{for} \quad n \geq 1$$

$$\hookrightarrow \qquad a_k = \frac{2(k-5)}{k(k-1)}a_{k-2} \qquad \text{for} \quad k \geq 3 \quad .$$

The last is the recursion formula. Comparing the above formula for a_2 with what the recursion formula yields when $k = 2$, we see that, in fact, this recursion formula holds for $k \geq 2$.

Repeatedly using the recursion formula:

$$a_2 = \frac{2(2-5)}{2(2-1)}a_{2-2} = \frac{2(-3)}{2\cdot 1}a_0 \quad ,$$

$$a_3 = \frac{2(3-5)}{3(3-1)}a_{3-2} = \frac{3(-2)}{2\cdot 2}a_1 \quad ,$$

$$a_4 = \frac{2(4-5)}{4(4-1)}a_{4-2} = \frac{2(-1)}{4\cdot 3}a_2 = \frac{2(-1)}{4\cdot 3}\left[\frac{2(-3)}{2\cdot 1}a_0\right] = \frac{2^2(-1)(-3)}{4\cdot 3\cdot 2\cdot 1}a_0 \quad ,$$

$$a_5 = \frac{2(5-5)}{4(4-1)}a_{4-2} = 0 \quad ,$$

$$a_6 = \frac{2(6-5)}{6(6-1)}a_{6-2} = \frac{2(1)}{6\cdot 5}\left[\frac{2^2(-1)(-3)}{4\cdot 3\cdot 2\cdot 1}a_0\right] = \frac{2^3(1)(-1)(-3)}{6!}a_0 \quad ,$$

$$a_7 = \frac{2(7-5)}{7(7-1)}a_{7-2} = \frac{2(2)}{7\cdot 6}\,[0] = 0 \quad ,$$

$$a_8 = \frac{2(8-5)}{8(8-1)}a_{8-2} = \frac{2(3)}{8\cdot 7}\left[\frac{2^3(1)(-1)(-3)}{6!}a_0\right] = \frac{2^4(3)(1)(-1)(-3)}{8!}a_0 \quad ,$$

$$\vdots$$

Plugging into the power series for y then yields

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$= \cdots = a_0y_1(x) + a_1y_2(x)$$

where

$$y_1(x) = 1 + \frac{2(-3)}{2!}x^2 + \frac{2^2[(-1)(-3)]}{4!}x^4 + \frac{2^3[(1)(-1)(-3)]}{6!}x^6$$
$$+ \frac{2^3[(3)(1)(-1)(-3)]}{8!}x^8 + \cdots$$

and

$$y_2(x) \;=\; x \;-\; \frac{2}{3}x^3 \quad .$$

30.5 i. Since $x_0 = -2 \neq 0$, set $Y(X) = y(x)$ with

$$X \;=\; x \;-\; x_0 \;=\; x \;-\; (-2) \;=\; x \;+\; 2 \quad .$$

The differential equation then becomes

$$Y'' \;+\; XY' \;+\; 2Y \;=\; 0 \quad .$$

Using the power series about $X_0 = 0$ for Y, Y' and Y'', we get

$$\begin{aligned}
0 \;&=\; Y'' \;+\; XY' \;+\; 2Y \\
&=\; \sum_{k=2}^{\infty} k(k-1)a_k X^{k-2} \;+\; X\sum_{k=1}^{\infty} ka_k X^{k-1} \;+\; 2\sum_{k=0}^{\infty} a_k X^k \\
&=\; \underbrace{\sum_{k=2}^{\infty} k(k-1)a_k X^{k-2}}_{n=k-2} \;+\; \underbrace{\sum_{k=1}^{\infty} ka_k X^k}_{n=k} \;+\; \underbrace{\sum_{k=0}^{\infty} 2a_k X^k}_{n=k} \\
&=\; \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}X^n \;+\; \sum_{n=1}^{\infty} na_n X^n \;+\; \sum_{n=0}^{\infty} 2a_n X^n \\
&=\; \left[(0+2)(0+1)a_{0+2}X^0 + (1+2)(1+1)a_{1+2}X^1 + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}X^n\right] \\
&\qquad +\; \left[1a_1X^1 + \sum_{n=2}^{\infty} na_n X^n\right] \;+\; \left[2a_0X^0 + 2a_1X^1 + \sum_{n=0}^{\infty} 2a_n X^n\right] \\
&=\; [2a_2 + 2a_0]\,X^0 \;+\; [3\cdot 2a_3 + 3a_1]\,X^1 \\
&\qquad +\; \sum_{n=2}^{\infty}\big[(n+2)(n+1)a_{n+2} + (n+2)a_n\big]\,X^n \quad .
\end{aligned}$$

So,

$$a_2 \;=\; -a_0 \qquad , \qquad a_3 \;=\; \frac{-1}{2}a_1$$

and

$$a_{n+2} \;=\; \frac{-1}{n+1}a_n \qquad \text{for} \quad n \geq 2$$

$$\hookrightarrow \qquad a_k \;=\; \frac{-1}{k-1}a_{k-2} \qquad \text{for} \quad k \geq 4 \quad .$$

The last equation gives the recursion formula. Checking the formulas for a_2 and a_3, we see that the recursion formula actually holds for $k \geq 2$.

Computing a_k for $k = 3,\ 4,\ \ldots$:

$$a_2 \;=\; -a_0 \quad ,$$

$$a_3 \;=\; \frac{-1}{2}a_1 \quad ,$$

$$a_4 = \frac{-1}{4-1}a_{4-2} = \frac{-1}{3}a_2 = \frac{-1}{3}[-a_0] = \frac{(-1)^2}{3}a_0 \quad ,$$

$$a_5 = \frac{-1}{5-1}a_{5-2} = \frac{-1}{4}a_3 = \frac{-1}{4}\left[\frac{-1}{2}a_1\right] = \frac{(-1)^3}{4\cdot 2}a_1 \quad ,$$

$$a_6 = \frac{-1}{6-1}a_{6-2} = \frac{-1}{5}a_4 = \frac{-1}{5}\left[\frac{(-1)^2}{3}a_0\right] = \frac{(-1)^3}{5\cdot 3}a_0 \quad ,$$

$$a_7 = \frac{-1}{7-1}a_{7-2} = \frac{-1}{6}a_5 = \frac{-1}{6}\left[\frac{(-1)^3}{4\cdot 2}a_1\right] = \frac{(-1)^4}{6\cdot 4\cdot 2}a_1$$

$$\vdots$$

Plugging this back into the power series for Y yields

$$Y(X) = a_0 + a_1X + a_2X^2 + a_3X^3 + \cdots = \cdots = a_0Y_1(X) + a_1Y_2(X)$$

with

$$Y_1(X) = 1 - X^2 + \frac{(-1)^2}{3}X^4 + \frac{(-1)^3}{5\cdot 3}X^6 + \cdots$$

and

$$Y_2(X) = X + \frac{(-1)}{2}X^3 + \frac{(-1)^3}{4\cdot 2}X^5 + \frac{(-1)^4}{6\cdot 4\cdot 2}X^7 + \cdots \quad .$$

Finally, since our answer $y(x)$ is given by $Y(X)$ with $X = x+2$,

$$y(x) = a_0y_1(x) + a_1y_2(x)$$

where

$$\begin{aligned} y_1(x) &= Y_1(x+2) \\ &= 1 - (x+2)^2 + \frac{(-1)^2}{3}(x+2)^4 + \frac{(-1)^3}{5\cdot 3}(x+2)^6 + \cdots \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= Y_2(x+2) \\ &= (x+2) + \frac{(-1)}{2}(x+2)^3 + \frac{(-1)^3}{4\cdot 2}(x+2)^5 + \frac{(-1)^4}{6\cdot 4\cdot 2}(x+2)^7 + \cdots \quad . \end{aligned}$$

By the way, it can be shown that

$$y_1(x) = \sum_{m=0}^{\infty}\frac{(-2)^m m!}{(2m)!}(x+2)^{2m} \qquad \text{and} \qquad y_2(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}(x+2)^{2m+1} \quad .$$

30.5 k. Using the power series about $x_0 = 0$ for y, y' and y'', we get

$$\begin{aligned} 0 &= y'' - 2y' - xy \\ &= \sum_{k=2}^{\infty}k(k-1)a_kx^{k-2} - 2\sum_{k=1}^{\infty}ka_kx^{k-1} - x\sum_{k=0}^{\infty}a_kx^k \\ &= \underbrace{\sum_{k=2}^{\infty}k(k-1)a_kx^{k-2}}_{n=k-2} + \underbrace{\sum_{k=1}^{\infty}(-2)ka_kx^{k-1}}_{n=k-1} + \underbrace{\sum_{k=0}^{\infty}(-1)a_kx^{k+1}}_{n=k+1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \;+\; \sum_{n=0}^{\infty}(-2)(n+1)a_{n+1}x^n \;+\; \sum_{n=1}^{\infty}(-1)a_{n-1}x^n \\
&= \left[(0+2)(0+1)a_{0+2}x^0 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n\right] \\
&\quad + \left[-2(0+1)a_{0+1}x^0 + \sum_{n=1}^{\infty}(-2)(n+1)a_{n+1}x^n\right] \;+\; \sum_{n=1}^{\infty}(-1)a_{n-1}x^n \\
&= [2a_2 - 2a_1]\,x^0 \;+\; \sum_{n=1}^{\infty}\big[(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} - a_{n-1}\big]\,x^n \quad .
\end{aligned}
$$

Thus,

$$a_2 \;=\; a_1$$

and

$$(n+2)(n+1)a_{n+2} \;-\; 2(n+1)a_{n+1} \;-\; a_{n-1} \;=\; 0 \qquad \text{for} \quad n \geq 1$$

$$\hookrightarrow \qquad a_{n+2} \;=\; \frac{2}{n+2}a_{n+1} \;+\; \frac{1}{(n+2)(n+1)}a_{n-1} \qquad \text{for} \quad n \geq 1$$

$$\hookrightarrow \qquad a_k \;=\; \frac{2}{k}a_{k-1} \;+\; \frac{1}{k(k-1)}a_{k-3} \qquad \text{for} \quad k \geq 3 \quad .$$

The last is the recursion formula.

Using the above (and not attempting to find any ‘pattern’):

$$
\begin{aligned}
a_2 &= a_1 \quad , \\
a_3 &= \frac{2}{3}a_{3-1} \;+\; \frac{1}{3(3-1)}a_{3-3} \\
&= \frac{2}{3}a_2 \;+\; \frac{1}{3\cdot 2}a_0 \\
&= \frac{2}{3}a_1 \;+\; \frac{1}{6}a_0 \;=\; \frac{1}{6}a_0 \;+\; \frac{2}{3}a_1 \quad , \\
a_4 &= \frac{2}{4}a_{4-1} \;+\; \frac{1}{4(4-1)}a_{4-3} \\
&= \frac{2}{4}a_3 \;+\; \frac{1}{12}a_1 \\
&= \frac{2}{4}\left[\frac{2}{3}a_1 + \frac{1}{6}a_0\right] \;+\; \frac{1}{12}a_1 \;=\; \frac{1}{12}a_0 \;+\; \frac{5}{12}a_1 \quad , \\
a_5 &= \frac{2}{5}a_{5-1} \;+\; \frac{1}{5(5-1)}a_{5-3} \\
&= \frac{2}{5}a_4 \;+\; \frac{1}{20}a_2 \\
&= \frac{2}{5}\left[\frac{1}{12}a_0 + \frac{5}{12}a_1\right] \;+\; \frac{1}{20}a_1 \;=\; \frac{1}{30}a_0 \;+\; \frac{13}{60}a_1 \\
&\;\;\vdots
\end{aligned}
$$

So,

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\ &= a_0 + a_1 x + [a_1] x^2 + \left[\frac{1}{6}a_0 + \frac{2}{3}a_1\right] x^3 + \left[\frac{1}{12}a_0 + \frac{5}{12}a_1\right] x^4 \\ &\quad + \left[\frac{1}{30}a_0 + \frac{13}{60}a_1\right] x^5 + \cdots \\ &= \cdots \\ &= a_0 y_1(x) + a_1 y_2(x) \end{aligned}$$

where

$$y_1(x) = 1 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + \cdots$$

and

$$y_2(x) = x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \frac{13}{60}x^5 + \cdots \quad .$$

30.6 a. The differential equation

$$(1 + x^2)y'' - 2y = 0$$

is in preferred form. So the singular points are the points on the complex plane where the first coefficient is zero:

$$1 + z^2 = 0 \quad \rightarrowtail \quad z_s = \pm i \quad .$$

Thus,

$$R = |x_0 - z_s| = |0 - (\pm i)| = 1$$

and

$$I = (x_0 - R, x_0 + I) = (0 - 1, 0 + 1) = (-1, 1) \quad .$$

30.6 c. The differential equation

$$\left(4 + x^2\right) y'' + 2xy' = 0$$

is in preferred form. So the singular points are the points on the complex plane where the first coefficient is zero:

$$4 + z^2 = 0 \quad \rightarrowtail \quad z_s = \pm 2i \quad .$$

Thus,

$$R = |x_0 - z_s| = |0 - (\pm 2i)| = 2$$

and

$$I = (x_0 - R, x_0 + I) = (0 - 2, 0 + 2) = (-2, 2) \quad .$$

30.6 e. The differential equation

$$\left(4 - x^2\right) y'' - 5xy' - 3y = 0$$

is in preferred form. So the singular points are the points on the complex plane where the first coefficient is zero:

$$4 - z^2 = 0 \quad \rightarrowtail \quad z_s = \pm 2 \quad .$$

Thus,

$$R = |x_0 - z_s| = |0 - (\pm 2)| = 2$$

and

$$I = (x_0 - R, x_0 + I) = (0-2, 0+2) = (-2, 2) \quad .$$

30.6 g. The differential equation is in preferred form, and the first coefficient is the constant 1 (which is never 0). So there are no singular points, $R = \infty$ and

$$I = (x_0 - R, x_0 + I) = (-\infty, \infty) \quad .$$

30.6 i. The differential equation is in preferred form, and the first coefficient is the constant 1 (which is never 0). So there are no singular points, $R = \infty$ and

$$I = (x_0 - R, x_0 + I) = (-\infty, \infty) \quad .$$

30.6 k. The differential equation is in preferred form, and the first coefficient is the constant 1 (which is never 0). So there are no singular points, $R = \infty$ and

$$I = (x_0 - R, x_0 + I) = (-\infty, \infty) \quad .$$

30.7 a. Starting with the first and using the hint:

$$6 \cdot 4 \cdot 2 = (2 \cdot 3)(2 \cdot 2)(2 \cdot 1) = [2 \cdot 2 \cdot 2][3 \cdot 2 \cdot 1] = 2^3 3! \quad .$$

For the second:

$$\begin{aligned} 8 \cdot 6 \cdot 4 \cdot 2 &= (2 \cdot 4)(2 \cdot 3)(2 \cdot 2)(2 \cdot 1) \\ &= [2 \cdot 2 \cdot 2 \cdot 2][4 \cdot 3 \cdot 2 \cdot 1] = 2^3 3! \quad . \end{aligned}$$

In general:

$$\begin{aligned} &(2m)(2m-2)(2m-4)\cdots 6 \cdot 4 \cdot 2 \\ &\qquad = (2m)(2[m-1])(2[m-2])\cdots(2\cdot 3)(2 \cdot 2)(2 \cdot 1) \\ &\qquad = [\underbrace{2 \cdot 2 \cdots 2 \cdot 2}_{m \text{ times}}][m \cdot (m-1) \cdot (m-2) \cdots 3 \cdot 2 \cdot 1] = 2^m m! \quad . \end{aligned}$$

30.8. This exercise set is similar to exercise set 30.9, which is worked out in detail.

30.9 a. Letting $y = y_\lambda(x) = \sum_{k=0}^{\infty} a_k x^k$,

$$\begin{aligned} 0 &= \left(1 - x^2\right) y'' - xy' + \lambda y \\ &= \left(1 - x^2\right) \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - x \sum_{k=1}^{\infty} k a_k x^{k-1} + \lambda \sum_{k=0}^{\infty} a_k x^k \\ &= 1 \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - x^2 \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - x \sum_{k=1}^{\infty} k a_k x^{k-1} + \lambda \sum_{k=0}^{\infty} a_k x^k \end{aligned}$$

$$= \underbrace{\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}}_{n=k-2} + \underbrace{\sum_{k=2}^{\infty} (-1)k(k-1)a_k x^k}_{n=k} + \underbrace{\sum_{k=1}^{\infty} (-1)k a_k x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} \lambda a_k x^k}_{n=k}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=2}^{\infty} (-1)n(n-1)a_n x^n + \sum_{n=1}^{\infty} (-1)n a_n x^n$$

$$+ \sum_{n=0}^{\infty} \lambda a_n x^n$$

$$= \left[(0+2)(0+1)a_{0+2}x^0 + (1+2)(1+1)a_{1+2}x^1 + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n\right]$$

$$+ \sum_{n=2}^{\infty} (-1)n(n-1)a_n x^n + \left[-1a_1x^1 + \sum_{n=1}^{\infty}(-1)n a_n x^n\right]$$

$$+ \left[\lambda a_0 x^0 + \lambda a_1 x^1 + \sum_{n=2}^{\infty} \lambda a_n x^n\right]$$

$$= [2a_2 + \lambda a_0]\, x^0 + [3 \cdot 2a_3 - (1-\lambda)a_1]\, x^1$$

$$+ \sum_{n=2}^{\infty} \Big[(n+2)(n+1)a_{n+2} - [n^2 - \lambda]a_n\Big] x^n \quad .$$

This means

$$a_2 = \frac{-\lambda}{2} a_0 \quad , \quad a_3 = \frac{1-\lambda}{3 \cdot 2} a_1$$

and

$$a_{n+2} = \frac{n^2 - \lambda}{(n+2)(n+1)} a_n \qquad \text{for} \quad n \ge 2$$

$$\hookrightarrow \qquad a_k = \frac{(k-2)^2 - \lambda}{k(k-1)} a_{k-2} \qquad \text{for} \quad k \ge 4 \quad .$$

However, if we plug $k = 2$ and $k = 3$ into the last equation, we find that we rederive the already derived formulas for a_2 and a_3. So our recursion formula is

$$a_k = \frac{(k-2)^2 - \lambda}{k(k-1)} a_{k-2} \qquad \text{for} \quad k \ge 3 \quad .$$

30.9 b. Because the recursion formula is

$$a_k = \underbrace{\frac{(k-2)^2 - \lambda}{k(k-1)}}_{\rho(k)} a_{k-2} \qquad \text{for} \quad k \ge 2 \quad ,$$

theorem 30.5 on page 612 immediately applies and tells us that

$$y_\lambda(x) = a_0 y_{\lambda,E}(x) + a_1 y_{\lambda,O}(x)$$

where $y_{\lambda,E}$ and $y_{\lambda,O}$ are, respectively, the even- and odd-termed series

$$y_{\lambda,E}(x) = \sum_{m=0}^{\infty} c_{2m} x^{2m} \qquad \text{and} \qquad y_{\lambda,O}(x) = \sum_{m=0}^{\infty} c_{2m+1} x^{2m+1}$$

with $c_0 = c_1 = 1$ and the other c_k's satisfying the recursion formula

$$c_k = \rho(k)\, c_{k-2} = \frac{(k-2)^2 - \lambda}{k(k-1)} c_{k-2} \qquad \text{for} \quad k \geq 2 \quad .$$

30.9 c. Using the recursion formula derived earlier,

$$a_{N+2} = \rho(N+2)\, a_N = \frac{([N+2]-2)^2 - \lambda}{[N+2]([N+2]-1)} a_N = 0 \cdot a_N$$

$$\hookrightarrow \quad \rho(N+2) = \frac{N^2 - \lambda}{(N+2)(N+1)} = 0 \quad \rightarrowtail \quad \lambda_N = \lambda = N^2 \quad .$$

30.9 d. Let ρ be as above,

$$\rho(k) = \frac{(k-2)^2 - \lambda}{k(k-1)} \quad .$$

Remember, for any positive integer K, the corresponding coefficients of $y_{\lambda,E}(x)$ and $y_{\lambda,O}(x)$ are given, respectively, by

$$c_{2K} = \rho(2K)\, c_{2K-2} = \rho(2K)\rho(2K-2)\, c_{2K-4}$$
$$= \cdots = \rho(2K)\rho(2K-2)\cdots\rho(2)\, c_0$$

and

$$c_{2K+1} = \rho(2K+1)\, c_{2K-1} = \rho(2K+1)\rho(2K-1)\, c_{2K-3}$$
$$= \cdots = \rho(2K+1)\rho(2K-1)\cdots\rho(3)\, c_1 \quad .$$

Since $c_0 = c_1 = 1$, this reduces to

$$c_{2K} = \rho(2K)\rho(2K-2)\cdots\rho(2)$$

and

$$c_{2K+1} = \rho(2K+1)\rho(2K-1)\cdots\rho(3) \quad .$$

From this, we see that

- $c_{2K} = 0$ if and only if there is an even nonnegative integer $m = 2n \leq 2K$ with $\rho(m) = \rho(2n) = 0$,

and

- $c_{2K+1} = 0$ if and only if there is an even nonnegative integer $m = 2n+1 \leq 2K+1$ with $\rho(m) = \rho(2n+1) = 0$,

Combining this with the fact (derived in earlier) that $\rho(k) = 0$ if and only if $\lambda = N^2$ and $k = N + 2$ for some nonnegative integer N, then yields the following:

1. If $\lambda = N^2$ for some nonnegative even integer $N = 2n$, then $\rho(m) = 0$ for $m = N + 2 = 2n + 2$, which, in turn, means that

$$c_{2m} = 0 \quad \text{for} \quad 2m = 2n+2,\ 2n+4,\ 2n+6,\ \dots \quad ,$$
$$c_{2m} \neq 0 \quad \text{for} \quad 2m = 0,\ 2,\ 4,\ \dots,\ 2n$$

and

$$c_{2m+1} \neq 0 \quad \text{for} \quad 2m+1 = 1,\ 3,\ 5,\ \dots \quad .$$

Hence,

$$\begin{aligned} y_{\lambda,E}(x) &= \sum_{m=0}^{\infty} c_{2m}x^{2m} \\ &= \sum_{m=0}^{n} c_{2m}x^{2m} + \sum_{m=n+1}^{\infty} c_{2m}x^{2m} = p_N(x) + 0 \end{aligned}$$

where, taking into account the facts that $c_0 = 1$ and $c_{2n} = c_N \neq 0$,

$$p_m(x) = 1 + c_2x^2 + c_4x^4 + \cdots + c_Nx^N$$

is an even N^{th} degree polynomial.

Moreover, because every coefficient in

$$y_{\lambda,O}(x) = \sum_{m=0}^{\infty} c_{2m+1}x^{2m+1}$$

is nonzero, this series does not reduce to a polynomial.

2. If $\lambda = N^2$ for some nonnegative odd integer $N = 2n + 1$, , then $\rho(m) = 0$ for $m = N + 2 = 2n + 3$, which, in turn, means that

$$c_{2m+1} = 0 \quad \text{for} \quad 2m+1 = 2n+3,\ 2n+5,\ 2n+7,\ \dots \quad ,$$
$$c_{2m+1} \neq 0 \quad \text{for} \quad 2m+1 = 1,\ 3,\ 5,\ \dots,\ 2n+1$$

and

$$c_{2m} \neq 0 \quad \text{for} \quad 2m = 0,\ 2,\ 4,\ \dots \quad .$$

Hence,

$$\begin{aligned} y_{\lambda,)}(x) &= \sum_{m=0}^{\infty} c_{2m+1}x^{2m+1} \\ &= \sum_{m=0}^{n} c_{2m+1}x^{2m+1} + \sum_{j=n+1}^{\infty} c_{2m+1}x^{2m+1} = p_m(x) + 0 \end{aligned}$$

where, taking into account the facts that $c_1 = 1$ and $c_N = c_{2n+1} \neq 0$,

$$p_N(x) = x + c_3x^3 + c_5x^5 + \cdots + c_Nx^N$$

is an odd m^{th} degree polynomial.

Moreover, because every coefficient in

$$y_{\lambda,E}(x) = \sum_{m=0}^{\infty} c_{2m}x^{2m}$$

is nonzero, this series does not reduce to a polynomial.

3. If $\lambda \neq N^2$ for any nonnegative integer N, then for every nonnegative integer m

$$c_{2m} \neq 0 \qquad \text{and} \qquad c_{2m+1} \neq 0 \quad .$$

Hence, no term in either $y_{\lambda,E}(x)$ or $y_{\lambda,O}(x)$ is zero, and, thus, neither reduces to a polynomial.

30.9 e i. $\lambda_0 = 0^2 = 0$ and $p_0(x) = \sum_{m=1}^{0} c_{2m}x^{2m} = c_0 = 1 \quad .$

30.9 e iii. $\lambda_2 = 2^2 = 4$ and $p_2(x) = \sum_{m=1}^{1} c_{2m}x^{2m} = c_0 + c_2x^2 = 1 + c_2x^2$

where

$$c_2 = \rho(2)\,c_0 = \frac{(2-2)^2 - \lambda_2}{2(2-1)} \cdot 1 = \frac{0-4}{2} = -2 \quad .$$

So, $p_2(x) = 1 - 2x^2 \quad .$

30.9 e v. $\lambda_4 = 4^2 = 16$ and

$$p_4(x) = \sum_{m=1}^{2} c_{2m}x^{2m} = c_0 + c_2x^2 + c_4x^4 = 1 + c_2x^2 + c_4x^4$$

where

$$c_2 = \rho(2)\,c_0 = \frac{(2-2)^2 - \lambda_4}{2(2-1)} \cdot 1 = \frac{0-16}{2} = -8$$

and

$$c_4 = \rho(4)\,c_2 = \frac{(4-2)^2 - \lambda_4}{4(4-1)} c_2 = \frac{4-16}{12}(-8) = 8 \quad .$$

So, $p_4(x) = 1 - 8x^2 + 8x^4 \quad .$

30.9 f. The singular points for the Chebyshev equation

$$\left(1 - x^2\right) y'' - xy' + \lambda y = 0$$

are given by

$$1 - z^2 = 0 \quad \rightarrowtail \quad z_s = z = \pm 1 \quad .$$

So, $R = |x_0 - z_s| = |0 - (\pm 1)| = 1$ and the series are guaranteed to converge on

$$I = (x_0 - R, x_0 + R) = (-1, 1) \quad .$$

The two series solutions are

$$\sum_{m=0}^{\infty} a_{2m}x^{2m} \qquad \text{and} \qquad \sum_{m=0}^{\infty} a_{2m+1}x^{2m+1} \quad .$$

To apply the limit ratio test, we need to compute

$$\lim_{k\to\infty}\left|\frac{(k+1)^{\text{st}}\text{ term}}{k^{\text{th}}\text{ term}}\right|$$

For either series, this is

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+2}x^{k+2}}{a_kx^k}\right| &= \lim_{k\to\infty}\left|\frac{\rho(k+2)\,a_kx^{k+2}}{a_kx^k}\right| \\ &= \lim_{k\to\infty}\left|\rho(k+2)x^2\right| \\ &= \lim_{k\to\infty}\left|\frac{(k-2)^2-\lambda}{k(k-1)}x^2\right| = \cdots = \left|x^2\right| \quad ,\end{aligned}$$

provided the series does not reduce to a polynomial. By the limit ratio test, the series converges if

$$1 > \lim_{k\to\infty}\left|\frac{a_{k+2}x^{k+2}}{a_kx^k}\right| = \left|x^2\right|$$

and diverges if

$$1 < \lim_{k\to\infty}\left|\frac{a_{k+2}x^{k+2}}{a_kx^k}\right| = \left|x^2\right| \quad .$$

Thus, the series converges if $|x| < 1$ and diverges if $|x| > 1$. Consequently, each series that does not reduce to a polynomial has a radius of convergence of $R = 1$, and the largest interval of convergence is

$$I = (x_0 - R, x_0 + R) = (-1, 1) \quad .$$

30.9 g. Both ***i*** and ***ii*** follow immediately from the results given in exercise 30.9 d, while ***iii*** is the result obtained from exercise 30.9 f.

30.11 a. We need to find $\displaystyle\sum_{k=0}^{5}\frac{y^{(k)}(0)}{k!}x^k$ with the derivatives at 0 all in terms of two arbitrary constants a_0 and a_1.

We start by rewriting the equation as

$$y'' = -4y \quad , \tag{1}$$

and setting

$$y(0) = a_0 \qquad \text{and} \qquad y'(0) = a_1 \quad .$$

Thus,

$$y''(0) = -4y(0) = -4a_0 \quad .$$

Differentiating equation (1) gives us

$$y''' = \frac{d}{dx}\left[y''\right] = \frac{d}{dx}\left[-4y\right] = -4y' \quad .$$

So,

$$y^{(3)} = -4y' \quad . \tag{2}$$

In particular,

$$y^{(3)}(0) = -4y'(0) = -4a_1 \quad .$$

Differentiating equation (2) gives us

$$y^{(4)} = \frac{d}{dx}\left[y'''\right] = \frac{d}{dx}\left[-4y'\right] = -4y'' \quad .$$

So,

$$y^{(4)} = -4y'' \quad . \tag{3}$$

In particular,

$$y^{(4)}(0) = -4y''(0) = -4\left[-4a_0\right] = 4^2 a_0 \quad .$$

Differentiating equation (3) gives us

$$y^{(5)} = \frac{d}{dx}\left[y^{(4)}\right] = \frac{d}{dx}\left[-4y''\right] = -4y^{(3)} \quad .$$

So,

$$y^{(5)} = -4y^{(3)} \quad . \tag{4}$$

In particular,

$$y^{(5)}(0) = -4y^{(3)}(0) = -4\left[-4a_1\right] = 4^2 a_1 \quad .$$

Plugging the above values for the derivatives at 0 into the corresponding 5^{th} degree Taylor polynomial then yields

$$\begin{aligned}
\sum_{k=0}^{5} \frac{y^{(k)}(0)}{k!}x^k &= \frac{y(0)}{0!}x^0 + \frac{y'(0)}{1!}x^1 + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 \\
&\quad + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 \\
&= \frac{a_0}{0!}x^0 + \frac{a_1}{1!}x^1 + \frac{-4a_0}{2!}x^2 + \frac{-4a_1}{3!}x^3 \\
&\quad + \frac{4^2 a_0}{4!}x^4 + \frac{4^2 a_1}{5!}x^5 \\
&= a_0 + a_1 x - \frac{4a_0}{2!}x^2 - \frac{4a_1}{3!}x^3 + \frac{4^2 a_0}{4!}x^4 + \frac{4^2 a_1}{5!}x^5 \\
&= a_0\left[1 - \frac{4}{2!}x^2 + \frac{4^2}{4!}x^4\right] + a_1\left[x - \frac{4}{3!}x^3 + \frac{4^2}{5!}x^5\right] \quad .
\end{aligned}$$

Since this is a second-order linear equation with constant coefficients, there are no singular points, and theorem 30.7 on page 613 assures us that $I = (-\infty, \infty)$.

30.11 c. We need to find $\displaystyle\sum_{k=0}^{4} \frac{y^{(k)}(0)}{k!}x^k$ with the derivatives at 0 all in terms of two arbitrary constants a_0 and a_1 .

We start by rewriting the equation as

$$y'' = -e^{2x} y \quad , \tag{1}$$

and setting

$$y(0) = a_0 \qquad \text{and} \qquad y'(0) = a_1 \quad .$$

Thus,

$$y''(0) = -e^{2\cdot 0}y(0) = -a_0 \quad .$$

Differentiating equation (1) gives us

$$y^{(3)} = \frac{d}{dx}\left[y''\right] = \frac{d}{dx}\left[-e^{2x}y\right] = -2e^{2x}y - e^{2x}y' = -e^{2x}\left[2y + y'\right] \quad .$$

So,

$$y^{(3)} = -e^{2x}\left[2y + y'\right] \quad . \tag{2}$$

In particular,

$$y^{(3)}(0) = -e^{2\cdot 0}\left[2y(0) + y'(0)\right] = -2a_0 - a_1 \quad .$$

Differentiating equation (2) gives us

$$\begin{aligned} y^{(4)} &= \frac{d}{dx}\left[y^{(3)}\right] = \frac{d}{dx}\left[-e^{2x}\left[2y + y'\right]\right] \\ &= -2e^{2x}\left[2y + y'\right] - e^{2x}\left[2y' + y''\right] \quad , \end{aligned}$$

which simplifies to

$$y^{(4)} = -e^{2x}\left[4y + 4y' + y''\right] \quad . \tag{3}$$

In particular,

$$\begin{aligned} y^{(4)}(0) &= -e^{2\cdot 0}\left[4y(0) + 4y'(0) + y''(0)\right] \\ &= -4a_0 - 4a_1 - [-a_0] = -3a_0 - 4a_1 \quad . \end{aligned}$$

Plugging the above values for the derivatives at 0 into the corresponding 4^{th} degree Taylor polynomial then gives

$$\begin{aligned} \sum_{k=0}^{4} \frac{y^{(k)}(0)}{k!}x^k &= \frac{y(0)}{0!}x^0 + \frac{y'(0)}{1!}x^1 + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 \\ &= a_0 + a_1 x + \frac{-a_0}{2!}x^2 + \frac{-2a_0 - a_1}{3!}x^3 + \frac{-3a_0 - 4a_1}{4!}x^4 \\ &= a_0\left[1 - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 - \frac{3}{4!}x^4\right] + a_1\left[x - \frac{1}{3!}x^3 - \frac{4}{4!}x^4\right] \quad . \end{aligned}$$

30.11 e. Rewrite the equation as

$$y'' = \sin(x) - xy \quad , \tag{1}$$

and set

$$y(0) = a_0 \qquad \text{and} \qquad y'(0) = a_1 \quad .$$

Thus,

$$y''(0) = \sin(0) - 0y(0) = 0 \quad .$$

Differentiating equation (1) gives us

$$y^{(3)} = \frac{d}{dx}\left[y''\right] = \frac{d}{dx}\left[\sin(x) - xy\right] = \cos(x) - y - xy' \quad .$$

So,

$$y^{(3)} = \cos(x) - y - xy' \quad . \tag{2}$$

In particular,

$$y^{(3)}(0) = \cos(0) - y(0) - 0y'(0) = 1 - a_0 \quad .$$

Differentiating equation (2) gives us

$$y^{(4)} = \frac{d}{dx}\left[y^{(3)}\right] = \frac{d}{dx}\left[\cos(x) - y - xy'\right] = -\sin(x) - y' - y' - xy'' \quad .$$

So,

$$y^{(4)} = -\sin(x) - 2y' - xy'' \quad . \tag{3}$$

In particular,

$$y^{(4)}(0) = -\sin(0) - 2y'(0) - 0y''(0) = -2a_1 \quad .$$

Differentiating equation (3) gives us

$$\begin{aligned} y^{(5)} &= \frac{d}{dx}\left[y^{(4)}\right] = \frac{d}{dx}\left[-\sin(x) - 2y' - xy''\right] \\ &= -\cos(x) - 2y'' - y'' - xy''' \quad . \end{aligned}$$

So,

$$y^{(5)} = -\cos(x) - 3y'' - xy''' \quad . \tag{3}$$

In particular,

$$y^{(5)}(0) = -\cos(0) - 3y''(0) - 0y'''(0) = -1 - 3[0] = -1 \quad .$$

Thus,

$$\begin{aligned} \sum_{k=0}^{5} \frac{y^{(k)}(0)}{k!}x^k &= \frac{y(0)}{0!}x^0 + \frac{y'(0)}{1!}x^1 + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 \\ &\quad + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 \\ &= a_0 + a_1 x + \frac{0}{2!}x^2 + \frac{1 - a_0}{3!}x^3 + \frac{-2a_1}{4!}x^4 + \frac{-1}{5!}x^5 \\ &= a_0 + a_1 x + \frac{1 - a_0}{3!}x^3 - \frac{2a_1}{4!}x^4 - \frac{1}{5!}x^5 \quad . \end{aligned}$$

30.11 g. Rewrite the equation as

$$y'' = y^2 \quad , \tag{1}$$

and set

$$y(0) = a_0 \quad \text{and} \quad y'(0) = a_1 \quad .$$

Thus,

$$y''(0) = [y(0)]^2 = {a_0}^2 \quad .$$

Differentiating equation (1) gives us

$$y^{(3)} = \frac{d}{dx}\left[y''\right] = \frac{d}{dx}\left[y^2\right] = 2yy' \quad .$$

So,

$$y^{(3)} = 2yy' \quad . \tag{2}$$

In particular,

$$y^{(3)}(0) = 2y(0)y'(0) = 2a_0a_1 \quad .$$

Differentiating equation (2) gives us

$$y^{(4)} = \frac{d}{dx}\left[y^{(3)}\right] = \frac{d}{dx}\left[2yy'\right] = 2y'y' + 2yy'' \quad .$$

So,

$$y^{(4)} = 2\left[y'\right]^2 + 2yy'' \quad . \tag{3}$$

In particular,

$$y^{(4)}(0) = 2\left[y'(0)\right]^2 + 2y(0)y''(0) = 2a_1{}^2 + 2a_0\left[a_0{}^2\right] = 2\left[a_0{}^3 + a_1{}^2\right] \quad .$$

Differentiating equation (3) gives us

$$y^{(5)} = \frac{d}{dx}\left[y'''\right] = \frac{d}{dx}\left[2\left[y'\right]^2 + 2yy''\right] = 4y'y'' + 2y'y'' + 2yy^{(3)} \quad .$$

So,

$$y^{(5)} = 6y'y'' + 2yy^{(3)} \quad . \tag{3}$$

In particular,

$$y^{(5)}(0) = 5y'(0)y''(0) + 2y(0)y^{(3)}(0) = 6a_1\left[a_0{}^2\right] + 2a_0[2a_0a_1] = 10a_0{}^2a_1 \quad .$$

Thus,

$$\begin{aligned}\sum_{k=0}^{5}\frac{y^{(k)}(0)}{k!}x^k &= \frac{y(0)}{0!}x^0 + \frac{y'(0)}{1!}x^1 + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 \\ &\quad + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 \\ &= a_0 + a_1x + \frac{a_0{}^2}{2!}x^2 + \frac{2a_0a_1}{3!}x^3 + \frac{2\left[a_0{}^3 + a_1{}^2\right]}{4!}x^4 + \frac{10a_0{}^2a_1}{5!}x^5 \quad .\end{aligned}$$