38

General Solutions to Homogeneous Linear Systems

In this chapter, we will develop the basic "linear" theory regarding solutions to standard first-order homogeneous $N \times N$ linear systems of differential equations. Fortunately, this theory is very similar to that for single linear differential equations developed in chapters 12, 14 and 15. In fact, we may even use what we already know about general solutions to N^{th}-order linear differential equations to help guide our development here. We will also make heavy use of some of the results you learned in linear algebra regarding solutions to $N \times N$ linear systems of algebraic equations.

Will we finally actually solve a few systems in this chapter? No, not really, but we will need the theory developed here when we finally do start solving systems in the next chapter.

38.1 Basic Assumptions and Terminology

The System and Basic Assumptions

For the rest of this chapter, (α, β) is some interval, N is some positive integer, and

$$\mathbf{P} = \mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1N}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2N}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1}(t) & p_{N2}(t) & \cdots & p_{NN}(t) \end{bmatrix} \quad . \tag{38.1}$$

is an $N \times N$ matrix of functions, each of which is continuous over the interval (α, β).

For now, our interest is just in the possible solutions to the homogeneous system

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \tag{38.2}$$

over (α, β). For brevity, in our computations, we will often just refer to this as "our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$" with the implicit understanding that $\mathbf{P}$ is as just described. Along these same lines, let us simplify our verbage and agree that, in our discussion, the phrases "solution" and "solution to $\mathbf{x}' = \mathbf{P}\mathbf{x}$" both mean "solution over (α, β) to $\mathbf{x}' = \mathbf{P}\mathbf{x}$".

Also keep in mind that a solution $\mathbf{x}$ to this system is a vector-valued function on (α, β)

$$\mathbf{x}(t) = \left[x_1(t)\ ,\ x_2(t)\ ,\ \ldots\ ,\ x_N(t)\right]^{\mathsf{T}}$$

satisfying $\mathbf{x}' = \mathbf{P}\mathbf{x}$ at every point in the interval (α, β). Often, we will have several such vector-valued functions. When we do, we will use superscripts to distinguish the different vector-valued functions; that is, we will write the set of vector-valued functions as either

$$\left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^M \right\} \qquad \text{or} \qquad \left\{ \mathbf{x}^1(t) , \mathbf{x}^2(t) , \ldots , \mathbf{x}^M(t) \right\}$$

with

$$\mathbf{x}^1(t) = \begin{bmatrix} x_1^1(t) \\ x_2^1(t) \\ \vdots \\ x_N^1(t) \end{bmatrix} \quad , \quad \mathbf{x}^2(t) = \begin{bmatrix} x_1^2(t) \\ x_2^2(t) \\ \vdots \\ x_N^2(t) \end{bmatrix} \quad , \quad \ldots \quad \text{and} \quad \mathbf{x}^M(t) = \begin{bmatrix} x_1^M(t) \\ x_2^M(t) \\ \vdots \\ x_N^M(t) \end{bmatrix} \quad .$$

!►*Example 38.1:* *It is easily verified one pair of solutions* $\{\mathbf{x}^1, \mathbf{x}^2\}$ *to*

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \textit{with} \quad \mathbf{P} = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$$

is given by

$$\mathbf{x}^1(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \qquad \textit{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} -2e^{-4t} \\ 5e^{-4t} \end{bmatrix} \quad ,$$

which we may write more simply as

$$\mathbf{x}^1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \qquad \textit{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} \quad .$$

Linearity

Naturally, we should expect "linearity" to play a role in solving linear systems of differential equations, and to simplify our discussion, at least initially, let L be the operator

$$L = \frac{d}{dt} - \mathbf{P} \quad .$$

That is, for any differentiable vector-valued function $\mathbf{x}$,

$$L[\mathbf{x}] = \frac{d\mathbf{x}}{dt} - \mathbf{P}\mathbf{x} \quad .$$

Now let $\mathbf{x}^1$ and $\mathbf{x}^2$ be any pair of differentiable vector-valued functions, and c_1 and c_2 any pair of constants. Using the linearity of the derivative and matrix multiplication, we have

$$\begin{aligned} L\left[c_1\mathbf{x}^1 + c_2\mathbf{x}^2\right] &= \frac{d}{dt}\left[c_1\mathbf{x}^1 + c_2\mathbf{x}^2\right] - \mathbf{P}\left[c_1\mathbf{x}^1 + c_2\mathbf{x}^2\right] \\ &= c_1\frac{d\mathbf{x}^1}{dt} + c_2\frac{d\mathbf{x}^2}{dt} - c_1\mathbf{P}\mathbf{x}^1 - c_2\mathbf{P}\mathbf{x}^2 \\ &= c_1\left(\frac{d\mathbf{x}^1}{dt} - \mathbf{P}\mathbf{x}^1\right) + c_2\left(\frac{d\mathbf{x}^2}{dt} - \mathbf{P}\mathbf{x}^2\right) = c_1L[\mathbf{x}^1] + c_2L[\mathbf{x}^2] \quad . \end{aligned}$$

Of course, the above computations can easily be repeated using larger sets of functions and constants, giving us:

Theorem 38.1 (basic linearity property for systems)
Assume

$$L = \frac{d}{dt} - \mathbf{P}$$

and let, for some finite positive integer M *,*

$$\{c_1, c_2, \ldots, c_M\} \qquad \text{and} \qquad \{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\}$$

be sets, respectively, of constants and differentiable vector-valued functions on (α, β) *. Then*

$$L[c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M] = c_1 L[\mathbf{x}^1] + c_2 L[\mathbf{x}^2] + \cdots + c_M L[\mathbf{x}^M] \quad .$$

This naturally leads us to reintroduce some terminology first used in chapter 12: Given a finite set of vector-valued functions $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\}$ on the interval (α, β) , a *linear combination* of these $\mathbf{x}^k$'s (on the interval (α, β)) is any expression of the form

$$c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M$$

where the c_k's are constants. Remember, these are vector-valued functions on an interval (α, β) . So

$$\mathbf{x} = c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M$$

means

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_M\mathbf{x}^M(t) \qquad \text{for} \quad \alpha < t < \beta \quad .$$

Principle of Superposition, Linear Independence and Fundamental Solution Sets

Recall that any linear combination of solutions to a single homogeneous linear differential equation is another solution to that differential equation (see theorem 12.2 on page 266). Likewise, for homogeneous linear systems of differential equations, we have:

Lemma 38.2 (principle of superposition for systems)
If $\mathbf{x}^1$ *,* $\mathbf{x}^2$ *, ..., and* $\mathbf{x}^M$ *are all solutions to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *, then so is any linear combination of these* $\mathbf{x}_k$ *'s ,*

$$\mathbf{x} = c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M \quad .$$

The proof of this lemma is easy. You do it.

?►*Exercise 38.1:* *Use theorem 38.1 to verify the last lemma.*

!►*Example 38.2:* *We already know that*

$$\mathbf{x}^1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t}$$

are solutions to

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{P} = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} \quad .$$

The above lemma now assures us that, for any pair c_1 and c_2 of constants,

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t}$$

is also a solution to our homogeneous system.

The above lemma tells us that we can construct "more general" solution formulas from a small set of particular solutions by forming arbitrary linear combinations of these particular solutions. This naturally leads us to ask if we could find some "fundamental set" of solutions

$$\left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^M \right\}$$

from which we can construct a general solution of the form

$$\mathbf{x} = c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M$$

where the c_k's are arbitrary constants. And, naturally, we will want this set of solutions to be as small as possible. In particular, no $\mathbf{x}^k$ should be a linear combination of the others. After all, if, say, $\mathbf{x}^M$ is a linear combination of the other $\mathbf{x}^k$'s over (α, β)

$$\mathbf{x}^M = C_1\mathbf{x}^1 + C_2\mathbf{x}^2 + \cdots + C_{M-1}\mathbf{x}^{M-1} \quad ,$$

then

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_{M-1}\mathbf{x}^{M-1} + c_M\mathbf{x}^M \\ &= c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_{M-1}\mathbf{x}^{M-1} + c_M\left[C_1\mathbf{x}^1 + \cdots + C_{M-1}\mathbf{x}^{M-1}\right] \\ &= (c_1 + c_M C_1)\,\mathbf{x}^1 + (c_2 + c_M C_2)\,\mathbf{x}^2 + \cdots + (c_{M-1} + c_M C_{M-1})\,\mathbf{x}^{M-1} \quad , \end{aligned}$$

showing that, if $\mathbf{x}^M$ is a linear combination of the other $\mathbf{x}^k$'s , then we can then convert any linear combination of solutions from the set

$$\left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^{M-1} , \mathbf{x}^M \right\}$$

to a linear combination of solutions from the smaller set

$$\left\{ \mathbf{x}^1 , \mathbf{x}^2 , \ldots , \mathbf{x}^{M-1} \right\} \quad .$$

Accordingly, let us reintroduce some more terminology. We'll refer to a set $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\right\}$ of M vector-valued functions on (α, β) as being

1. *linearly independent* (over (α, β)) if none of the $\mathbf{x}^k$'s can be written as a linear combination of the other $\mathbf{x}^k$'s on (α, β) ,

and

2. *linearly dependent* (over (α, β)) if at least one of the $\mathbf{x}^k$'s can be written as a linear combination of the other $\mathbf{x}^k$'s on (α, β) .

Of course, for a pair of vector-valued functions $\{\mathbf{x}^1, \mathbf{x}^2\}$, the definition of linear independence reduces to this set being linearly independent if and only if neither $\mathbf{x}^1$ or $\mathbf{x}^2$ is a constant multiple of each other. Tests for determining the linear independence of larger sets will be discussed later.

While we are reviving old terminology, let us revive the term "fundamental set", by saying that a set $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\right\}$ of M vector-valued functions on (α, β) is a *fundamental set of solutions* for our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ (on (α, β)) if and only if all the following hold on (α, β):

1. Each $\mathbf{x}^k$ is a solution to $\mathbf{x}' = \mathbf{P}\mathbf{x}$.
2. The set is a linearly independent on (α, β).
3. Every solution to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ can be written as a linear combination of the $\mathbf{x}^k$'s.

So, do fundamental sets of solutions exist? And how might we recognize when a solution set is "fundamental"? These are the questions we'll deal with for the next several pages.

A "Matrix/Vector" Formula for Linear Combinations

To help apply what we know from linear algebra, it helps to first make the following observation regarding linear combinations of the $\mathbf{x}^k$'s:

$$c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M = c_1\begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_N^1 \end{bmatrix} + c_2\begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_N^2 \end{bmatrix} + \cdots + c_M\begin{bmatrix} x_1^M \\ x_2^M \\ \vdots \\ x_N^M \end{bmatrix}$$

$$= \begin{bmatrix} x_1^1c_1 + x_1^2c_2 + \cdots + x_1^Mc_M \\ x_2^1c_1 + x_2^2c_2 + \cdots + x_2^Mc_M \\ \vdots \\ x_N^1c_1 + x_N^2c_2 + \cdots + x_N^Mc_M \end{bmatrix}$$

$$= \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^M \\ x_2^1 & x_2^2 & \cdots & x_2^M \\ \vdots & \vdots & \ddots & \vdots \\ x_N^1 & x_N^2 & \cdots & x_N^M \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} .$$

That is, for $\alpha < t < \beta$,

$$c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_M\mathbf{x}^M(t) = [\mathbf{X}(t)]\mathbf{c}$$

where

$$\mathbf{X}(t) = \begin{bmatrix} x_1^1(t) & x_1^2(t) & \cdots & x_1^M(t) \\ x_2^1(t) & x_2^2(t) & \cdots & x_2^M(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_N^1(t) & x_N^2(t) & \cdots & x_N^M(t) \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} .$$

The above $N \times M$ matrix-valued function $\mathbf{X}$ will be important to us. Until we can come up with better terminology, we'll simply call it the *matrix whose* k^{th} *column is given by* $\mathbf{x}^k$.

!►Example 38.3: *The matrix whose k^{th} column is given by $\mathbf{x}^k$ when*

$$\mathbf{x}^1(t) = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}^2(t) = \begin{bmatrix} -2e^{-4t} \\ 5e^{-4t} \end{bmatrix}$$

is

$$\mathbf{X}(t) = \begin{bmatrix} e^{3t} & -2e^{-4t} \\ e^{3t} & 5e^{-4t} \end{bmatrix} \quad .$$

Observe that, indeed,

$$[\mathbf{X}(t)]\mathbf{c} = \begin{bmatrix} e^{3t} & -2e^{-4t} \\ e^{3t} & 5e^{-4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + c_2(-2)e^{-4t} \\ c_1 e^{3t} + 5c_2 e^{-4t} \end{bmatrix}$$

$$= c_1 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -2e^{-4t} \\ e^{-4t} \end{bmatrix} = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) \quad .$$

38.2 Deriving the Main Results

In this section we will derive the answers to our two questions of whether fundamental sets of solutions exist, and how we might recognize them (if they exist). We will derive these answers in stages, using mainly an existence theorem from the previous chapter, and well-known facts from linear algebra. As we go along, we will summarize the immediate results of our derivations in a series of lemmas, which, themselves, will be summarized in the two major theorems of this chapter, theorems 38.12 and 38.13 in section 38.3. Of course, if you are too impatient for this admittedly lengthy derivation, you could skip the rest of this section and go straight to page 38–13 and just read section 38.3, but you won't appreciate the results there nearly as well.

Throughout these derivations, keep in mind that $\mathbf{P}$ is always an $N \times N$ matrix of continuous functions on an interval (α, β), whether or not we remember to explicitly say so.

Immediate Results

The first two lemmas of this section should require almost no discussion. The first is simply a simplification of theorem 37.3 on page 37–17, and the second is an application of a test for linear independence that you should recall from your study of linear algebra.[1]

Lemma 38.3
Let t_0 be a point in the interval (α, β) and $\mathbf{a}$ a constant vector. Then the initial-value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \quad \text{with} \quad \mathbf{x} = \mathbf{a} \quad ,$$

has exactly one solution over the interval (α, β).

[1] If you don't recall this test, see exercise 38.3 at the end of the chapter.

Lemma 38.4 (test for linear independence)
A set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\}$ of vector-valued functions is linearly independent on (α, β) if and only if the only choice of constants c_1, c_2, ... and c_M such that

$$c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_M\mathbf{x}^M(t) = \mathbf{0} \qquad \text{for} \quad \alpha < t < \beta$$

is

$$c_1 = c_2 = \cdots = c_M = 0 \quad .$$

"Fundamental Sets" for Initial-Value Problems

Suppose we have a set

$$\left\{ \mathbf{x}^1(t), \mathbf{x}^2(t), \ldots, \mathbf{x}^M(t) \right\}$$

of particular solutions to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$, and we want to use this set to solve the initial-value problem

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

for some t_0 in (α, β) and some constant vector $\mathbf{a} = [a_1, a_2, \ldots, a_N]^{\mathsf{T}}$. To find this solution, it should seem logical to set

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_M\mathbf{x}^M(t)$$

and then try to determine the constants c_1, c_2, ... and c_M so that the initial condition, $\mathbf{x}(t_0) = \mathbf{a}$, is satisfied. That is, we try to solve the vector equation

$$c_1\mathbf{x}^1(t_0) + c_2\mathbf{x}^2(t_0) + \cdots + c_M\mathbf{x}^M(t_0) = \mathbf{a} \tag{38.3}$$

for the M unknowns c_1, c_2, ... and c_M.

Keep in mind that equation (38.3) is equivalent to the algebraic system of N equations and M unknowns

$$\begin{aligned}
x_1^1(t_0)c_1 + x_1^2(t_0)c_2 + \cdots + x_1^M(t_0)c_M &= a_1 \\
x_2^1(t_0)c_1 + x_2^2(t_0)c_2 + \cdots + x_2^M(t_0)c_M &= a_2 \\
&\vdots \\
x_N^1(t_0)c_1 + x_N^2(t_0)c_2 + \cdots + x_N^M(t_0)c_M &= a_N
\end{aligned} \quad , \tag{38.4}$$

which can also be written as the matrix/vector equation

$$[\mathbf{X}(t_0)]\,\mathbf{c} = \mathbf{a} \tag{38.5}$$

where $\mathbf{c} = [c_1, c_2, \ldots, c_M]^{\mathsf{T}}$ and $\mathbf{X}(t)$ is the $N \times M$ matrix whose k^{th} column is given by $\mathbf{x}^k(t)$.

But solving algebraic system (38.4) or equation (38.5) is a classic problem in linear algebra, and from linear algebra we know that there is one and only solution $\mathbf{c}$ for each $\mathbf{a}$ if and only if both of the following hold:

1. $M = N$.
2. *The $N \times N$ matrix $\mathbf{X}(t_0)$ is invertible.*

If these conditions are satisfied, then $\mathbf{c}$ can be determined from each $\mathbf{a}$ by

$$\mathbf{c} = [\mathbf{X}(t_0)]^{-1}\mathbf{a}$$

where $[\mathbf{X}(t_0)]^{-1}$ is the inverse of matrix $\mathbf{X}(t_0)$. (In practice, though, a "row reduction" method may be a more efficient way to find $\mathbf{c}$.)

So, ideally, to solve an arbitrary $N \times N$ initial-value problem of the form

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a} \quad ,$$

we want a set of N solutions to the system of differential equations

$$\left\{ \mathbf{x}^1(t) , \mathbf{x}^2(t) , \ldots , \mathbf{x}^N(t) \right\}$$

such that the matrix $\mathbf{X}(t)$ formed from this set is invertible when $t = t_0$. And to make life easier, recall that there is a relatively simple test for determining if a given square matrix $\mathbf{M}$ is invertible[2] based on the matrix's determinant, $\det(\mathbf{M})$; namely,

$$\mathbf{M} \text{ is invertible} \quad \iff \quad \det(\mathbf{M}) \neq 0 \quad .$$

Combined with lemma 37.3, telling us that each initial-value problem has exactly one solution, the above gives us

Lemma 38.5
Assume $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\right\}$ *is a set of* M *solutions to our system* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *, and* $\mathbf{X}$ *is the* $M \times N$ *matrix whose* k^{th} *column is given by* $\mathbf{x}^k$ *. Then, for each constant vector* $\mathbf{a}$ *, there is exactly one choice of constants* c_1 *,* c_2 *, ... and* c_M *such that*

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_M\mathbf{x}^M(t)$$

is the one and only solution to the initial-value problem

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \textit{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

if and only if both of the following hold:

1. $M = N$.

2. $\det(\mathbf{X}(t_0)) \neq 0$.

!►*Example 38.4:* *Consider the* 2×2 *initial-value problem*

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}\mathbf{x} \qquad \textit{with} \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad .$$

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[2] More terminology you should recall:

$$\mathbf{M} \text{ is singular} \quad \iff \quad \mathbf{M} \text{ is not invertible}$$
$$\mathbf{M} \text{ is nonsingular} \quad \iff \quad \mathbf{M} \text{ is invertible} \quad .$$

From previous examples, we know that two solutions to the system of differential equations in this problem are

$$\mathbf{x}^1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} \quad .$$

Letting $\mathbf{X}$ *be the* 2×2 *matrix whose* k^{th} *column is given by* $\mathbf{x}^k$ *, we have*

$$\det(\mathbf{X}(0)) = \det \begin{bmatrix} e^{3\cdot 0} & -2e^{-4\cdot 0} \\ e^{3\cdot 0} & 5e^{-4\cdot 0} \end{bmatrix} = \det \begin{bmatrix} 1 & -2 \\ 1 & 5 \end{bmatrix} = (1\cdot 5) - (-2\cdot 1) \neq 0 \quad .$$

The last lemma now assures us that there is exactly one choice for constants c_1 *and* c_2 *such that*

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t}$$

is a solution to the given initial-value problem.

?►*Exercise 38.2:* *Verify that the solution to the initial-value problem given in the last example is*

$$\mathbf{x}(t) = \frac{11}{7} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \frac{2}{7} \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t}$$

But what if our set of solutions does not satisfy the conditions given in the last lemma? There are then three possibilities: $M < N$, $M > N$, and $M = N$ with $\det(\mathbf{X}(t_0)) = 0$ for some t_0 in (α, β) . And, after recalling our linear algebra, we know that:

1. If $M < N$, then we have more equations than unknowns in system (38.4) (i.e., the system is "overdetermined"), and there is at least one $\mathbf{a}^0 = \left[a_1^0, \ldots, a_N^0\right]^{\mathsf{T}}$ such that matrix/vector equation (38.5) has no solution $\mathbf{c} = [c_1, \ldots, c_M]^{\mathsf{T}}$ when $\mathbf{a} = \mathbf{a}^0$. Consequently, the solution to
$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a}^0 \quad ,$$
which lemma 38.3 assures us exists, cannot be written as
$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_M\mathbf{x}^M(t) \qquad \text{for} \quad \alpha < t < \beta$$
for any choice of constants c_1 , c_2 , ... and c_M .

2. If $M > N$, then we have more unknowns than equations in system (38.4) (i.e., the system is "underdetermined"), and there is at least one *nonzero* $\mathbf{c} = [c_1, \ldots, c_M]^{\mathsf{T}}$ — call it $\mathbf{c}^0 = \left[c_1^0, \ldots, c_M^0\right]^{\mathsf{T}}$ — such that matrix/vector equation (38.5) has solution $\mathbf{c}^0$ when $\mathbf{a} = \mathbf{0}$. Thus
$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{0}$$
has solution
$$\mathbf{x}(t) = c_1^0\mathbf{x}^1(t) + c_2^0\mathbf{x}^2(t) + \cdots + c_M^0\mathbf{x}^M(t) \qquad \text{for} \quad \alpha < t < \beta$$
with at least one of the c_k^0's being nonzero. But, clearly,
$$\mathbf{x}(t) = \mathbf{0} \qquad \text{for} \quad \alpha < t < \beta$$

is also a solution to the above initial-value problem, and since lemma 38.3 tells us that each initial-value problem only has one solution, the two above solutions must be the same; that is,

$$c_1^0\mathbf{x}^1(t) + c_2^0\mathbf{x}^2(t) + \cdots + c_M^0\mathbf{x}^M(t) = \mathbf{0} \qquad \text{for} \quad \alpha < t < \beta \quad , \tag{38.6}$$

which tells us (via the test for linear independence in lemma 38.4) that $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\}$ is not linearly independent.

3. If $M = N$ with $\det(\mathbf{X}(t_0)) = 0$, then

(a) there is at least one $\mathbf{a}^0$ such that matrix/vector equation (38.5) has no solution $\mathbf{c}$ when $\mathbf{a} = \mathbf{a}^0$,

and

(b) there is at least one *nonzero* $\mathbf{c}^0$ such that linear system (38.4) has solution $\mathbf{c}^0$ when $\mathbf{a} = \mathbf{0}$.

Consequently, all the issues described above that arise when $M < N$ or $M > N$ also arise when $M = N$ if $\det(\mathbf{X}(t_0)) = 0$.

Summarizing:

Lemma 38.6
Assume $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\right\}$ is a set of M solutions to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$, and let $\mathbf{X}(t)$ be the $M \times N$ matrix whose k^{th} column is given by $\mathbf{x}^k$. Then:

1. *If $M < N$ or if $M = N$ with $\det(\mathbf{X}(t_0)) = 0$, then there is a solution to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on (α, β) that is not a linear combination of these $\mathbf{x}^k$'s.*
2. *If $M > N$ or if $M = N$ with $\det(\mathbf{X}(t_0)) = 0$, then the given set of $\mathbf{x}^k$'s is not linearly independent.*

This lemma tells us, among other things, that if $M = N$ and $\det(\mathbf{X}(t)) = 0$ for some t_0 in (α, β), then the set $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\}$ can*not* be a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$. That, along with what it tells us about fundamental sets, is certainly worth noting.

Lemma 38.7
Assume $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\}$ is a set of N solutions to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$, and let $\mathbf{X}(t)$ be the $M \times N$ matrix whose k^{th} column is given by $\mathbf{x}^k$. Then:

1. *If $\det(\mathbf{X}(t_0)) = 0$ for one t_0 in (α, β), then $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\}$ is not a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$.*
2. *If $\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\}$ is a fundamental seto of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$, then $\det(\mathbf{X}(t)) \neq 0$ for every t in (α, β)*

General Solutions

It immediately follows from our last few lemmas that a set of M solutions

$$\left\{ \mathbf{x}^1(t)\,,\ \mathbf{x}^2(t)\,,\ \ldots\,,\ \mathbf{x}^M(t) \right\}$$

to our $N \times N$ system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ can be a fundamental set of solutions only if

$$M = N \qquad \text{and} \qquad \det(\mathbf{X}(t_0)) \neq 0 \quad \text{for some } t_0 \text{ in } (\alpha, \beta) \qquad .$$

But do these two conditions ensure that our set is a fundamental set?

Well, suppose we have a set of solutions

$$\left\{ \mathbf{x}^1(t)\,,\ \mathbf{x}^2(t)\,,\ \dots\,,\ \mathbf{x}^N(t) \right\}$$

with $\det(\mathbf{X}(t_0)) \neq 0$ for some t_0 in (α, β). To show this is a fundamental set of solutions we must show both that each solution to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ is a linear combination of these $\mathbf{x}^k$'s, and that this set is linearly independent.

So consider any single solution $\mathbf{x}(t) = \widehat{\mathbf{x}}(t)$ to $\mathbf{x}' = \mathbf{P}\mathbf{x}$. That vector-valued function is then certainly a solution to the initial-value problem

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \widehat{\mathbf{x}}(t_0) \quad .$$

Now, according to lemma 38.3, there is one and only one choice of constants c_1, c_2, ... and c_N such that

$$c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_N\mathbf{x}^N(t)$$

is the one and only solution to this initial-value problem. But the only way for both this linear combination and $\widehat{\mathbf{x}}$ to be the one and only solution is for the two to be the same,

$$\widehat{\mathbf{x}}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_N\mathbf{x}^N(t) \quad ,$$

verifying that, indeed, each solution to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ is a linear combination of our $\mathbf{x}^k$'s.

Verifying the linear independence of our set is now easy: Just consider the particular initial-value problem

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{0} \quad .$$

The solution to this is clearly the constant solution $\mathbf{x}^0 = \mathbf{0}$, which is

$$\mathbf{x}^0(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_N\mathbf{x}^N(t) \qquad \text{for} \quad \alpha < t < \beta$$

with

$$c_1 = c_2 = \cdots = c_N = 0 \quad .$$

Lemma 38.5 assures us that no other choices for c_k's will yield this answer. The test for linear independence from lemma 38.4 then informs us that our set of $\mathbf{x}^k$'s is, as we hoped, linearly independent. In summary:

Lemma 38.8

Every fundamental set of solutions for our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ contains exactly N solutions. Moreover, a set of N solutions $\left\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\right\}$ to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ is a fundamental set of solutions if and only if

$$\det(\mathbf{X}(t_0)) \neq 0 \qquad \text{for some } t_0 \text{ in } (\alpha, \beta)$$

where $\mathbf{X}$ is the $N \times N$ matrix whose k^{th} column is given by $\mathbf{x}^k$.

Look at what combining the last lemma with lemma 38.7 gives us:

$$\begin{aligned} &\det(\mathbf{X}(t_0)) \neq 0 \text{ for some } t_0 \text{ in } (\alpha, \beta) \\ &\qquad \Longrightarrow \quad \{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\} \text{ is a fundamental set of solutions} \\ &\qquad \Longrightarrow \quad \det(\mathbf{X}(t)) \neq 0 \text{ for every } t \text{ in } (\alpha, \beta) \quad , \end{aligned}$$

giving us one more lemma worth noting:

Lemma 38.9
Let $\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\}$ be a set of N solutions to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ and let $\mathbf{X}(t)$ be the $N \times N$ matrix whose k^{th} column is given by $\mathbf{x}^k(t)$. Then

$$\det(\mathbf{X}(t)) \neq 0 \text{ for one } t \text{ in } (\alpha, \beta) \iff \det(\mathbf{X}(t)) \neq 0 \text{ for every } t \text{ in } (\alpha, \beta) \quad .$$

Equivalently,

$$\det(\mathbf{X}(t)) = 0 \text{ for one } t \text{ in } (\alpha, \beta) \iff \det(\mathbf{X}(t)) = 0 \text{ for every } t \text{ in } (\alpha, \beta) \quad .$$

Linear Independence

In our discussions, it may seem that we've replaced the idea of a set $\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\}$ of N solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ being "linearly independent" with the condition that $\det(\mathbf{X}(t_0)) \neq 0$ for some t_0. But remember, lemma 38.6 explicitly tells us that

$$\det(\mathbf{X}(t_0)) = 0 \implies \{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\} \text{ is not linearly independent} \quad ,$$

while lemma 38.8 and the definition of fundamental solution sets gives us

$$\begin{aligned} \det(\mathbf{X}(t_0)) \neq 0 &\iff \{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\} \text{ is fundamental set of solutions} \\ &\implies \{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\} \text{ is linearly independent} \quad . \end{aligned}$$

With a little thought, you will realize that the above implications tell us that we have linear independence if and only if the corresponding determinant is nonzero. This gives us another lemma for future reference.

Lemma 38.10
Let $\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\}$ be a set of N solutions to our $N \times N$ system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on (α, β), and let $\mathbf{X}$ be the $N \times N$ matrix whose k^{th} column is given by $\mathbf{x}^k$. Then the following three statements are equivalent; that is, if one holds, they all hold:

1. $\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\}$ *is a fundamental set of solutions for* $\mathbf{x}' = \mathbf{P}\mathbf{x}$.
2. $\det(\mathbf{X}(t_0)) \neq 0$ *for some* t_0 *in* (α, β).
3. $\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\}$ *is a linearly independent set of functions on* (α, β).

Existence of Fundamental Sets

We now know how to recognize fundamental sets. But can we be sure there are any fundamental sets of solutions to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$? Of course we can. To see this, just take any $N \times N$

matrix of constants

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{bmatrix} \quad \text{with} \quad \det \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{bmatrix} \neq 0 \quad ,$$

set $\mathbf{a}^k$ equal to the k^{th} column in this matrix, and consider the N initial-value problems

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a}^k \qquad \text{for} \quad k = 1, 2, \ldots, N \quad .$$

Existence lemma 38.3 assures us that, for each $\mathbf{a}^k$, we have a corresponding solution $\mathbf{x}^k(t)$. This gives us the set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ of solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$, such that

$$\det \begin{bmatrix} x_1^1(t_0) & x_1^2(t_0) & \cdots & x_1^N(t_0) \\ x_2^1(t_0) & x_2^2(t_0) & \cdots & x_2^N(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ x_N^1(t_0) & x_N^2(t_0) & \cdots & x_N^N(t_0) \end{bmatrix} = \det \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{bmatrix} \neq 0 \quad .$$

Lemma 38.8 then assures us that $\{\mathbf{x}^1, \ldots, \mathbf{x}^N\}$ is indeed a fundamental set of solutions.

Keeping in mind that there are many $N \times N$ matrices with nonzero determinant, we now have:

Lemma 38.11 (existence of fundamental solution sets)
Fundamental sets of solutions to our system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ exist. In fact, there are many of these sets — one for each $N \times N$ matrix with nonzero determinant.

38.3 The Main Results, Summarized
General Solutions to Homogeneous Systems

Combining lemmas 38.11, 38.10, 38.8 and 38.5, along with our definitions, gives us our first major theorem for homogeneous linear systems of differential equations:

Theorem 38.12 (general solutions to homogenous systems)
Let $\mathbf{P}$ be an $N \times N$ matrix of functions continuous on an interval (α, β), and consider the system of differential equations $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Then all the following statements hold:

1. *Fundamental sets of solutions over (α, β) for this system exist.*
2. *Every fundamental set of solutions contains exactly N solutions.*
3. *If $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is any linearly independent set of N solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on (α, β), then*
   (a) $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ *is a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on (α, β).*

(b) *A general solution to* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *on* (α, β) *is given by*

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_N\mathbf{x}^N(t)$$

where c_1 *,* c_2 *,* $\ldots$ *and* c_N *are arbitrary constants.*

(c) *Given any single point* t_0 *in* (α, β) *and any constant vector* $\mathbf{a}$ *, there is exactly one ordered set of constants* $\{c_1, c_2, \ldots, c_N\}$ *such that*

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_N\mathbf{x}^N(t)$$

satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{a}$ *.*

This theorem is the systems analog of theorem 14.2 on page 304 concerning general solutions to single N^{th}-order homogeneous linear differential equations. In fact, theorem 14.2 can be considered a corollary to the above.

!►*Example 38.5:* *We know that*

$$\mathbf{x}^1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \qquad \text{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} \quad .$$

are two solutions to the 2×2 *linear homogeneous system*

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{P} = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} \quad .$$

Clearly, $\mathbf{x}^1$ *and* $\mathbf{x}^2$ *are not constant multiples of each other; so,* $\{\mathbf{x}^1, \mathbf{x}^2\}$ *is a linearly independent set of solutions to the above* 2×2 *homogeneous linear system of differential equations. The above theorem now assures us that* $\{\mathbf{x}^1, \mathbf{x}^2\}$ *is a fundamental set of solutions to the system, and that the general solution is given by*

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} \quad .$$

Moreover, the above theorem assures us that the one particular solution found in example 36.3 on page 36–3 to the initial-value problem

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{with} \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad ,$$

namely,

$$x(t) = \frac{2}{7} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - \frac{1}{7} \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{-4t} \quad ,$$

is the only solution to that initial-value problem.

Fundamental Matrices, Wronskians and Testing for Linear Independence

In section 38.2, we made heavy use of a matrix $\mathbf{X}(t)$ and its determinant. It's now time to describe some standard terminology associated with these entities, and review their role in testing a set of solutions for linear independence.

Assume, as usual, that $\mathbf{P}$ is an $N \times N$ matrix of continuous functions on an interval (α, β). Let $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ be any set of N vector-valued functions on (α, β) with

$$\mathbf{x}^k(t) = \begin{bmatrix} x_1^k(t) \\ x_2^k(t) \\ \vdots \\ x_N^k(t) \end{bmatrix} \qquad \text{for} \quad k = 1, 2, \ldots, N \quad ,$$

and let $\mathbf{X}$ be the $N \times N$ matrix of functions on (α, β) whose k^{th} column is given by $\mathbf{x}^k$,

$$\mathbf{X}(t) = \begin{bmatrix} x_1^1(t) & x_1^2(t) & \cdots & x_1^N(t) \\ x_2^1(t) & x_2^2(t) & \cdots & x_2^N(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_N^1(t) & x_N^2(t) & \cdots & x_N^N(t) \end{bmatrix} \quad .$$

Given this,

1. The *Wronskian*, usually denoted by W, for the set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is simply the determinant of the matrix $\mathbf{X}$,
$$W(t) = \det(\mathbf{X}(t)) \quad .$$

2. The matrix $\mathbf{X}$ is said to be a *fundamental matrix* for the system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ (on (α, β)) if and only if the corresponding set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on (α, β).

The significance of the determinant of $\mathbf{X}$ was particularly described in lemmas 38.9 and 38.10. Those lemmas (and our new definitions) immediately give us simple ways of testing whether a prospective set of N vector-valued solutions is linearly independent (and, hence, a fundamental set of solutions).

Theorem 38.13 (tests for linear independence)
Let $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ be any set of N solutions to $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on an interval (α, β), where $\mathbf{P}$ is some $N \times N$ matrix of functions continuous on (α, β). Also, let $\mathbf{X}$ be the $N \times N$ matrix whose k^{th} column is given by $\mathbf{x}^k$. Then, if any one of the following statements holds, they all hold:

1. *The set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$ on (α, β).*

2. *The set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is linearly independent on (α, β).*

3. *The Wronskian, W, of $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is nonzero at one point in (α, β).*

4. *The Wronskian, W, of $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ is nonzero at every point in (α, β).*

5. *The matrix* $\mathbf{X}$ *is a fundamental matrix for* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *on* (α, β) *.*

6. *The determinant of* $\mathbf{X}$ *is nonzero at one point in* (α, β) *.*

7. *The determinant of* $\mathbf{X}$ *is nonzero at every point in* (α, β) *.*

!►*Example 38.6:* *It is not hard to verify that three solutions to*

$$\mathbf{x}' = \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix} \mathbf{x}$$

are

$$\mathbf{x}^1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{2t} \quad , \quad \mathbf{x}^2 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} e^{-2t} \quad \text{and} \quad \mathbf{x}^3 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} e^{2t} \quad .$$

The corresponding matrix is

$$\mathbf{X}(t) = \begin{bmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{bmatrix} \quad .$$

According to the above theorem we can determine if $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$ *is a linearly independent set of solutions to our* 3×3 *system by computing this set's Wronskian at some convenient point, say,* $t = 0$ *:*

$$\begin{aligned} W(0) = \det(\mathbf{X}(0)) &= \det \begin{bmatrix} e^{2\cdot 0} & 2e^{-2\cdot 0} & 3e^{2\cdot 0} \\ e^{2\cdot 0} & 3e^{-2\cdot 0} & e^{2\cdot 0} \\ 3e^{2\cdot 0} & -e^{-2\cdot 0} & 3e^{2\cdot 0} \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \\ &= 1 \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} \\ &= 1[9+1] - 2[3-3] + 3[-1-9] \\ &= 20 \quad . \end{aligned}$$

Since $W(0) \neq 0$ *, we know the set* $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$ *is a linearly independent set of three solutions for our above* 3×3 *system of differential equations and, hence, is a fundamental set of solutions for the above system of differential equations. Hence, also, the* 3×3 *matrix* $\mathbf{X}$ *is a fundamental matrix for the system.*

Additional Exercises

38.3. *Consider the two equations*

$$\mathbf{x}^M = C_1\mathbf{x}^1 + C_2\mathbf{x}^2 + \cdots + C_{M-1}\mathbf{x}^{M-1} \quad . \tag{38.7}$$

and

$$c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_M\mathbf{x}^M = \mathbf{0} \quad . \tag{38.8}$$

where $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M\}$ *is a set of vector-valued functions on an interval* (α, β) *.*

a. *Using simple algebra, show that equation (38.7) holds for some constants* C_1 *,* C_2 *, ... and* C_{M-1} *if and only if equation (38.8) holds for some constants* c_1 *,* c_2 *, ... and* c_M *with* $c_M \neq 0$ *.*

b. *Expand on the above and explain how it follows that at least one of the* $\mathbf{x}^k$*'s must be a linear combination of the other* $\mathbf{x}^k$*'s if and only if equation (38.8) holds with at least one of the* c_k*'s being nonzero.*

c. *Finish proving lemma 38.4 on page 38–7.*

38.4. *Consider the system*

$$\begin{aligned} x' &= y \\ y' &= -4t^{-2}x + 3t^{-1}y \end{aligned} \quad .$$

a. *Rewrite this system in matrix/vector form.*

b. *What are the largest intervals over which we are sure solutions to this system exist?*

c. *Verify that*

$$\mathbf{x}^1(t) = \begin{bmatrix} t^2 \\ 2t \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^2(t) = \begin{bmatrix} t^2 \ln|t| \\ t(1 + 2\ln|t|) \end{bmatrix}$$

are both solutions to this system.

d. *Compute the Wronskian* $W(t)$ *of the set of the above* $\mathbf{x}^k$*'s at some convenient point* $t = t_0$ *(part of this problem is to choose a convenient point). What does this value of* $W(t_0)$ *tell you?*

e. *Using the above, find the solution to the above system satisfying*

i. $\mathbf{x}(1) = [1, 0]^{\mathsf{T}}$ ***ii.*** $\mathbf{x}(1) = [0, 1]^{\mathsf{T}}$

38.5. *Consider the system*

$$\begin{aligned} x' &= 0x + 2y - 2z \\ y' &= -2x + 4y - 2z \\ z' &= 2x + 2y - 4z \end{aligned} \quad .$$

a. *Rewrite this system in matrix/vector form.*

b. *What is the largest interval over which we are sure solutions to this system exist?*

c. *Verify that*

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad \mathbf{y}(t) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{-2t} \quad \text{and} \quad \mathbf{z}(t) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t}$$

are all solutions to this system.

d. *Compute the Wronskian $W(t)$ of the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ some convenient point $t = t_0$ (choosing a convenient point is part of the problem), and verify that the above $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ a fundamental set of solutions to the above system of differential equations.*

38.6. *Four solutions to*

$$\mathbf{x}' = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \mathbf{x}$$

are

$$\mathbf{x}^1(t) = \begin{bmatrix} \cos(2t) \\ \sin(2t) \\ \cos(2t) \end{bmatrix} \quad , \quad \mathbf{x}^2(t) = \begin{bmatrix} \sin(2t) \\ -\cos(2t) \\ \sin(2t) \end{bmatrix} \quad , \quad \mathbf{x}^3(t) = \begin{bmatrix} -\sin^2(t) \\ \sin(t)\cos(t) \\ \cos^2(t) \end{bmatrix} \quad ,$$

and

$$\mathbf{x}^4(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad .$$

Given this, determine which of the following are fundamental sets of solutions to the given system:

a. $\{\mathbf{x}^1, \mathbf{x}^2\}$ **b.** $\{\mathbf{x}^1, \mathbf{x}^4\}$ **c.** $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$

d. $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^4\}$ **e.** $\{\mathbf{x}^1, \mathbf{x}^3, \mathbf{x}^4\}$ **f.** $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4\}$

38.7. *Four solutions to*

$$\mathbf{x}' = \begin{bmatrix} -1 & -1 & 2 \\ -8 & 1 & 4 \\ -4 & -1 & 5 \end{bmatrix} \mathbf{x}$$

are

$$\mathbf{x}^1(t) = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^{3t} \quad , \quad \mathbf{x}^2(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^{3t} \quad , \quad \mathbf{x}^3(t) = \begin{bmatrix} 3 \\ -4 \\ 4 \end{bmatrix} e^{3t}$$

and

$$\mathbf{x}^4(t) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-t} \quad .$$

Given this, determine which of the following are fundamental sets of solutions to the given system:

a. $\{\mathbf{x}^1, \mathbf{x}^2\}$ ***b.*** $\{\mathbf{x}^1, \mathbf{x}^4\}$ ***c.*** $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$

d. $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^4\}$ ***e.*** $\{\mathbf{x}^1, \mathbf{x}^3, \mathbf{x}^4\}$ ***f.*** $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4\}$

38.8. *Traditionally (i.e., in most other texts), lemma 38.9 on page 38–12 is usually proven by showing that the Wronskian* W *of a set of* N *solutions to an* $N \times N$ *system* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *satisfies the differential equation*

$$W' = \left[p_{1,1} + p_{2,2} + \cdots + p_{N,N}\right] W \quad ,$$

and then solving this differential equation and verifying that the solution is nonzero over the interval of interest if and only if it is nonzero at one point in the interval. Do this yourself for the case where $N = 2$ *.*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

4a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4t^{-2} & 3t^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

4b. $(-\infty, 0)$ and $(0, \infty)$

4d. $W(1) = 1 \neq 0$ (Hence $\{\mathbf{x}^1, \mathbf{x}^2\}$ is a fundamental set of solutions.)

4e i. $\mathbf{x}(t) = \mathbf{x}^1(t) - 2\mathbf{x}^2(t) = \begin{bmatrix} t^2(1 - 2\ln|t| \\ -4t\ln|t| \end{bmatrix}$

4e ii. $\mathbf{x}(t) = \mathbf{x}^2(t) = \begin{bmatrix} t^2 \ln|t| \\ t(1 + 2\ln|t|) \end{bmatrix}$

5a. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 4 & -2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

5b. $(-\infty, \infty)$

5d. $W(0) = -1$

6a. It is not a fundamental set since – the set is too small.

6b. It is not a fundamental set since – the set is too small.

6c. It is a fundamental set – there are three solutions in the set, and $W(0) \neq 0$.

6d. Is a fundamental set – there are three solutions in the set, and $W(0) \neq 0$.

6e. Is not a fundamental set – there are three solutions in the set, but $W(0) = 0$.

6f. Is not a fundamental set – the set is too large.

7a. It is not a fundamental set – the set is too small.

7b. It is not a fundamental set – the set is too small.

7c. It is not a fundamental set – there are three solutions in the set, but $W(0) = 0$.

7d. It is a fundamental set – there are three solutions in the set, and $W(0) \neq 0$.

7e. It is a fundamental set – there are three solutions in the set, and $W(0) \neq 0$.

7f. It is not a fundamental set – the set is too large.