41

Miscellaneous Topics Involving Homogeneous Constant Matrix Systems

In this chapter we will discuss a variety of topics, all more-or-less related to the constant matrix systems discussed in the previous two chapters. Some of this material is of interest for its own sake, and some is developed here for use either in the chapter on nonhomogeneous systems or for use in discussing nonlinear systems.

41.1 Phase Portraits for Large Constant Matrix Systems

In the previous two chapters, we pretty well demonstrated how the phase portrait of a 2×2 constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ depends on the eigenvalues and eigenvectors of $\mathbf{A}$. We won't attempt an analogous development when $\mathbf{A}$ is $N \times N$ with $N > 2$. There are just too many cases to consider, and the two-dimensional medium of this text is not adequate for representing the corresponding phase portraits. Nonetheless, the basic ideas developed assuming $\mathbf{A}$ is 2×2 still apply, and you can use what we developed to help visualize the possible trajectories when $\mathbf{A}$ is, say, 3×3.

!►*Example 41.1:* *Consider the rather simple 3×3 constant matrix system*

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -2 & 3 & 0 \\ -3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad .$$

The matrix for this system has three distinct eigenvalues. Two are complex, $r_{\pm} = -2 \pm 3i$, and the third is $r_3 = -1$ (with corresponding eigenvector $[0, 0, 1]^{\mathsf{T}}$). Together, they lead to the system's general solution

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + c_3\mathbf{x}^3(t) \tag{41.1}$$

with

$$\mathbf{x}^1(t) = \begin{bmatrix} \cos(3t) \\ -\sin(3t) \\ 1 \end{bmatrix} e^{-2t} \quad , \quad \mathbf{x}^2(t) = \begin{bmatrix} \sin(3t) \\ \cos(3t) \\ 1 \end{bmatrix} e^{-2t}$$

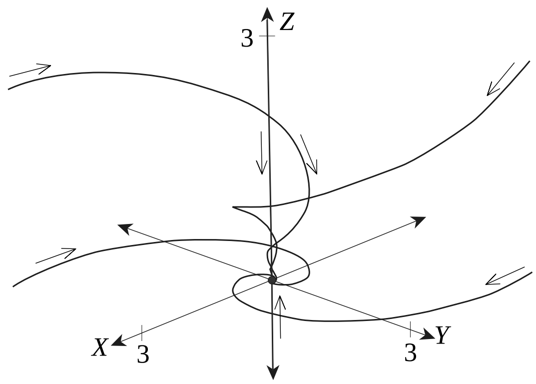

Figure 41.1: Seven trajectories for the 3×3 constant matrix system in example 41.1. Two are straight line trajectories along the Z–axis, two are spirals in the XY–plane, and two are "three-dimensional spirals" about the Z–axis. The seventh is the critical point $(0, 0, 0)$.

and

$$\mathbf{x}^3(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \quad .$$

The first two terms in the general solution, which came from the complex eigenvalues $-2 \pm 3i$ *, trace out spirals "spiralling in" towards the origin in the* XY*–plane. That's what you get from general solution (41.1) if* $c_3 = 0$ *but at least one of the other two constants in formula (41.1) is nonzero.*

The last term, corresponding to the eigenpair $\left(-1, [0, 0, 1]^{\mathsf{T}}\right)$*, traces out straight line trajectories along the* Z*–axis with the direction of travel being towards the origin. That's what you get from general solution (41.1) if* $c_1 = c_2 = 0$ *.*

Finally, if $c_3 \neq 0$ *and at least one of the other two constants in formula (41.1) is nonzero, then* $\mathbf{x}(t)$ *traces out a "three-dimensional spiral" about the* Z*–axis heading into the origin.*

Examples of these trajectories have been sketched in figure 41.1. Note that $\mathbf{x}(t) = \mathbf{0}$ *is still an equilibrium solution, and that, whatever the values of* c_1 *,* c_2 *and* c_3 *,*

$$\lim_{t \to \infty} \left[c_1 \mathbf{x}^1(t) + c_2 \mathbf{x}^2(t) + c_3 \mathbf{x}^3(t) \right] = 0 \quad .$$

So $\mathbf{x}(t) = \mathbf{0}$ *is an asymptotically stable equilibrium solution for this system.*

41.2 Shifted Constant Matrix Systems

A *shifted constant matrix system* is simply a system of differential equations that can be written as

$$\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$$

where $\mathbf{A}$ is a constant $N \times N$ matrix and $\mathbf{x}^0 = \left[x_1^0, x_2^0, \ldots, x_N^0\right]^{\mathsf{T}}$ is a constant vector. Such systems will later be important in approximating nonlinear systems about critical points.

The above shifted system looks a lot like the constant matrix systems in the last two chapters, and we can make it look even more like such a system by defining a new vector-valued functions $\hat{\mathbf{x}}(t)$ by

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}^0 \quad .$$

This is equivalent to introducing a new coordinate system that is just the original coordinate system shifted so that the new origin $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N) = (0, 0, \ldots, 0)$ is at the point given by $(x_1, x_2, \ldots, x_N) = \left(x_1^0, x_2^0, \ldots, x_N^0\right)$ in the old coordinate system. Since

$$\frac{d\hat{\mathbf{x}}}{dt} = \frac{d}{dt}[\mathbf{x}(t) - \mathbf{x}^0] = \frac{d\mathbf{x}}{dt} - \mathbf{0} = \frac{d\mathbf{x}}{dt}$$

and

$$\mathbf{A}[\mathbf{x} - \mathbf{x}^0] = \mathbf{A}\hat{\mathbf{x}} \quad ,$$

our system of differential equations reduces, in the shifted coordinate system, to the basic constant matrix system

$$\hat{\mathbf{x}}' = \mathbf{A}\hat{\mathbf{x}} \quad .$$

So everything we learned about solving basic constant matrix systems applies here provided we take into account the “shift by $\mathbf{x}^0$” . In particular,

1. The point $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ is a critical point for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$, and is the only critical point if $\det(\mathbf{A}) \neq 0$.

2. All solutions to $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ can be obtained by just adding $\mathbf{x}^0$ to all solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

3. The stability of the equilibrium solution $\mathbf{x}(t) = \mathbf{x}^0$ for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ is the same as the stability of the equilibrium soluton $\mathbf{x}(t) = \mathbf{0}$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$

4. A phase portrait for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ can be obtained by just “shifting” a phase portrait of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ so that the trajectories of the shifted system about $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ match the trajectories of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ about $(0, 0, \ldots, 0)$.

!►*Example 41.2:* *Consider the shifted system*

$$\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad .$$

The corresponding “unshifted” system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ was considered in example 40.2 on page 40–8. There, we saw that a general solution for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by $C\mathbf{x}^R(t - t_0)$ where

$$\mathbf{x}^R(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(t) \quad .$$

We also constructed the phase portrait for this system (redrawn in figure 41.2a), and observed that the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ is stable, but not asymptotically stable.

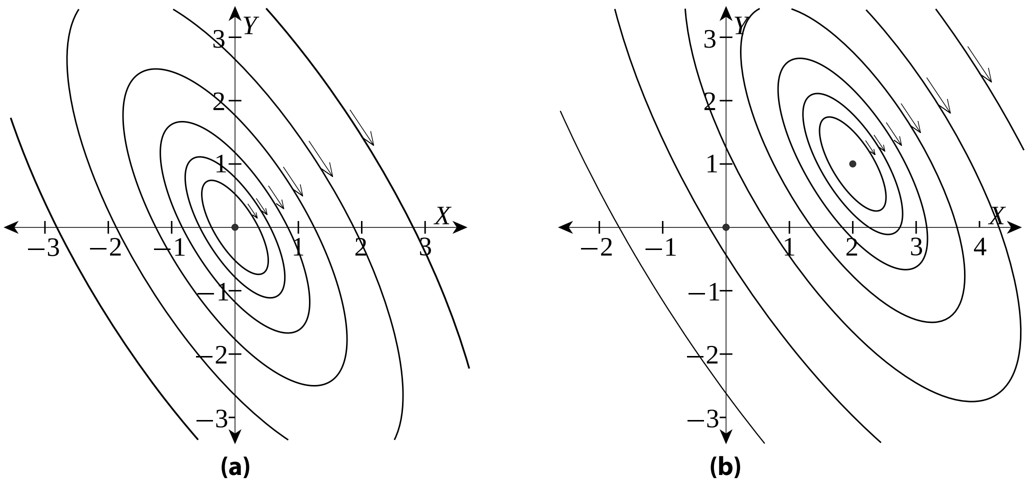

Figure 41.2: Phase portraits for **(a)** the basic system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and **(b)** the shifted system $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ from example 41.2.

By simply adding the "shift" to the above formula, we then obtain the general solution

$$\begin{aligned}\mathbf{x}(t) &= C\mathbf{x}^R(t - t_0) + \mathbf{x}^0 \\ &= C\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos(t - t_0) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(t - t_0)\right) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\end{aligned}$$

for the shifted system, $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ *, and, by suitably shifting the phase portrait of* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *in figure 41.2a, we get the phase portrait of the shifted system in figure 41.2b. Again, it is clear that the equilibrium solution* $\mathbf{x}(t) = \mathbf{x}^0$ *is stable, but not asymptotically stable.*

41.3 Classifying Critical Points for 2×2 Systems

Later, we will use what we've developed in the last few chapters for constant matrix systems to help analyze solutions to 2×2 nonlinear systems of differential equations. So, for future reference, let us now

1. summarize some of what we've derived regarding the stability, and

2. give definitions for some of the terms introduced in the previous two chapters.

In this discussion, we will assume $\mathbf{A}$ is a constant 2×2 matrix with real components, and with eigenvalues r_1 and r_2 (possibly with $r_1 = r_2$). If r_1 and r_2 are complex, then we know they are complex conjugates of each other, and we'll denote the real and imaginary parts, respectively, by λ and ω,

$$r_1 = \lambda + i\omega \qquad \text{and} \qquad r_2 = \lambda - i\omega \quad .$$

Stability in Constant Matrix Systems

If you go back and review the possible cases, you will see that, whenever either r_1 or r_2 is positive, or whenever r_1 and r_2 are complex with λ positive, then $\mathbf{x}(t) = \mathbf{0}$ is an unstable equilibrium for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (and $\mathbf{x}(t) = \mathbf{x}^0$ is an unstable equilibrium for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$). Otherwise, $\mathbf{x}(t) = \mathbf{0}$ is a stable equilibrium for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (and $\mathbf{x}(t) = \mathbf{x}^0$ is a stable equilibrium for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$).

Furthermore, we have that $\mathbf{x}(t) = \mathbf{0}$ is an asymptotically stable equilibrium for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (and $\mathbf{x}(t) = \mathbf{x}^0$ is an asymptotically stable equilibrium for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$) if and only if both r_1 and r_2 are negative, or are complex with $\lambda < 0$.

Nodes, Saddle Points, Centers and Spiral Points in General

Let (x_0, y_0) be a critical point for any 2×2 system of differential equations. Then:

1. The critical point (x_0, y_0) is called a *node* if, in a region near (x_0, y_0), all of the nonequilibrium trajectories are either straight half-lines with (x_0, y_0) as an endpoint, or become tangent to such half-lines at (x_0, y_0). For the basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a node if and only if r_1 and r_2 are both positive or are both negative.

   Sometimes, nodes are further subdivided into being either "proper" or "improper", with the node being *proper* if and only if, for every straight half-line with (x_0, y_0) as an endpoint, there is a trajectory which is that half line or which becomes tangent to that half line at (x_0, y_0). For the basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a proper node if and only if $r_1 = r_2$, in which case we may also refer to (x_0, y_0) as a *star node*.

2. The critical point (x_0, y_0) is called a *saddle point* if there are two nonequilibrium solutions $\mathbf{x}^1(t)$ and $\mathbf{x}^2(t)$ such that

$$\lim_{t \to -\infty} \mathbf{x}^1(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \text{and} \qquad \lim_{t \to \infty} \mathbf{x}^2(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \quad .$$

   For our basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a saddle point if and only if r_1 and r_2 are both real, but have opposite signs.

3. The critical point (x_0, y_0) is called a *center* if all the nearby nonequilibrium trajectories are closed loops about (x_0, y_0). For our basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a center if and only if the eigenvalues of $\mathbf{A}$ are purely imaginary; that is, $r_1 = i\omega$ and $r_2 = -i\omega$ with $\omega \neq 0$.

4. The critical point (x_0, y_0) is called a *spiral point* if all the nearby nonequilibrium trajectories are spirals about that point. For our basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a spiral point if and only if the eigenvalues of $\mathbf{A}$ are complex with both real and imaginary parts being nonzero; that is, $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ with both $\lambda \neq 0$ and $\omega \neq 0$.

You you check the literature, you may find other terms used in classifying critical points. For example, the terms "sink node" and "source node" (or just "sink" and "source") are often used as synonyms for stable and unstable nodes, respectively.

41.4 Phase Portraits for Imprecisely Known Systems

Note that the basic nature of a phase portrait for a 2×2 constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ depends strongly on the whether the real and imaginary parts of the eigenvalues of $\mathbf{A}$ are positive, negative or zero. This can be a significant issue when the matrix $\mathbf{A}$ is only approximately known, such as when the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ arises from a "real-world application" and the components of $\mathbf{A}$ are determined by "real-world measurements". Such measurements are invariably approximate. As a result, the eigenvalues obtained when solving the characteristic equation $\det[\mathbf{A} - r\mathbf{I}] = 0$ will only be approximations of the true eigenvalues for the system of real interest. And, of course the corresponding computed eigenvectors will also only be approximations of the true eigenvectors for the system.

So let use consider some of the possibilities when $\mathbf{A}$ is 2×2, and our computed eigenvalues are are known to be approximations of the true eigenvalues. For convenience, we will denote the computed eigenvalues by r_1 and r_2 if they are real, and by $\lambda \pm i\omega$ if they are complex. We will assume each computed r_1, r_2, λ and ω is known to be "within ϵ" of the true value, where ϵ is some (hopefully small) positive value. To simplify notation, let us write, say, $r_1 \approx r_2$ or $\omega \approx 0$ whenever these computed values are close enough that it is possible for the corresponding equalities to hold for the true values.

$\epsilon < r_1 < r_2$ with $r_1 \not\approx r_2$

According to the computed eigenvalues, the origin is an unstable node. In this case, the true values of the eigenvalues still must both be positive and different. So, using the true eigenvalues, the origin is still an unstable node. Moreover, (assuming the errors are reasonably small) the computed eigenvectors will be reasonably close to the true eigenvectors. Consequently, the phase portraits generated by the computed values will be good approximations of the true phase portraits, and will all look something like that sketched in figure 39.4a on page 39–21.

$\epsilon < r_1 < r_2$ with $r_1 \approx r_2$

Again, according to the computed eigenvalues, the origin is an unstable node, and any phase portrait based on the computed eigenvalues will be similar to that sketched in figure 39.4a on page 39–21. In this case, however, there are four general possibilities for the true values of the eigenvalues:

1. The true eigenvalues are two different positive numbers. In this case, the origin is truly an unstable node, and the phase portraits drawn using the computed eigenvalues and eigenvectors will be good approximations of the true phase portraits.

2. The true eigenvalues are equal and real. In this case there two additional possibilities:
   - *(a)* If the true eigenvalue has geometric multiplicity two, then the true trajectories are all straight half-lines, and the origin is an unstable star node.
   - *(b)* If the true eigenvalue has geometric multiplicity one, then the origin is still an unstable node, but the true trajectories will be similar to those in figure 40.4a on page 40–16.

3. The true eigenvalues are complex, with the same real parts and small (but nonzero) imaginary parts. In this case, the origin is a unstable spiral point, and the true phase portrait will be somewhat similar to that in figure 40.2a on page 40–9.

Observe that, no matter what, we can be sure that the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ is an unstable equilibrium solution. However, the actual trajectories can vary radically from case to case.

$r = \lambda \pm i\omega$ with $\epsilon < \lambda$ and $\epsilon < \omega$

According to the computed eigenvalues, the origin is an unstable spiral point, and the phase portrait is similar to that in figure 40.2a on page 40–9. In this case, the computed values of λ and ω are large enought to assure us that the true eigenvalues are complex with similar values for the real and imaginary parts. In particular, the real part of the true eigenvalues will be a single positive value. Consequently, the origin must be an unstable spiral point, and a phase portrait based on the true eigenvalues and eigenvectors will be similar to that based on the computed values.

$r = \lambda \pm i\omega$ with $0 < \lambda < \epsilon$ and $\epsilon < \omega$

According to the computed eigenvalues, the origin is an unstable spiral point, and the phase portrait is similar to that in figure 40.2a on page 40–9. Here, the imaginary parts of the computed eigenvalues are large enough to ensure that the true eigenvalues have nonzero imaginary parts, but the real parts of the computed eigenvalues are so close to 0 that we have three possibilities:

1. The real parts of the true eigenvalues are positive. In this case the origin is an unstable spiral point and a true phase portrait will be somewhat similar to that of the computed phase portrait.

2. The real part of the true eigenvalues is zero. In this case the origin is a center, $\mathbf{x}(t) = \mathbf{0}$ is a stable equilibrium solution, and a true phase portrait will consist of a bunch of ellipses centered at the origin, and not the spirals drawn using the computed eigenvalues.

3. The real part of the true eigenvalues is negative. In this case the origin is a stable spiral point, $\mathbf{x}(t) = \mathbf{0}$ is an asymptotically stable equilibrium solution, and a true phase portrait will consist of spirals, just as in a phase portrait drawn using the computed eigenvalues, but with the direction of travel being towards the origin instead of away.

Other Cases

In exercise 41.3, you will briefly go through some of the other possible cases, comparing what the computed eigenvalues tell us with what the true eigenvalues would have told us. One thing to observe: If the eigenvalues or the real parts of the eigenvalues of the computed eigenvalues are close to zero, then you have little idea as to whether the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ is asymptotically stable, stable, or unstable.

41.5 Using Fundamental and Exponential Matrices

Fundamental Matrices

Let's go back to a fairly general linear system of the form

$$\mathbf{x}' = \mathbf{P}\mathbf{x}$$

where $\mathbf{P}$ is an $N \times N$ matrix of functions on some interval (α, β). Recall that a fundamental matrix for this system is any $N \times N$ matrix of functions on (α, β)

$$\mathbf{X} = \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^N \\ x_2^1 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \ddots & \vdots \\ x_N^1 & x_N^2 & \cdots & x_N^N \end{bmatrix}$$

whose columns

$$\mathbf{x}^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_N^1 \end{bmatrix} \quad , \quad \mathbf{x}^2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_N^2 \end{bmatrix} \quad , \quad \cdots \quad \text{and} \quad \mathbf{x}^N = \begin{bmatrix} x_1^N \\ x_2^N \\ \vdots \\ x_N^N \end{bmatrix}$$

make up a fundamental set of solutions for our linear system $\mathbf{x}' = \mathbf{P}\mathbf{x}$.

Here are some simple observations and recollections regarding the above fundamental matrix $\mathbf{X}$ and corresponding fundamental set of solutions $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$:

1. Because each column $\mathbf{x}$ satisfies $\mathbf{x}' = \mathbf{P}\mathbf{x}$, it is easy to verify that any fundamental matrix $\mathbf{X}$ satisfies the "matrix/matrix" system of differential equations

$$\mathbf{X}' = \mathbf{P}\mathbf{X}$$

where $\mathbf{X}'$, the derivative of $\mathbf{X}$, is simply the matrix obtained by differentiating each component of $\mathbf{X}$.

Moreover, it should be clear that the theory discussed for the matrix/vector system of differential equations $\mathbf{x} = \mathbf{P}\mathbf{x}$ extends naturally to the matrix/matrix system. This includes the facts regarding the existence and uniqueness of solutions.

2. It is also easy to verify that, if $\mathbf{F}$ and $\mathbf{G}$ are any matrices such that the product $\mathbf{F}\mathbf{G}$ exists, then the standard product rule

$$(\mathbf{F}\mathbf{G})' = \mathbf{F}'\mathbf{G} + \mathbf{F}\mathbf{G}'$$

holds. Moreover, if $\mathbf{G}$ is a constant $N \times N$ matrix, then

$$\mathbf{G}' = \mathbf{O}$$

where $\mathbf{O}$ is the constant $N \times N$ matrix whose every component is 0, and

$$(\mathbf{X}\mathbf{G})' = \mathbf{X}'\mathbf{G} = (\mathbf{P}\mathbf{X})\,\mathbf{G} = \mathbf{P}\,(\mathbf{X}\mathbf{G}) \quad .$$

3. If $\mathbf{c} = [c_1, c_2, \ldots, c_N]^{\mathsf{T}}$ then (as derived in *A "Matrix/Vector" Formula for Linear Combinations* on page 38–5)

$$[\mathbf{X}(t)]\mathbf{c} = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_N\mathbf{x}^N \quad .$$

That is,

$$\mathbf{x}(t) = [\mathbf{X}(t)]\mathbf{c}$$

is a general solution for $\mathbf{x}' = \mathbf{P}\mathbf{x}$.

Let's now include an initial condition $\mathbf{x}(t_0) = \mathbf{a}$ for some suitable real number t_0 and constant vector $\mathbf{a}$. From the above observations, we know the solution to the resulting initial-value problem

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

is given by

$$\mathbf{x}(t) = [\mathbf{X}(t)]\mathbf{c} \tag{41.2}$$

where $\mathbf{c} = [c_1, c_2, \ldots, c_N]^{\mathsf{T}}$ is chosen so that

$$[\mathbf{X}(t_0)]\mathbf{c} = \mathbf{a} \quad .$$

But, as noted in chapter 38, fundamental matrices are invertible. So we can easily solve the above for the $\mathbf{c}$ using the inverse of $\mathbf{X}(t_0)$,

$$\mathbf{c} = [\mathbf{X}(t_0)]^{-1}\mathbf{a} \quad .$$

Combining this with formula (41.2) for $\mathbf{x}$ (and recalling the requirements we made on $\mathbf{P}$ in chapter 38) gives us:

Theorem 41.1

Let $\mathbf{X}(t)$ be a fundamental matrix for an $N \times N$ system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ over an interval on which each component of $\mathbf{P}(t)$ is continuous. Let t_0 be in this interval and $\mathbf{a}$ any column vector. Then the solution to

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \textit{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

is given by

$$\mathbf{x}(t) = [\mathbf{X}^0(t)]\mathbf{a} \qquad \textit{where} \quad \mathbf{X}^0(t) = [\mathbf{X}(t)][\mathbf{X}(t_0)]^{-1} \quad .$$

From our observations and basic linear algebra, it follows that the $\mathbf{X}^0$ described in the last theorem satisfies both

$$\frac{d\mathbf{X}^0}{dt} = \mathbf{P}\mathbf{X}^0$$

and

$$\mathbf{X}^0(t_0) = [\mathbf{X}(t_0)][\mathbf{X}(t_0)]^{-1} = \mathbf{I} \quad .$$

So $\mathbf{X}^0$ is the one solution to the matrix/matrix initial-value problem

$$\mathbf{x}' = \mathbf{P}\mathbf{X} \qquad \text{with} \quad \mathbf{X}(t_0) = \mathbf{I} \quad .$$

From this, our discussions in chapter 38, and the fact $\det(\mathbf{X}(t_0)) = \det(\mathbf{I}) = 1$, it quickly follows that $\mathbf{X}^0$ is a fundamental matrix for $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Clearly, it is the one we would want if we had to solve

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

for several different choices of $\mathbf{a}$ (but the same t_0 for each).

At this point, we have two ways of finding this $\mathbf{X}^0$:

1. We first solve the N initial-value problems

$$\frac{d\mathbf{x}^k}{dt} = \mathbf{P}\mathbf{x}^k \qquad \text{with} \quad \mathbf{x}^k(t_0) = \mathbf{e}^k$$

   where $\mathbf{e}^k$ is the $N \times 1$ column matrix whose components are all 0 except for the k^{th} component which is 1. Then we use each $\mathbf{x}^k(t)$ just found as the k^{th} column of $\mathbf{X}^0(t)$.

2. We take any fundamental matrix $\mathbf{X}(t)$ already found for $\mathbf{x}' = \mathbf{P}\mathbf{x}$, compute $\mathbf{X}(t_0)$ and its inverse and finally compute the product $\mathbf{X}^0(t) = [\mathbf{X}(t)][\mathbf{X}(t_0)]^{-1}$.

!►*Example 41.3:* *In example 38.6 on page 38–16, we saw that one fundamental matrix for*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \textit{with} \quad \mathbf{A} = \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix}$$

is

$$\mathbf{X}(t) = \begin{bmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{bmatrix} \quad .$$

To find the fundamental matrix $\mathbf{X}^0$ *such that*

$$\frac{d\mathbf{X}^0}{dt} = \mathbf{A}\mathbf{X}^0 \qquad \textit{with} \quad \mathbf{X}^0(0) = \mathbf{I} \quad ,$$

we first find the inverse of $\mathbf{X}(0)$ *(using whichever method you prefer),*

$$[\mathbf{X}(0)]^{-1} = \begin{bmatrix} e^{2\cdot 0} & 2e^{-2\cdot 0} & 3e^{2\cdot 0} \\ e^{2\cdot 0} & 3e^{-2\cdot 0} & e^{2\cdot 0} \\ 3e^{2\cdot 0} & -e^{-2\cdot 0} & 3e^{2\cdot 0} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}^{-1} = \cdots = \begin{bmatrix} -\frac{1}{2} & \frac{9}{20} & \frac{7}{20} \\ 0 & \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{2} & -\frac{7}{20} & -\frac{1}{20} \end{bmatrix}$$

Then

$$\mathbf{X}^0(t) = [\mathbf{X}(t)][\mathbf{X}(t_0)]^{-1} = \begin{bmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{9}{20} & \frac{7}{20} \\ 0 & \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{2} & -\frac{7}{20} & -\frac{1}{20} \end{bmatrix}$$

$$= \cdots = \begin{bmatrix} e^{2t} & -\frac{3}{5}e^{2t} + \frac{3}{5}e^{-2t} & \frac{1}{5}e^{2t} - \frac{1}{5}e^{-2t} \\ 0 & \frac{1}{10}e^{2t} + \frac{9}{10}e^{-2t} & \frac{3}{10}e^{2t} - \frac{3}{10}e^{-2t} \\ 0 & \frac{3}{10}e^{2t} - \frac{3}{10}e^{-2t} & \frac{9}{10}e^{2t} + \frac{1}{10}e^{-2t} \end{bmatrix} \quad .$$

While the formula for $\mathbf{X}^0$ *is not as simple as what we had originally obtained for* $\mathbf{X}$*, this more complicated formula will simplify solving*

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \qquad \textit{with} \quad \mathbf{x}(0) = \mathbf{a} \quad .$$

For example, if $\mathbf{a} = [1, 2, 3]^{\mathsf{T}}$*, then*

$$\mathbf{x}(t) = [\mathbf{X}^0(t)]\mathbf{a} = \begin{bmatrix} e^{2t} & -\frac{3}{5}e^{2t} + \frac{3}{5}e^{-2t} & \frac{1}{5}e^{2t} - \frac{1}{5}e^{-2t} \\ 0 & \frac{1}{10}e^{2t} + \frac{9}{10}e^{-2t} & \frac{3}{10}e^{2t} - \frac{e}{10}e^{-2t} \\ 0 & \frac{3}{10}e^{2t} - \frac{3}{10}e^{-2t} & \frac{9}{10}e^{2t} + \frac{1}{10}e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \cdots = \begin{bmatrix} \frac{2}{5}e^{2t} + \frac{3}{5}e^{-2t} \\ \frac{11}{10}e^{2t} + \frac{9}{10}e^{-2t} \\ \frac{33}{10}e^{2t} - \frac{3}{10}e^{-2t} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 \\ 11 \\ 33 \end{bmatrix} e^{2t} + \frac{1}{10}\begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix} e^{-2t} \quad .$$

And if $\mathbf{a} = [20, 0, 30]^{\mathsf{T}}$*, then*

$$\mathbf{x}(t) = [\mathbf{X}^0(t)]\mathbf{a} = \begin{bmatrix} e^{2t} & -\frac{3}{5}e^{2t} + \frac{3}{5}e^{-2t} & \frac{1}{5}e^{2t} - \frac{1}{5}e^{-2t} \\ 0 & \frac{1}{10}e^{2t} + \frac{9}{10}e^{-2t} & \frac{3}{10}e^{2t} - \frac{e}{10}e^{-2t} \\ 0 & \frac{3}{10}e^{2t} - \frac{3}{10}e^{-2t} & \frac{9}{10}e^{2t} + \frac{1}{10}e^{-2t} \end{bmatrix} \begin{bmatrix} 20 \\ 0 \\ 30 \end{bmatrix}$$

$$= \cdots = \begin{bmatrix} 26e^{2t} - 6e^{-2t} \\ 9e^{2t} - 9e^{-2t} \\ 27e^{2t} + 3e^{-2t} \end{bmatrix} = \begin{bmatrix} 26 \\ 9 \\ 27 \end{bmatrix} e^{2t} + \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix} e^{-2t} \quad .$$

The Exponential Matrix

Let us now limit ourselves to the cases we've been considering in the last few chapters; namely, where $\mathbf{P}(t) = \mathbf{A}$, and $\mathbf{A}$ is a constant real $N \times N$ matrix. If $t_0 = 0$, then we would be particularly interest in the fundamental matrix $\mathbf{X} = \mathbf{X}^0$ satisfying

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \qquad \text{with} \quad \mathbf{X}(0) = \mathbf{I} \quad . \tag{41.3}$$

Observe the similarity between this initial-value problem and the first-order initial-value problem

$$x' = ax \qquad \text{with} \quad x(0) = 1$$

where a is some constant. This is a simple problem with a simple solution:

$$x(t) = e^{at} \quad .$$

In analogy to this, we often refer to the solution $\mathbf{X}^0$ of initial-value problem (41.3) as the (matrix) exponential (of $\mathbf{A}t$), writing

$$\mathbf{X}^0(t) = e^{\mathbf{A}t} = \exp(\mathbf{A}t) \quad .$$

We can then write the solution to

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(0) = \mathbf{a}$$

as

$$x(t) = e^{\mathbf{A}t}\mathbf{a} \quad .$$

More generally, we can define the exponential of any $N \times N$ matrix $\mathbf{M}$ by the exponential Taylor series

$$e^{\mathbf{M}} = \exp(\mathbf{M}) = \sum_{k=0}^{\infty} \frac{\mathbf{M}^k}{k!}$$

where

$$\mathbf{M}^0 = \mathbf{I} \quad , \quad \mathbf{M}^1 = \mathbf{M} \quad , \quad \mathbf{M}^2 = \mathbf{M}\mathbf{M} \quad , \quad \mathbf{M}^3 = \mathbf{M}\mathbf{M}\mathbf{M} \quad , \quad \ldots \quad . \tag{41.4}$$

In particular,

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} t^k \quad . \tag{41.5}$$

The theory for power series of square matrices is a straightforward extension of the theory for power series discussed in chapter 30, and we can safely use the "matrix" versions of the results discussed in 30 (with one warning to be mentioned in a moment). From that, we know the series for $e^{\mathbf{M}}$ converges for every square matrix $\mathbf{M}$. Moreover, for any constant $N \times N$ matrix $\mathbf{A}$,

$$\frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} t^k\right] = \sum_{k=1}^{\infty} \frac{\mathbf{A}^k}{k!} k t^{k-1} = \mathbf{A} \sum_{k=1}^{\infty} \frac{\mathbf{A}^{k-1}}{(k-1)!} t^{k-1} = \mathbf{A} \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{(n)!} t^n$$

So

$$\frac{d}{dt}\left[e^{\mathbf{A}t}\right] = \mathbf{A} e^{\mathbf{A}t} \tag{41.6}$$

and

$$e^{\mathbf{A}0} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} 0^k = \mathbf{A}^0 = \mathbf{I} \quad ,$$

verifying that the solution to initial-value problem (41.3) is, indeed, given by $\mathbf{X}^0(t) = e^{\mathbf{A}t}$ using the more general definition of the exponential.

Formula (41.5) provides another way for computing the fundamental matrix satisfying initial-value problem (41.3). However, unless $\mathbf{A}$ is particularly simple, it may be easier to compute formula (41.5) for a given $\mathbf{A}$ by solving initial-value problem (41.3) as discussed earlier in this section.

!►*Example 41.4:* *Let α be any constant or function of t, and set*

$$\mathbf{P} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{Q} = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$$

By basic matrix computations,

$$\mathbf{P}^2 = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ 0 & 0 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{Q}^2 = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad .$$

Continuing these calculations, we see that

$$\mathbf{P}^k = \begin{bmatrix} \alpha^k & 0 \\ 0 & 0 \end{bmatrix} \qquad \textit{for} \quad k = 1, 2, 3, \dots \quad ,$$

but that

$$\mathbf{Q}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O} \qquad \textit{for} \quad k = 2, 3, \dots \quad .$$

So

$$\begin{aligned} e^{\mathbf{P}} &= \sum_{k=0}^{\infty} \frac{\mathbf{P}^k}{k!} = \mathbf{P}^0 + \sum_{k=1}^{\infty} \frac{\mathbf{P}^k}{k!} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{1}{k!} \begin{bmatrix} \alpha^k & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\alpha} & 0 \\ 0 & 1 \end{bmatrix} \quad , \end{aligned}$$

while

$$\begin{aligned} e^{\mathbf{Q}} &= \sum_{k=0}^{\infty} \frac{\mathbf{Q}^k}{k!} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{1!} \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{3!} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \quad . \end{aligned}$$

In exercise 41.4 c, you will extend the computations done in the above exercise for $e^{\mathbf{P}}$ to show that, if $\mathbf{P}$ is any diagonal matrix

$$\mathbf{P} = \begin{bmatrix} r_1 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & 0 & \cdots & 0 \\ 0 & 0 & r_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & 0 & \cdots & r_N \end{bmatrix} \quad ,$$

then

$$e^{\mathbf{P}} = \begin{bmatrix} e^{r_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{r_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & 0 & \cdots & e^{r_N} \end{bmatrix} \quad .$$

However, the above computation of $e^{\mathbf{Q}}$ shows that, in general, the entries in the exponential of a given matrix are not simply the exponentials of the corresponding entries of the given matrix.

Inverses and Limitations of Matrix Exponentials

In extending the theory of power series to a theory of power series of matrices, it is important to remember that matrix multiplication is not commutative; that is, $\mathbf{AB} \neq \mathbf{BA}$ in general. Because of this, it turns out that the matrix versions of some of the standard exponential identities are not generally valid. For example, while we know $e^{a+b} = e^a e^b$ for any two numbers a and b, it can also be demonstrated that there are $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}} e^{\mathbf{B}}$$

(see exercise 41.5 a). That said, we do have some commutativity that will be helpful in verifying at least one identity we would hope to hold. From formula (41.5) and the almost trivial fact that

$$\left(\mathbf{A}^k\right)\mathbf{A} \;=\; \mathbf{A}^{k+1} \;=\; \mathbf{A}\left(\mathbf{A}^k\right) \quad ,$$

you can easily confirm that

$$e^{\mathbf{A}}\mathbf{A} \;=\; \mathbf{A}e^{\mathbf{A}} \tag{41.7}$$

for any square matrix $\mathbf{A}$, and that will help us prove one favorite identity for exponentials:

Lemma 41.2

Let $\mathbf{A}$ be any $N \times N$ matrix. Then the exponential of $\mathbf{A}t$, $e^{\mathbf{A}t}$, is invertible, and its inverse is $e^{-\mathbf{A}t}$.

PROOF: Let

$$\mathbf{X}(t) \;=\; e^{\mathbf{A}t}e^{-\mathbf{A}t} \quad ,$$

and observe that we can verify this lemma by showing that $\mathbf{X}(t) \;=\; \mathbf{I}$ for every t.

For $t = 0$, this is easy,

$$\mathbf{X}(0) \;=\; e^{\mathbf{A}0}e^{-\mathbf{A}0} \;=\; \mathbf{I}e^{-\mathbf{A}0} \;=\; e^{-\mathbf{A}0} \;=\; \mathbf{I} \quad .$$

To extend this, we will be computing $\mathbf{X}'$ using the product rule. But first, observe that, by equation (41.6)

$$\frac{de^{\mathbf{A}t}}{dt}e^{-\mathbf{A}t} \;=\; \left(\mathbf{A}e^{\mathbf{A}t}\right)e^{-\mathbf{A}t} \;=\; \mathbf{A}\left(e^{\mathbf{A}t}e^{-\mathbf{A}t}\right) \;=\; \mathbf{AX} \quad ,$$

and by equations (41.6) and (41.7),

$$e^{\mathbf{A}t}\frac{de^{-\mathbf{A}t}}{dt} \;=\; e^{\mathbf{A}t}\left(-\mathbf{A}e^{-\mathbf{A}t}\right) \;=\; -\left(e^{\mathbf{A}t}\mathbf{A}\right)e^{-\mathbf{A}t} \;=\; -\mathbf{A}e^{\mathbf{A}t}e^{-\mathbf{A}t} \;=\; -\mathbf{AX} \quad .$$

From this and the product rule, we see that

$$\frac{d\mathbf{X}}{dt} \;=\; \frac{d}{dt}\left[e^{\mathbf{A}t}e^{-\mathbf{A}t}\right] \;=\; \frac{de^{\mathbf{A}t}}{dt}e^{-\mathbf{A}t} \;+\; e^{\mathbf{A}t}\frac{de^{-\mathbf{A}t}}{dt} \;=\; \mathbf{AX} \;-\; \mathbf{AX} \;=\; [\mathbf{A}-\mathbf{A}]\mathbf{X} \;=\; \mathbf{0X} \quad .$$

So $\mathbf{X}$ is the solution to the initial-value problem

$$\mathbf{X}' \;=\; \mathbf{0X} \qquad \text{with} \quad \mathbf{X}(0) \;=\; \mathbf{I} \quad .$$

But it is trivial to verify that the one solution to this simple initial-value problem is given by $\mathbf{X}(t) = \mathbf{I}$ for all t. Thus,

$$e^{\mathbf{A}t}e^{-\mathbf{A}t} \;=\; \mathbf{X}(t) \;=\; \mathbf{I} \qquad \text{for all} \quad t \quad .$$

∎

Another formula we would probably like to extend is the differentiation formula $\left(e^{m(t)}\right)' = e^{m(t)}m'(t)$, which is valid whenever m is a differentiable function of one variable. Unfortunately, this formula does not extend to the matrix exponential. It is easy to come up with an $N \times N$ matrix $\mathbf{M}$ with differentiable elements such that

$$\frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] \;\neq\; e^{\mathbf{M}(t)}\frac{d\mathbf{M}}{dt} \qquad \text{and} \qquad \frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] \;\neq\; \frac{d\mathbf{M}}{dt}e^{\mathbf{M}(t)}$$

(see exercise 41.7 a). The best we can say is that

$$\textit{sometimes} \quad \frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] = e^{\mathbf{M}(t)}\frac{d\mathbf{M}}{dt} \quad .$$

For example, from our derivations of formulas (41.6) and (41.7), we know that

$$\frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] = e^{\mathbf{M}(t)}\frac{d\mathbf{M}}{dt} \qquad \text{if} \quad \mathbf{M}(t) = \mathbf{A}t$$

for a constant $N \times N$ matrix $\mathbf{A}$. Other cases for which we can so differentiate $e^{\mathbf{M}(t)}$ are described in exercise 41.7 b.

41.6 Using Similarity Transforms

Two constant $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be related by a *similarity transform* if and only if there is an invertible $N \times N$ matrix $\mathbf{T}$ such that

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \quad . \tag{41.8}$$

Note that this is completely equivalent to saying

$$\mathbf{A} = \mathbf{T}\mathbf{B}\mathbf{T}^{-1}$$

(just multiply both sides of equation (41.8) on the left by $\mathbf{T}$ and on the right by $\mathbf{T}^{-1}$).

Similarity transforms are quite important in linear algebra. For example, from the theory of linear algebra, we have the following:

Lemma 41.3
Let $\mathbf{A}$ be a constant $N \times N$ matrix. Then there is an invertible matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{B}$$

where all the entries of $\mathbf{B}$ are 0 except, possibly, for those on the main diagonal and immediately above the main diagonal. In fact,

$$\mathbf{B} = \begin{bmatrix} r_1 & s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_2 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & r_2 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & r_{N-1} & s_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & r_N \end{bmatrix}$$

where the r_k's are the eigenvalues of $\mathbf{A}$, and each s_k is either 0 or 1. Moreover:

1. *The number of times a particular eigenvalue appears in the main diagonal is equal to the algebraic multiplicity of that eigenvalue (as an eigenvalue of $\mathbf{A}$).*

2. $\mathbf{A}$ *has a complete set of eigenvectors if and only if*

$$s_k = 0 \qquad \textit{for} \quad k = 1, 2, \ldots, N-1 \quad .$$

This lemma tells us that every $N \times N$ matrix is related by a similarity transform to a particularly simple matrix. To see the importance of that to us, let $\mathbf{A}$ and $\mathbf{B}$ be two constant $N \times N$ matrices related by

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \qquad (\text{equivalently, } \mathbf{T}\mathbf{B}\mathbf{T}^{-1} = \mathbf{A})$$

for some invertible matrix $\mathbf{T}$. Also let $\mathbf{x}$ and $\mathbf{y}$ be two vector-valued functions on the real line related to each other via

$$\mathbf{y} = \mathbf{T}^{-1}\mathbf{x} \qquad (\text{equivalently, } \mathbf{T}\mathbf{y} = \mathbf{x}) \quad .$$

Now suppose $\mathbf{x}$ satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Observe that (using the above, the product rule for matrices, and the fact that $\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{T}\mathbf{T}^{-1}\mathbf{x}$),

$$\mathbf{y}' = \left(\mathbf{T}^{-1}\mathbf{x}\right)' = \mathbf{T}^{-1}\mathbf{x}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{x} = \mathbf{T}^{-1}\mathbf{A}\left(\mathbf{T}\mathbf{T}^{-1}\mathbf{x}\right) = \left(\mathbf{T}^{-1}\mathbf{A}\mathbf{T}\right)\left(\mathbf{T}^{-1}\mathbf{x}\right) = \mathbf{B}\mathbf{y} \quad ,$$

showing that $\mathbf{y}$ satisfies the constant matrix system $\mathbf{y}' = \mathbf{B}\mathbf{y}$. And if we had instead assumed $\mathbf{y}' = \mathbf{B}\mathbf{y}$, then we would have

$$\mathbf{x}' = (\mathbf{T}\mathbf{y})' = \mathbf{T}\mathbf{y}' = \mathbf{T}\mathbf{B}\mathbf{y} = \mathbf{T}\mathbf{B}\mathbf{T}^{-1}\mathbf{T}\mathbf{y} = \mathbf{A}\mathbf{x} \quad ,$$

showing that $\mathbf{x}$ satisfies the constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. And if you then take into account the fact that, because matrix multiplication is linear,

$$\mathbf{T}\left[c_1\mathbf{y}^1(t) + c_2\mathbf{y}^2(t) + \cdots + c_N\mathbf{y}^N(t)\right] = c_1\mathbf{T}\mathbf{y}^1(t) + c_2\mathbf{T}\mathbf{y}^2(t) + \cdots + c_N\mathbf{T}\mathbf{y}^N(t) \quad ,$$

you can easily finish proving the next theorem.

Theorem 41.4
Let $\mathbf{A}$ *and* $\mathbf{B}$ *be two constant* $N \times N$ *matrices related by a similarity transform*

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \quad ,$$

and let

$$\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\} \qquad \textit{and} \qquad \left\{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N\right\}$$

be two sets of N *vector-valued functions on the real line with*

$$\mathbf{x}^k = \mathbf{T}\mathbf{y}^k \qquad \textit{for} \quad k = 1, 2, \ldots, N \quad .$$

Then

$$\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\}$$

is a fundamental set of solutions for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *if and only if*

$$\left\{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N\right\}$$

is a fundamental set of solutions for $\mathbf{y}' = \mathbf{B}\mathbf{y}$.

So, if we can find a $\mathbf{T}$ so that $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is of the form described in lemma 41.3, then we can find a fundamental set of solutions $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ by first finding a fundamental set of solutions $\{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N\}$ for the much simpler system $\mathbf{y}' = \mathbf{B}\mathbf{y}$, and then setting

$$\mathbf{y}^k = \mathbf{T}^{-1}\mathbf{x}^k \qquad \text{for} \quad k = 1, 2, \ldots, N \quad .$$

Do observe how simple this system $\mathbf{y}' = \mathbf{B}\mathbf{y}$ is: If $\mathbf{A}$ has a complete set of eigenvectors, then $\mathbf{B}$ is diagonal, $\mathbf{y}' = \mathbf{B}\mathbf{y}$ is completely uncoupled, and a fundamental set can be found almost by inspection. Otherwise, it is still a very simple, "weakly coupled" system which is still easily solved.

Doubtlessly, you are wondering how we find that matrix $\mathbf{T}$. Here is part of the answer:

?►*Exercise 41.1:* *Let $\mathbf{A}$ be a constant $N \times N$ matrix with a complete set of eigenvectors. Show the following:*

a: *If $\mathbf{T}$ is as in lemma 41.3, then the k^{th} column of $\mathbf{T}$ is an eigenvector for $\mathbf{A}$ corresponding to r_k.*

b: *Let $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ where $\mathbf{T}$ is any matrix whose k^{th} column is given by $\mathbf{u}^k$ where $\{\mathbf{u}^1, \mathbf{u}^2, \cdots, \mathbf{u}^N\}$ is any complete set of eigenvectors for $\mathbf{A}$. Then $\mathbf{B}$ is of the form described in lemma 41.3.*

Thus, to find that matrix $\mathbf{T}$ when $\mathbf{A}$ has a complete set of eigenvectors, we first need to find a complete set of eigenvectors for $\mathbf{A}$. And if $\mathbf{A}$ does not have a complete set of eigenvectors, then you can show that $\mathbf{T}$ is constructed from the $\mathbf{w}^{k,j}$'s described in lemma 40.7 on page 40–18.

This relation between the $\mathbf{T}$ in lemma 41.3 and the eigenvectors of $\mathbf{A}$ (or the "$\mathbf{w}^{k,j}$'s" from lemma 40.7) rather lowers the value of similarity transformations as a practical tool for solving homogeneous constant coefficient systems of differential equations. After all, if we have already found those vectors, then it is easier to finish solving $\mathbf{x}' = \mathbf{A}\mathbf{x}$ via the methods discussed in the last two chapters. Still, similarity transforms has theoretical value. In fact, lemmas 40.6 and 40.7 in the last chapter are really corollaries of lemma 41.3. In addition, we may find some use for similarity transformation when dealing with nonhomogeneous constant matrix systems.

Let's just do one example, and move on to other topics.

!►*Example 41.5:* *Let us reconsider (again) the system in example 41.3,*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \text{with} \quad \mathbf{A} = \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix} \quad .$$

We already know that $\mathbf{A}$ has eigenpairs

$$\left(2, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right) \quad , \quad \left(-2, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right) \quad \text{and} \quad \left(2, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}\right)$$

with the three above eigenvectors forming a complete set of eigenvectors for $\mathbf{A}$. So it immediately follows that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{2t} , \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} e^{-2t} , \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} e^{2t} \right\}$$

is a fundamental set of solutions for $\mathbf{x}' = \mathbf{Ax}$ *, and*

$$\mathbf{X}(t) \;=\; \mathbf{X}(t) \;=\; \begin{bmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{bmatrix}$$

is the corresponding fundamental matrix.

Ignoring the fact that we can so readily find the solutions from the eigenpairs of $\mathbf{A}$ *let's find the matrix* $\mathbf{T}$ *using the above eigenvectors,*

$$\mathbf{T} \;=\; \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \quad .$$

(Observe that, in this case at least, $\mathbf{T} = \mathbf{X}(0)$ *!) Computing the inverse of* $\mathbf{T}$ *however you wish, you will find that*

$$\mathbf{T}^{-1} \;=\; \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}^{-1} \;=\; \cdots \;=\; \frac{1}{20}\begin{bmatrix} -10 & 9 & 7 \\ 0 & 6 & -2 \\ 10 & -7 & -1 \end{bmatrix}$$

So,

$$\begin{aligned} \mathbf{B} \;&=\; \mathbf{T}^{-1}\mathbf{AT} \\ &=\; \frac{1}{20}\begin{bmatrix} -10 & 9 & 7 \\ 0 & 6 & -2 \\ 10 & -7 & -1 \end{bmatrix} \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \;=\; \cdots \;=\; \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

Clearly $\mathbf{B}$ *has eigenpairs*

$$\left(2\,,\,\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad , \quad \left(-2\,,\,\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad \left(2\,,\,\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

with the three above eigenvectors forming a complete set of eigenvectors for $\mathbf{B}$*. Hence, a fundamental set of solutions for* $\mathbf{y}' = \mathbf{By}$ *is* $\{\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3\}$ *with*

$$\mathbf{y}^1(t) \;=\; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} \quad , \quad \mathbf{y}^2(t) \;=\; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} \quad \text{and} \quad \mathbf{y}^3(t) \;=\; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} \quad .$$

Theorem 41.4 now assures us that a fundamental set of solution to $\mathbf{x}' = \mathbf{Ax}$ *is* $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$ *with*

$$\mathbf{x}^1(t) \;=\; \mathbf{Ty}^1(t) \;=\; \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} \;=\; \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{2t} \quad ,$$

$$\mathbf{x}^2(t) = \mathbf{T}\mathbf{y}^1(t) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-2t} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} e^{-2t}$$

and

$$\mathbf{x}^3(t) = \mathbf{T}\mathbf{y}^3(t) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} e^{2t} \quad ,$$

just as we had noted near the start of this example.

41.7 Euler Systems

What Is An Euler System?

We will refer to any $N \times N$ system of differential equations as an *Euler* system if it can be given by

$$\frac{d\mathbf{x}}{dt} = \frac{1}{t}\mathbf{A}\mathbf{x} \qquad \text{for} \quad 0 < t < \infty$$

where $\mathbf{A}$ is constant $N \times N$ matrix with real components. Such a system is a natural first-order system extension of the Euler equations discussed in chapter 19 (see exercises 36.11 and 36.12 on page 36.11). As with the Euler equations, the domains of our solutions will not include $t = 0$. For convenience, we will insist that $0 < t < \infty$.

Our interest in Euler systems is mainly so that we can have systems that we can 'easily' solve other than constant matrix systems. And Euler systems can be solved just as easily as constant matrix systems. In fact, with only a few hints, you should be able to figure out how to solve Euler systems by building on what you know about Euler equations and what we've learned about solving constant matrix systems.

Direction Fields and Trajectories

Do observe that an Euler system is a homogeneous linear system, but, because of the $1/t$ factor, it is not an autonomous system. So the "velocity" $\mathbf{x}'(t)$ of a solution as it goes through a given position will depend both on that position and on when (i.e., t) it goes through that point. Still, as illustrated in the next example, changes in t only affects the magnitude of $\mathbf{x}'(t)$, not its direction. Consequently, as we'll see, we can still construct a direction field for an Euler system that does not depend on t.

!►*Example 41.6:* *Let consider the direction arrows at various points on the XY–plane for both*

$$\frac{d\mathbf{x}}{dt} = \frac{1}{t}\mathbf{A}\mathbf{x} \qquad \textit{and} \qquad \frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \qquad \textit{with} \quad \mathbf{A} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix} \quad .$$

The first is an Euler system, while the second is a constant matrix system.

Clearly, $\mathbf{x}(t) = \mathbf{0}$ is a constant solution for each.

Now, let $\mathbf{x}^E(t)$ *and* $\mathbf{x}^{CM}$ *be solutions to the Euler system and the constant matrix system, respectively, that satisfy*

$$\mathbf{x}^E(t_0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{CM}(t_0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for some $t_0 > 0$ *.*

For the constant matrix system, the direction arrow that we would sketch at $(1, 2)$ *would be a short arrow in the same direction as*

$$\left.\frac{d\mathbf{x}^{CM}}{dt}\right|_{t=t_0} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ -7 \end{bmatrix} \quad ,$$

which, as should be expected, does not depend on t_0 *.*

However, for the Euler system, the direction arrow that we would sketch at $(1, 2)$ *would be a short arrow in the same direction as*

$$\left.\frac{d\mathbf{x}^{E}}{dt}\right|_{t=t_0} = \frac{1}{2t_0}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2t_0}\begin{bmatrix} -1 \\ -7 \end{bmatrix} \quad ,$$

which does depend on t_0 *, but only very simple manner: Its dependence is in a factor that scales the vector by some positive quantity. But that does not affect the direction. In fact,*

$$\left.\frac{d\mathbf{x}^{E}}{dt}\right|_{t=t_0} = \frac{1}{t_0}\left.\frac{d\mathbf{x}^{CM}}{dt}\right|_{t=t_0} \quad .$$

So, the direction arrow of the Euler system at $(x, y) = (1, 2)$

1. *does not change with* t *, and*

2. *is the same as the direction arrow at* $(x, y) = (1, 2)$ *for the constant matrix system.*

Of course, there was nothing special about the point $(1, 2)$ *. If* (x_0, y_0) *is any point on the* XY*–plane, and* $\mathbf{x}^E(t)$ *and* $\mathbf{x}^{CM}(t)$ *are solutions to the Euler system and the constant matrix system, respectively, that satisfy*

$$\mathbf{x}^E(t_0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{CM}(t_0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

for some $t_0 > 0$ *. Then*

$$\left.\frac{d\mathbf{x}^{E}}{dt}\right|_{t=t_0} = \frac{1}{2t_0}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{1}{t_0}\left.\frac{d\mathbf{x}^{CM}}{dt}\right|_{t=t_0} \quad ,$$

telling us that the direction arrow of the Euler system at (x_0, y_0)

1. *does not change with* t *, and*

2. *is the same as the direction arrow at* (x_0, y_0) *for the constant matrix system.*

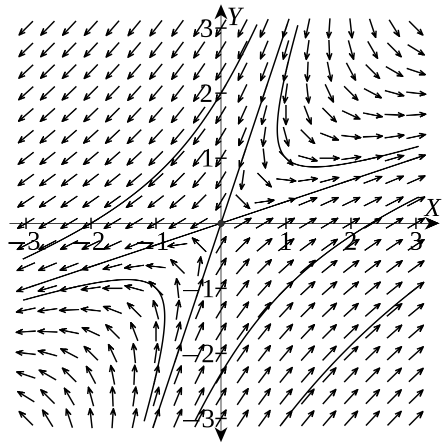

Figure 41.3: A direction field and some trajectories for the Euler system in example 41.6.

Thus, we can construct a well-defined direction field for our Euler system, and this direction field

1. *does not change with t, and*
2. *is also a direction field for the above constant matrix system.*

Moreover, since the trajectories of the solutions can be determined from the direction field, the trajectories of the solutions to the above Euler system

1. *do not vary with t, and*
2. *are the same as the trajectories for the above constant matrix system.*

A direction field for our Euler system, along with a few trajectories have been sketched in figure 41.3.

It should be clear that the observations made in the example hold for any Euler system. In summary:

Lemma 41.5
Let $\mathbf{A}$ be a constant $N \times N$ matrix with real components. Then direction fields and trajectories for the Euler system

$$\frac{d\mathbf{x}}{dt} = \frac{1}{t}\mathbf{A}\mathbf{x} \qquad \text{for} \quad 0 < t < \infty$$

are well-defined and do not depend on t. Moreover these direction fields and trajectories are also direction fields and trajectories for the constant matrix system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \qquad \text{for} \quad 0 < t < \infty \quad .$$

Keep in mind, however, that the trajectories of an Euler system are traced out by the solutions as t goes from 0 to ∞, unlike the trajectories of a constant matrix system that are traced out by the solutions as t goes from $-\infty$ to ∞.

Solving Euler Systems

Our solving of a constant matrix system began with the derivation that $\mathbf{x}(t) = \mathbf{u}e^{rt}$ satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$ if $(r, \mathbf{u})$ is an eigenpair for $\mathbf{A}$, and the e^{rt} factor in this solution was inspired by the basic solutions to the linear homogeneous differential equations with constant coefficients. From our study of Euler equations in chapter 19 you probably suspect that, for Euler systems, we will want to use t^r instead of e^{rt}. Well, you are correct.

Theorem 41.6
Let $\mathbf{A}$ be a constant $N \times N$ matrix with real components. If $(r, \mathbf{u})$ is an eigenpair for $\mathbf{A}$, then $\mathbf{x}(t) = \mathbf{u}t^r$ is a solution to the Euler system

$$\frac{d\mathbf{x}}{dt} = \frac{1}{t}\mathbf{A}\mathbf{x} \qquad \textit{for} \quad 0 < t < \infty \quad .$$

?►*Exercise 41.2:* *Verify the above theorem.*

As an immediate corollary, we have:

Corollary 41.7
Let $\mathbf{A}$ be a constant $N \times N$ matrix with real components. Assume $\mathbf{A}$ has a complete set of eigenvectors $\{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N\}$, and, for $k = 1, 2, \ldots, N$, let r_k be the eigenvalue corresponding to $\mathbf{u}^k$. Then the Euler system

$$\frac{d\mathbf{x}}{dt} = \frac{1}{t}\mathbf{A}\mathbf{x} \qquad \textit{for} \quad 0 < t < \infty$$

has

$$\left\{\mathbf{u}^1 t^{r_1},\ \mathbf{u}^2 t^{r_2},\ \cdots,\ \mathbf{u}^N t^{r_N}\right\}$$

as a fundamental set of solutions, and

$$\mathbf{x}(t) = c_1\mathbf{u}^1 t^{r_1} + c_2\mathbf{u}^2 t^{r_2} + \cdots + c_N\mathbf{u}^N t^{r_N}$$

as a general solution.

!►*Example 41.7:* *You can easily verify that*

$$\left(-2, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) \qquad \textit{and} \qquad \left(2, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$$

are eigenpairs for the matrix

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix} \quad .$$

So, according to the above, the Euler system

$$\frac{d\mathbf{x}}{dt} = \frac{1}{t}\mathbf{A}\mathbf{x}$$

has

$$\left\{\mathbf{x}^1(t),\ \mathbf{x}^2(t)\right\} = \left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-2},\ \begin{bmatrix} 3 \\ 1 \end{bmatrix} t^2\right\}$$

as a fundamental set of solutions, and

$$\mathbf{x}(t) \;=\; c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-2} \;+\; c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} t^2$$

as a general solution.

In addition, from this fundamental set of solutions, we can construct the corresponding fundamental matrix for the Euler system,

$$\mathbf{X}(t) \;=\; \begin{bmatrix} t^{-2} & 3t^2 \\ 3t^{-2} & t^2 \end{bmatrix}$$

and the fundamental matrix $\mathbf{X}^0$ *that also satisfies* $\mathbf{X}^0(1) \;=\; \mathbf{I}$ *is*

$$\begin{aligned} \mathbf{X}^0(t) \;&=\; [\mathbf{X}(t)][\mathbf{X}(1)]^{-1} \\ &=\; \begin{bmatrix} t^{-2} & 3t^2 \\ 3t^{-2} & t^2 \end{bmatrix} \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}^{-1} \right) \\ &=\; \begin{bmatrix} t^{-2} & 3t^2 \\ 3t^{-2} & t^2 \end{bmatrix} \left(\frac{1}{8} \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} \right) \;=\; \frac{1}{8} \begin{bmatrix} -t^{-2} + 9t^2 & 3t^{-2} - 3t^2 \\ -3t^{-2} + 3t^2 & 9t^{-2} - t^2 \end{bmatrix} \quad . \end{aligned}$$

As with the constant matrix systems, there are two particular complications that may arise:

1. One is that $\mathbf{A}$ may have a complex eigenvalue r . If so, then taking the real and imaginary parts of the corresponding solution $\mathbf{u}t^r$ will yield real-valued solutions to the Euler system, just as taking the real and imaginary parts of solutions of the form $\mathbf{u}e^{rt}$ yielded real-valued solutions to the constant matrix system in section 40.1.

2. The other is that $\mathbf{A}$ might not have a complete set of eigenvectors. If so, then an adaptation of the development discussed in sections 40.3 and 40.5 is in order.

The details will be left to the interested reader.

Additional Exercises

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Exercises for section 41.1 TBW
Exercises for section 41.2 TBW
Exercises for section 41.3 TBW
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41.3. *Assume* $\mathbf{A}$ *is* 2×2 *constant matrix with real components, and let*

$$r_1 \;=\; \lambda_1 \;+\; i\omega \qquad \text{and} \qquad r_2 \;=\; \lambda_2 \;-\; i\omega$$

be approximations of the true eigenvalues, with $\lambda_1 = \lambda_2$ *if* $\omega \neq 0$ *and* $r_1 \leq r_2$ *if* $\omega = 0$ *. Assume each computed* λ *and* ω *is known to be "within* ϵ *" of the true* *value, where ϵ is some (hopefully small) positive value. To simplify notation, $r_1 \approx r_0$ or $\omega \approx 0$ means these computed values are close enough that it is possible for the corresponding equalities to hold for the true values.*

Several choices for r_1 and r_2 are given in each of the following. For each, state what these eigenvalues tell you about the critical point $(0,0)$, the stability of the equilibrium solution $\mathbf{x}(t) \equiv \mathbf{0}$ and the phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ both assuming the computed eigenvalues are correct, and for all the other possible values of the eigenvalues (see the discussion of Phase Portraits for Imprecisely Known Systems*).*

a. $r_1 < r_2 < -\epsilon$ *with* $r_1 \not\approx r_2$

b. $r_1 < r_2 < -\epsilon$ *with* $r_1 \approx r_2$

c. $r_1 < -\epsilon$ *and* $\epsilon < r_2$

d. $\lambda = 0$ *and* $\epsilon < \omega$

e. $r_1 \approx 0$ *and* $r_2 \approx 0$

41.4. *In the following, you will compute matrix exponentials using "basic definitions". When they appear, a, b and c denote arbitrary numbers or functions. (Note: Some of your results will be used in later exercises.)*

a. *Compute $e^{\mathbf{M}}$ for each of the following choices of $\mathbf{M}$, using formula either (41.4) or formula (41.5):*

i. $\begin{bmatrix} 2t & 0 \\ 0 & 3t \end{bmatrix}$ **ii.** $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ **iii.** $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ **iv.** $\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$

v. $\begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}$ **vi.** $\begin{bmatrix} t & 1 \\ 0 & 0 \end{bmatrix}$

b. *By solving the appropriate initial-value problem, find $e^{\mathbf{A}t}$ for each of the following choices of $\mathbf{A}$:*

i. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **ii.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **iii.** $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$

c. *Show that*

$$e^{\mathbf{M}} = \begin{bmatrix} e^{r_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{r_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{r_N} \end{bmatrix} \quad \textit{when} \quad \mathbf{M} = \begin{bmatrix} r_1 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & 0 & \cdots & 0 \\ 0 & 0 & r_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_N \end{bmatrix} \quad .$$

41.5. *In the following, we will explore the validity (and NONvalidity) of the identity $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$.*

a. *Let*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

i. *Using results from exercise 41.4, compute $e^{\mathbf{A}}$, $e^{\mathbf{B}}$ and $e^{\mathbf{A}+\mathbf{B}}$.*

ii. *By direct computation, show that*

$$e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{\mathbf{B}}e^{\mathbf{A}} \quad , \quad e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{\mathbf{A}+\mathbf{B}} \quad \text{and} \quad e^{\mathbf{B}}e^{\mathbf{A}} \neq e^{\mathbf{A}+\mathbf{B}} \quad .$$

for the above choices for $\mathbf{A}$ *and* $\mathbf{B}$*.*

b. *Now assume* $\mathbf{A}$ *and* $\mathbf{B}$ *are any two* $N \times N$ *matrices that commute; that is,* $\mathbf{A}$ *and* $\mathbf{B}$ *satisfy* $\mathbf{AB} = \mathbf{BA}$*.*

i. *Using series formula (41.5) for* $e^{\mathbf{A}t}$ *and the fact that* $\mathbf{AB} = \mathbf{BA}$*, show that*

$$\mathbf{B}e^{\mathbf{A}t} \;=\; e^{\mathbf{A}t}\mathbf{B} \quad .$$

ii. *Show that*

$$e^{\mathbf{A}+\mathbf{B}} \;=\; e^{\mathbf{A}}e^{\mathbf{B}} \tag{41.9}$$

by verifying that $\mathbf{X}(t) = e^{\mathbf{A}t}e^{\mathbf{B}t}$ *satisfies*

$$\frac{d\mathbf{X}}{dt} \;=\; [\mathbf{A}+\mathbf{B}]\mathbf{X} \qquad \text{with} \quad \mathbf{X}(0) \;=\; \mathbf{I} \quad .$$

Be sure to explain why this verification also confirms that equality (41.9) holds. (Hint: Take a look at the proof of lemma 41.2 on page 41–14.)

41.6. *In these exercises, we will explore the validity (and NONvalidity) of a standard differentiation formula.*

a. *Let*

$$\mathbf{M}(t) \;=\; \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix} \quad ,$$

and verify, by direct computation, the following:

i. $\mathbf{M}\dfrac{d\mathbf{M}}{dt} \;\neq\; \dfrac{d\mathbf{M}}{dt}\mathbf{M}$ ***ii.*** $\dfrac{d}{dt}\left[\mathbf{M}^2\right] \;\neq\; 2\mathbf{M}\dfrac{d\mathbf{M}}{dt}$ ***iii.*** $\dfrac{d}{dt}\left[\mathbf{M}^2\right] \;\neq\; 2\dfrac{d\mathbf{M}}{dt}\mathbf{M}$

b. *Now let* $\mathbf{M}(t)$ *be any* $N \times N$ *matrix of differentiable functions that commutes with its derivative,* $\mathbf{M}\mathbf{M}' = \mathbf{M}'\mathbf{M}$*. Show that, in this case,*

i. $\dfrac{d}{dt}\left[\mathbf{M}^2\right] \;=\; 2\mathbf{M}\dfrac{d\mathbf{M}}{dt} \quad .$

ii. $\dfrac{d}{dt}\left[\mathbf{M}^k\right] \;=\; k\dfrac{d\mathbf{M}}{dt}\mathbf{M}^{k-1} \qquad \text{for} \quad k = 2,\ 3,\ \dots \quad .$

41.7. *In these exercises, we will explore the validity (and NONvalidity) of another standard differentiation formula.*

a. *In exercise 41.4 you found* $e^{\mathbf{M}(t)}$ *when*

$$\mathbf{M}(t) \;=\; \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix} \quad .$$

Now verify, by direct computation, that

$$\frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] \;\neq\; e^{\mathbf{M}(t)}\frac{d\mathbf{M}}{dt} \qquad \text{and} \qquad \frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] \;\neq\; \frac{d\mathbf{M}}{dt}e^{\mathbf{M}(t)}$$

when $\mathbf{M}$ *is as above.*

b. *Now verify that*

$$\frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] = e^{\mathbf{M}(t)}\frac{d\mathbf{M}}{dt} \qquad \text{and} \qquad \frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] = \frac{d\mathbf{M}}{dt}e^{\mathbf{M}(t)}$$

when

i. $\mathbf{M}$ *and* $\mathbf{M}'$ *commute (Hint: See exercise 41.6 b).*

ii. $\mathbf{M}(t) = \mathbf{A}f(t)$ *for some constant* $N \times N$ *matrix* $\mathbf{A}$ *and some differentiable function* f .

Exercises for section 41.6 TBW

Exercises for section 41.7 TBW

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

4a i. $\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$

4a ii. $\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$

4a iii. $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

4a iv. $\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$

4a v. $\begin{bmatrix} e & t(e-1) \\ 0 & 1 \end{bmatrix}$

4a vi. $\begin{bmatrix} e^t & t^{-1}(e^t - 1) \\ 0 & 1 \end{bmatrix}$

4b i. $\begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix}$

4b ii. $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$

4b iii. $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$