31

Power Series Solutions I: Basic Computational Methods

When a solution to a differential equation is analytic at a point, then that solution can be represented by a power series about that point. In this and the next chapter, we will discuss when this can be expected, and how we might use this fact to obtain usable power series formulas for the solutions to various differential equations. In this chapter, we will concentrate on two basic methods — an "algebraic method" and a "Taylor series method" — for computing our power series. Our main interest will be in the algebraic method. It is more commonly used and is the method we will extend in chapter 33 to obtain "modified" power series solutions when we do not quite have the desired analyticity. But the algebraic method is not well suited for solving all types of differential equations, especially when the differential equations in question are not linear. For that reason (and others) we will also introduce the Taylor series method near the end of this chapter.

31.1 Basics

General Power Series Solutions

If it exists, a *power series solution* for a differential equation is just a power series formula

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

for a solution y to the given differential equation in some open interval containing x_0. The series is a *general power series solution* if it describes all possible solutions in that interval.

As noted in the last chapter (corollary 30.10 on page 30–16), if $y(x)$ is given by the above power series, then

$$a_0 = y(x_0) \qquad \text{and} \qquad a_1 = y'(x_0) \quad .$$

Because a general solution to a first-order differential equation normally has one arbitrary constant, we should expect a general power series solution to a first-order differential equation to also have a single arbitrary constant. And since that arbitrary constant can be determined by a given initial value $y(x_0)$, it makes sense to use a_0 as that arbitrary constant.

On the other hand, a general solution to a second-order differential equation usually has two arbitrary constants, and they are normally determined by initial values $y(x_0)$ and $y'(x_0)$. Consequently, we should expect the first two coefficients, a_0 and a_1, to assume the roles of the arbitrary constants in our general power series solutions for second-order differential equations.

The Two Methods, Briefly

The basic ideas of both the "algebraic method" and the "Taylor series method" are fairly simple.

The Algebraic Method

The *algebraic method* starts by assuming the solution y can be written as a power series

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

with the a_k's being constants to be determined. This formula for y is then plugged into the differential equation. By using a lot of algebra and only a little calculus, we then "simplify" the resulting equation until it looks something like

$$\sum_{n=0}^{\infty} \left[n^{\text{th}} \text{ formula of the } a_k\text{'s} \right] x^n = 0 \quad .$$

As we saw in the last chapter, this means

$$n^{\text{th}} \text{ formula of } a_k\text{'s} = 0 \qquad \text{for} \quad n = 0, 1, 2, 3, \dots \quad ,$$

which (as we will see) can be used to determine all the a_k's in terms of one or two arbitrary constants. Plugging these a_k's back into the series then gives the power series solution to our differential equation about the point x_0.

We will outline the details for this method in the next two sections for first- and second-order homogeneous linear differential equations

$$a(x)y' + b(x)y = 0 \qquad \text{and} \qquad a(x)y'' + b(x)y' + c(x)y = 0$$

in which the coefficients are rational functions. These are the equations for which the method is especially well suited.[1] For pedagogic reasons, we will deal with first-order equations first, and then expand our discussion to include second-order equations. It should then be clear that this approach can easily be extended to solve higher-order analogs of the equations discussed here.

The Taylor Series Method

The basic ideas behind the *Taylor series method* are even easier to describe. We simply use the given differential equation to find the values of all the derivatives of the solution $y(x)$ when $x = x_0$, and then plug these values into the formula for the Taylor series for y about x_0 (see corollary 30.11 on page 30–16). Details will be laid out in section 31.6.

[1] Recall that a rational function is a function that can be written as one polynomial divided by another polynomial. Actually, in theory at least, the algebraic method is "well suited" for a somewhat larger class of first- and second-order linear differential equations. We'll discuss this in the next chapter.

31.2 The Algebraic Method with First-Order Equations

Details of the Method

Here are the detailed steps of our algebraic method for finding a general power series solution to

$$a(x)y' + b(x)y = 0$$

assuming $a(x)$ and $b(x)$ are rational functions. To illustrate these steps, we'll find a general power series solution to

$$y' + \frac{2}{x-2}y = 0 \quad . \tag{31.1}$$

Admittedly you could solve this differential equation easier using methods from either chapter 4 or 5, but it is a good equation for our first example.[2]

Before actually starting the method, there are two preliminary steps:

Pre-step 1: Rewrite your differential equation in the form

$$A(x)y' + B(x)y = 0$$

where A and B are polynomials, preferably without common factors.

To get differential equation (31.1) into the form desired, we multiply the equation by $x-2$. That gives us

$$(x-2)y' + 2y = 0 \quad . \tag{31.2}$$

Pre-step 2: If not already specified, choose a value for x_0. For reasons we will discuss later, x_0 should be chosen so that $A(x_0) \neq 0$. If initial conditions are given for $y(x)$ at some point, then use that point for x_0 (provided $A(x_0) \neq 0$). Otherwise, choose x_0 as convenient — which usually means choosing $x_0 = 0$.[3]

For our example, we have no initial values at any point, and the first coefficient, $x-2$, is zero only if $x_0 = 2$. So let us choose x_0 as simply as possible; namely, $x_0 = 0$.

Now for the basic method:

Step 1: Assume a power series solution

$$y = y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k \tag{31.3}$$

[2] Truth is, power series are rarely used to solve first-order differential equations because these equations are often more easily solved using the more direct methods developed earlier in this text. In fact, many texts don't even mention using power series with first-order equations. We're doing first-order equations here because this author like to start as simple as possible.

[3] The requirement that $A(x_0) \neq 0$ is a slight simplification of requirements we'll develop in the next section. But "$A(x_0) \neq 0$" will suffice for now, especially if A and B are polynomials with no common factors.

with a_0 being arbitrary and the other a_k's "to be determined", and then compute/write out the corresponding first derivative

$$
\begin{aligned}
y' &= \frac{d}{dx}\sum_{k=0}^{\infty} a_k (x-x_0)^k \\
&= \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k (x-x_0)^k\right] = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1} \quad .
\end{aligned}
$$

(Remember that the derivative of the a_0 term is zero. Explicitly dropping this zero term in the series for y' is not necessary, but can simplify bookkeeping, later.)

Since we've already decided $x_0 = 0$, *we assume*

$$
y = y(x) = \sum_{k=0}^{\infty} a_k x^k \quad , \tag{31.4}
$$

and compute

$$
y' = \frac{d}{dx}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^k\right] = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad .
$$

Step 2: Plug the series for y and y' back into the differential equation and "multiply things out". (If $x_0 \neq 0$, see the comments on page 31–11.)

Some notes:

i. Absorb any x's from $A(x)$ and $B(x)$ into the series.

ii. Your goal is to get an equation in which zero equals the sum of a few power series about x_0.

Using the above series with the given differential equation, we get

$$
\begin{aligned}
0 &= (x-2)y' + 2y \\
&= (x-2)\sum_{k=1}^{\infty} k a_k x^{k-1} + 2\sum_{k=0}^{\infty} a_k x^k \\
&= \left[x\sum_{k=1}^{\infty} k a_k x^{k-1} - 2\sum_{k=1}^{\infty} k a_k x^{k-1}\right] + 2\sum_{k=0}^{\infty} a_k x^k \\
&= \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=1}^{\infty} (-2) k a_k x^{k-1} + \sum_{k=0}^{\infty} 2 a_k x^k \quad .
\end{aligned}
$$

Step 3: For each series in your last equation, do a change of index[4] so that each series looks like

$$
\sum_{n=\text{something}}^{\infty} \left[\text{something not involving } x\right](x-x_0)^n \quad .
$$

Be sure to appropriately adjust the lower limit in each series.

[4] see *Changing the Index* on page 30–11

In all but the second series in the example, the "change of index" is trivial ($n = k$). In the second series, we set $n = k - 1$ (equivalently, $k = n + 1$):

$$
\begin{aligned}
0 &= \underbrace{\sum_{k=1}^{\infty} k a_k x^k}_{n = k} + \underbrace{\sum_{k=1}^{\infty} (-2) k a_k x^{k-1}}_{n = k-1} + \underbrace{\sum_{k=0}^{\infty} 2 a_k x^k}_{n = k} \\
&= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n+1=1}^{\infty} (-2)(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2 a_n x^n \\
&= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} (-2)(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2 a_n x^n \quad .
\end{aligned}
$$

Step 4: Convert the sum of series in your last equation into one big series. The first few terms will probably have to be written separately. Go ahead and simplify what can be simplified.

Since one of the series in the last equation begins with $n = 1$, we need to separate out the terms corresponding to $n = 0$ in the other series before combining series:

$$
\begin{aligned}
0 &= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} (-2)(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2 a_n x^n \\
&= \sum_{n=1}^{\infty} n a_n x^n + \left[(-2)(0+1) a_{0+1} x^0 + \sum_{n=1}^{\infty} (-2)(n+1) a_{n+1} x^n \right] \\
&\qquad + \left[2 a_0 x^0 + \sum_{n=1}^{\infty} 2 a_n x^n \right] \\
&= [-2a_1 + 2a_0] x^0 + \sum_{n=1}^{\infty} [n a_n - 2(n+1) a_{n+1} + 2 a_n] x^n \\
&= 2[a_0 - a_1] x^0 + \sum_{n=1}^{\infty} \big[(n+2) a_n - 2(n+1) a_{n+1} \big] x^n \quad .
\end{aligned}
$$

Step 5: At this point, you have an equation basically of the form

$$
\sum_{n=0}^{\infty} \left[n^{\text{th}} \text{ formula of the } a_k\text{'s} \right] (x - x_0)^n = 0 \quad ,
$$

which is possible only if

$$
n^{\text{th}} \text{ formula of the } a_k\text{'s} = 0 \qquad \text{for} \quad n = 0, 1, 2, 3, \ldots \quad .
$$

Using this last equation:

(a) Solve for the a_k with the highest index, obtaining

$$a_{\text{highest index}} = \text{formula of } n \text{ and lower-indexed coefficients} \quad .$$

A few of these equations may need to be treated separately, but you should obtain one relatively simple formula that holds for all indices above some fixed value. This formula is a *recursion formula* for computing each coefficient from the previously computed coefficients.

(b) To simplify things just a little, do another change of indices so that the recursion formula just derived is rewritten as

$$a_k = \text{formula of } k \text{ and lower-indexed coefficients} \quad .$$

From the last step in our example, we have

$$2[a_0 - a_1]x^0 + \sum_{n=1}^{\infty} \big[(n+2)a_n - 2(n+1)a_{n+1}\big]x^n = 0 \quad .$$

So,

$$2[a_0 - a_1] = 0 \quad , \tag{31.5a}$$

and, for $n = 1, 2, 3, 4, \ldots,$

$$(n+2)a_n - 2(n+1)a_{n+1} = 0 \quad . \tag{31.5b}$$

In equation (31.5a), a_1 *is the highest indexed* a_k *; solving for it in terms of the lower-indexed* a_k*'s (i.e.,* a_0 *) yields*

$$a_1 = a_0 \quad .$$

Equation (31.5b) also just contains two a_k*'s:* a_n *and* a_{n+1} *. Since* $n+1 > n$ *, we solve for* a_{n+1} *,*

$$a_{n+1} = \frac{n+2}{2(n+1)}a_n \qquad \text{for} \quad n = 1, 2, 3, 4, \ldots \quad .$$

Letting $k = n+1$ *(equivalently,* $n = k-1$ *), this becomes*

$$a_k = \frac{k+1}{2k}a_{k-1} \qquad \text{for} \quad k = 2, 3, 4, 5, \ldots \quad . \tag{31.6}$$

This is the recursion formula we will use.

Step 6: Use the recursion formula (and any corresponding formulas for the lower-order terms) to find all the a_k's in terms of a_0. Look for patterns!

In the last step, we saw that

$$a_1 = a_0 \quad .$$

Using this and the recursion formula (31.6) with $k = 2, 3, 4, \ldots$ *(and looking for patterns), we obtain the following:*

$$a_2 = \frac{2+1}{2\cdot 2}a_{2-1} = \frac{3}{2\cdot 2}a_1 = \frac{3}{2\cdot 2}a_0 \quad ,$$

$$\begin{aligned}
a_3 &= \frac{3+1}{2\cdot 3}a_{3-1} = \frac{4}{2\cdot 3}a_2 = \frac{4}{2\cdot 3}\cdot\frac{3}{2\cdot 2}a_0 = \frac{4}{2^3}a_0 \quad ,\\
a_4 &= \frac{4+1}{2\cdot 4}a_{4-1} = \frac{5}{2\cdot 4}a_3 = \frac{5}{2\cdot 4}\cdot\frac{4}{2^3}a_0 = \frac{5}{2^4}a_0 \quad ,\\
a_5 &= \frac{5+1}{2\cdot 5}a_{5-1} = \frac{6}{2\cdot 5}a_4 = \frac{6}{2\cdot 5}\cdot\frac{5}{2^5}a_0 = \frac{6}{2^5}a_0 \quad ,\\
&\vdots
\end{aligned}$$

The pattern here is obvious:

$$a_k = \frac{k+1}{2^k}a_0 \qquad \text{for} \quad k = 2,\ 3,\ 4,\ \dots \quad .$$

Note that this formula even gives us our $a_1 = a_0$ *equation,*

$$a_1 = \frac{1+1}{2^1}a_0 = \frac{2}{2}a_0 = a_0 \quad ,$$

and is even valid with $k = 0$ *,*

$$a_0 = \frac{0+1}{2^2}a_0 = a_0 \quad .$$

So, in fact,

$$a_k = \frac{k+1}{2^k}a_0 \qquad \text{for} \quad k = 0,\ 1,\ 2,\ 3,\ 4,\ \dots \quad . \tag{31.7}$$

Step 7: Using the formulas just derived for the coefficients, write out the resulting series for $y(x)$. Try to simplify it and factor out the a_0 .

Plugging the formulas just derived for the a_k*'s into the power series assumed for* y *,*

$$y(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{k+1}{2^k}a_0 x^k = a_0 \sum_{k=0}^{\infty} \frac{k+1}{2^k}x^k \quad .$$

So we have

$$\begin{aligned}
y(x) &= a_0 \sum_{k=0}^{\infty} \frac{k+1}{2^k}x^k \\
&= a_0 \left[\frac{0+1}{2^0}x^0 + \frac{1+1}{2^1}x^1 + \frac{2+1}{2^2}x^2 + \frac{3+1}{2^3}x^3 + \cdots \right] \\
&= a_0 \left[1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \cdots \right]
\end{aligned}$$

as the series solution for our first-order differential equation (assuming it converges).

Last Step: See if you recognize the series derived as the series for some well-known function (you probably won't!).

By an amazing stroke of luck, in exercise 30.9 a on page 30–24 we saw that

$$\frac{2}{(2-x)^2} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k+1}{2^k}x^k \quad .$$

So our formula for y simplifies considerably:

$$y(x) = a_0\sum_{k=0}^{\infty}\frac{k+1}{2^k}x^k = a_0\left[2\cdot\frac{2}{(2-x)^2}\right] = \frac{4a_0}{(2-x)^2} \quad .$$

Practical Advice on Using the Method

General Comments

The method just described is a fairly straightforward procedure, at least up to the point where you are trying to "find a pattern" for the a_k's. The individual steps are, for the most part, simple and only involve elementary calculus and algebraic computations — but there are a lot of these elementary computations, and an error in just one can throw off the subsequent computations with disastrous consequences for your final answer. So be careful, write neatly, and avoid shortcuts and doing too many computations in your head. It may also be a good idea to do your work with your paper turned sideways, just to have enough room for each line of formulas.

On Finding Patterns

In computing the a_k's, we usually want to find some "pattern" described by some reasonably simple formula. In our above example, we found formula (31.7),

$$a_k = \frac{k+1}{2^k}a_0 \qquad \text{for} \quad k = 0, 1, 2, 3, 4, \ldots \quad .$$

Using this formula, it was easy to write out the power series solution.

More generally, we will soon verify that the a_k's obtained by this method can all be simply related to a_0 by an expression of the form

$$a_k = \alpha_k a_0 \qquad \text{for} \quad k = 0, 1, 2, 3, 4, \ldots$$

where $\alpha_0 = 1$ and the other α_k's are fixed numbers (hopefully given by some simple formula of k). In the example cited just above,

$$\alpha_k = \frac{k+1}{2^k} \qquad \text{for} \quad k = 0, 1, 2, 3, 4, \ldots \quad .$$

Finding that pattern and its formula (i.e., the above mentioned α_k's) is something of an art, and requires a skill that improves with practice. One suggestion is to avoid multiplying factors out. It was the author's experience that, in deriving formula (31.7), led him to leave 2^2 and 2^3 as 2^2 and 2^3, instead of as 4 and 8 — he suspected a pattern would emerge. Another suggestion is to compute "many" of the a_k's using the recursion formula before trying to identify the pattern. And once you believe you've found that pattern and derived that formula, say,

$$a_k = \frac{k+1}{2^k}a_0 \qquad \text{for} \quad k = 0, 1, 2, 3, 4, \ldots \quad ,$$

test it by computing a few more a_k's using both the recursion formula directly and using your newly found formula. If the values computed using both methods don't agree, your formula is wrong. Better yet, if you are acquainted with the method of induction, use that to rigorously confirm your formula.[5]

Unfortunately, in practice, it may not be so easy to find such a pattern for your a_k's. In fact, it is quite possible to end up with a three (or more) term recursion formula, say,

$$a_n = \frac{1}{n^2+1}a_{n-1} + \frac{2}{3n(n+3)}a_{n-2} \quad ,$$

which can make "finding patterns" quite difficult.

Even if you do see a pattern, it might be difficult to describe. In these cases, writing out a relatively simple formula for all the terms in the power series solution may not be practical. What we can still do, though, is to use the recursion formula to compute (or have a computer compute) as many terms as we think are needed for a reasonably accurate partial sum approximation.

Terminating Series

It's worth checking your recursion formula

$$a_k = \text{formula of } k \text{ and lower-indexed coefficients}$$

to see if the right side becomes zero for some value of k, say $k = K$. Then, of course,

$$a_K = 0$$

and the computation of the subsequent a_k's may become especially simple. In fact, you may well have

$$a_k = 0 \qquad \text{for all} \quad k \geq K \quad .$$

This, essentially, "terminates" the series and gives you a polynomial solution — something that's usually easier to handle that a true infinite series solution.

!►*Example 31.1:* *Consider finding the power series solution to*

$$\left(x^2+1\right)y' - 4xy = 0$$

about $x_0 = 0$.

It is already in the right form. So, following the procedure, we let

$$y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k = \sum_{k=0}^{\infty} a_k x^k \quad ,$$

and 'compute'

$$y'(x) = \frac{d}{dx}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^k\right] = \sum_{k=1}^{\infty} a_k k x^{k-1} \quad .$$

[5] And to learn about using induction, see section 31.7.

Plugging this into the differential equation and carrying out the index manipulation and algebra of our method:

$$
\begin{aligned}
0 &= \left(x^2+1\right)y' - 4xy \\
&= \left(x^2+1\right)\sum_{k=1}^{\infty} a_k k x^{k-1} - 4x\sum_{k=0}^{\infty} a_k x^k \\
&= x^2\sum_{k=1}^{\infty} a_k k x^{k-1} + 1\sum_{k=1}^{\infty} a_k k x^{k-1} - 4x\sum_{k=0}^{\infty} a_k x^k \\
&= \underbrace{\sum_{k=1}^{\infty} a_k k x^{k+1}}_{n\,=\,k+1} + \underbrace{\sum_{k=1}^{\infty} a_k k x^{k-1}}_{n\,=\,k-1} - \underbrace{\sum_{k=0}^{\infty} 4a_k x^{k+1}}_{n\,=\,k+1} \\
&= \sum_{n=2}^{\infty} a_{n-1}(n-1)x^n + \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=1}^{\infty} 4a_{n-1}x^n \\
&= \sum_{n=2}^{\infty} a_{n-1}(n-1)x^n + \left[a_{0+1}(0+1)x^0 + a_{1+1}(1+1)x^1 + \sum_{n=2}^{\infty} a_{n+1}(n+1)x^n\right] \\
&\qquad - \left[4a_{1-1}x^1 + \sum_{n=2}^{\infty} 4a_{n-1}x^n\right] \\
&= a_1x^0 + [2a_2 - 4a_0]\,x^1 + \sum_{n=2}^{\infty}\left[a_{n-1}(n-1) + a_{n+1}(n+1) - 4a_{n-1}\right]x^n \\
&= a_1x^0 + [2a_2 - 4a_0]\,x^1 + \sum_{n=2}^{\infty}\left[(n+1)a_{n+1} + (n-5)a_{n-1}\right]x^n \quad .
\end{aligned}
$$

Remember, the coefficient in each term must be zero. From the x^0 *term, we get*

$$a_1 = 0 \quad .$$

From the x^1 *term, we get*

$$2a_2 - 4a_0 = 0 \quad .$$

And for $n \geq 2$, *we have*

$$(n+1)a_{n+1} + (n-5)a_{n-1} = 0 \quad .$$

Solving each of the above for the a_k *with the highest index, we get*

$$
\begin{aligned}
a_1 &= 0 \quad , \\
a_2 &= 2a_0
\end{aligned}
$$

and

$$a_{n+1} = \frac{5-n}{n+1}a_{n-1} \qquad \text{for} \quad n = 2, 3, 4, \ldots \quad .$$

Letting $k = n + 1$ *then converts the last equation to the recursion formula*

$$a_k = \frac{6-k}{k} a_{k-2} \qquad \text{for} \quad k = 3, 4, 5, \ldots \quad .$$

Now, using our recursion formula, we see that

$$\begin{aligned}
a_3 &= \frac{6-3}{3} a_{3-2} = \frac{3}{3} a_1 = \frac{1}{2} \cdot 0 = 0 \quad , \\
a_4 &= \frac{6-4}{4} a_{4-2} = \frac{2}{4} a_2 = \frac{1}{2} \cdot 2a_0 = a_0 \quad , \\
a_5 &= \frac{6-5}{5} a_{5-2} = \frac{1}{5} a_3 = \frac{1}{5} \cdot 0 = 0 \quad , \\
a_6 &= \frac{6-6}{6} a_{6-2} = \frac{0}{6} a_4 = 0 \quad , \\
a_7 &= \frac{6-7}{7} a_{7-2} = -\frac{1}{7} a_5 = -\frac{1}{7} \cdot 0 = 0 \quad , \\
a_8 &= \frac{6-8}{8} a_{8-2} - \frac{2}{8} a_6 = -\frac{1}{4} \cdot 0 = 0
\end{aligned}$$

Clearly, the vanishing of both a_5 *and* a_6 *means that the recursion formula will give us*

$$a_k = 0 \qquad \text{whenever} \quad k > 4 \quad .$$

Thus,

$$\begin{aligned}
y(x) &= \sum_{k=0}^{\infty} a_k x^k \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots \\
&= a_0 + 0x + 2a_0 x^2 + 0x^3 + a_0 x^4 + 0x^5 + 0x^6 + 0x^7 + \cdots \\
&= a_0 \qquad + 2a_0 x^2 \qquad + a_0 x^4 \quad .
\end{aligned}$$

That is, the power series for y *reduces to the polynomial*

$$y(x) = a_0 \left[1 + 2x^2 + x^4\right] \quad .$$

If $x_0 \neq 0$

The computations in our procedure (and the others we'll develop) tend to get a little messier when $x_0 \neq 0$, and greater care needs to be taken. In particular, before you "multiply things out" in step 2, you should rewrite your polynomials $A(x)$ and $B(x)$ in terms of $(x - x_0)$ instead of x to better match the terms in the series. For example, if

$$A(x) = x^2 + 2 \qquad \text{and} \qquad x_0 = 1 \quad ,$$

then rewrite $A(x)$ as follows:

$$\begin{aligned}
A(x) &= [(x-1)+1]^2 + 2 \\
&= \left[(x-1)^2 + 2(x-1) + 1\right] + 2 = (x-1)^2 + 2(x-1) + 3 \quad .
\end{aligned}$$

Alternatively (and probably better), just convert the differential equation

$$A(x)y' + B(x)y = 0 \tag{31.8a}$$

using the change of variables $X = x - x_0$. That is, first set

$$Y(X) = y(x) \qquad \text{with} \quad X = x - x_0$$

and then rewrite the differential equation for $y(x)$ in terms of Y and X. After noting that $x = X + x_0$ and that (via the chain rule)

$$y'(x) = \frac{d}{dx}[y(x)] = \frac{d}{dx}[Y(X)] = \frac{dY}{dX}\frac{dX}{dx} = \frac{dY}{dX}\frac{d}{dx}[x-2] = \frac{dY}{dX} = Y'(X) \quad ,$$

we see that this converted differential equation is simply

$$A(X + x_0)Y' + B(X + x_0)Y = 0 \quad . \tag{31.8b}$$

Consequently, if we can find a general power series solution

$$Y(X) = \sum_{k=0}^{\infty} a_k X^k$$

to the converted differential equation (equation (31.8b)), then we can generate the corresponding general power series to the original equation (equation (31.8a)) by rewriting X in terms of x,

$$y(x) = Y(X) = Y(x - x_0) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad .$$

!▶*Example 31.2:* *Consider the problem of finding the power series solution about $x_0 = 3$ for*

$$\left(x^2 - 6x + 10\right) y' + (12 - 4x)y = 0 \quad .$$

Proceeding as suggested, we let

$$Y(X) = y(x) \qquad \textit{with} \quad X = x - 3 \quad .$$

Then $x = X + 3$, and

$$\left(x^2 - 6x + 10\right) y' + (12 - 4x)y = 0$$

$$\hookrightarrow \qquad \left([X + 3]^2 - 6[X + 3] + 10\right) Y' + (12 - 4[X + 3])Y = 0$$

After a bit of simple algebra, this last equation simplifies to

$$\left(X^2 + 1\right) Y' - 4XY = 0 \quad ,$$

which, by an amazing stroke of luck, is the differential equation we just dealt with in example 31.1 (only now written using capital letters). From that example, we know

$$Y(X) = a_0\left[1 + 2X^2 + X^4\right] \quad .$$

Thus,

$$y(x) = Y(X) = Y(x - 5) = a_0\left[1 + 2(x - 3)^2 + (x - 3)^4\right] \quad .$$

Initial-Value Problems (and Finding Patterns, Again)

The method just described yields a power series solution

$$y(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

in which a_0 is an arbitrary constant. Remember,

$$y(x_0) = a_0 \quad .$$

So the general series solution $y(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ obtained by the algebraic method for

$$A(x)y' + B(x)y = 0$$

becomes the solution to the initial-value problem

$$A(x)y' + B(x)y = 0 \qquad \text{with} \quad y(x_0) = y_0$$

if we simply replace the arbitrary constant a_0 with the value y_0.

Along these lines, it is worth recalling that we are dealing with first-order, homogeneous linear differential equations, and that the general solution to any such equation can be given as an arbitrary constant times any nontrivial solution. In particular, we can write the general solution y to any given first-order, homogeneous linear differential equation as

$$y(x) = a_0 y_1(x)$$

where a_0 is an arbitrary constant and y_1 is the particular solution satisfying the initial condition $y(x_0) = 1$. So if our solutions can be written as power series about x_0, then there is a particular power series solution

$$y_1(x) = \sum_{k=0}^{\infty} \alpha_k(x-x_0)^k$$

where $\alpha_0 = y_1(x_0) = 1$ and the other α_k's are fixed numbers (hopefully given by some simple formula of k). It then follows that the general solution y is given by

$$y(x) = a_0 y_1(x_0) = a_0 \sum_{k=0}^{\infty} \alpha_k(x-x_0)^k = \sum_{k=0}^{\infty} a_k(x-x_0)^k$$

where

$$a_k = \alpha_k a_0 \qquad \text{for} \quad k = 0,\ 1,\ 2,\ 3,\ \ldots \quad ,$$

just as was claimed a few pages ago when we discussed "finding patterns". (This also confirms that we will always be able to factor out the a_0 in our series solutions.)

One consequence of these observations is that, instead of assuming a solution of the form

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k \qquad \text{with} \quad a_0 \text{ arbitrary}$$

in the first step of our method, we could assume a solution of the form

$$\sum_{k=0}^{\infty} \alpha_k(x-x_0)^k \qquad \text{with} \quad \alpha_0 = 1 \quad ,$$

and then just multiply the series obtained by an arbitrary constant a_0. In practice, though, this approach is no simpler than that already outlined in the steps of our algebraic method.

31.3 Validity of of the Algebraic Method for First-Order Equations

Our algebraic method will certainly lead to a general solution of the form

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with } a_0 \text{ arbitrary} \qquad ,$$

provided such a general solution exists. But what assurance do we have that that such solutions exist? And what about the radius of convergence? What good is a formula for a solution if we don't know the interval over which that formula is valid? And while we are asking these sorts of question, why do we insist that $A(x_0) \neq 0$ in pre-step 2?

Let's see if we can at least partially answer these questions.

Non-Existence of Power Series Solutions

Let a and b be functions on an interval (α, β) containing some point x_0, and let y be any function on that interval satisfying

$$a(x)y' + b(x)y = 0 \qquad \text{with} \quad y(x_0) \neq 0 \quad .$$

For the moment, assume this differential equation has a general power series solution about x_0 valid on (α, β). This means there are finite numbers a_0, a_1, a_2, ... such that

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{for} \quad \alpha < x < \beta \quad .$$

In particular, there are finite numbers a_0 and a_1 with

$$y(x_0) = a_0 \neq 0 \qquad \text{and} \qquad y'(x_0) = a_1 \quad .$$

Also observe that we can algebraically solve our differential equation for y', obtaining

$$y'(x) = -\frac{b(x)}{a(x)} y(x) \quad .$$

Thus,

$$a_1 = y(x_0) = -\frac{b(x_0)}{a(x_0)} y(x_0) = -\frac{b(x_0)}{a(x_0)} a_0 \quad , \tag{31.9}$$

provided the above fraction is a finite number — which will certainly be the case if $a(x)$ and $b(x)$ are polynomials with $a(x_0) \neq 0$.

More generally, the fraction in equation (31.9) might be indeterminant. To get around this minor issue, we'll take limits:

$$\begin{aligned} a_1 = y'(x_0) = \lim_{x \to x_0} y'(x) &= \lim_{x \to x_0} \left[-\frac{b(x)}{a(x)} y(x) \right] \\ &= -\lim_{x \to x_0} \left[\frac{b(x)}{a(x)} \right] y(x_0) = -\lim_{x \to x_0} \left[\frac{b(x)}{a(x)} \right] a_0 \quad . \end{aligned}$$

Solving for the limit, we then have

$$\lim_{x\to x_0} \frac{b(x)}{a(x)} = -\frac{a_1}{a_0} \quad .$$

This means the above limit must exist and be a well-defined finite number *whenever* the solution y can be given by the above power series. And if you think about what this means when the above limit does *not* exist as a finite number, you get:

Lemma 31.1 (nonexistence of a power series solution)
Let a and b be two functions on some interval containing a point x_0. If

$$\lim_{x\to x_0} \frac{b(x)}{a(x)}$$

does not exist as a finite number, then

$$a(x)y' + b(x)y = 0$$

does not have a general power series solution about x_0 with arbitrary constant term.

!►*Example 31.3:* *Consider the differential equation*

$$(x-2)y' + 2y = 0 \quad .$$

This equation is

$$a(x)y' + b(x)y = 0$$

with

$$a(x) = (x-2) \qquad \text{and} \qquad b(x) = 2 \quad .$$

Note that these are polynomials without common factors but with $a(2) = 0$. Consequently,

$$\lim_{x\to 2} \frac{b(x)}{a(x)} = \lim_{x\to 2} \frac{2}{(x-2)} = \frac{2}{0} \quad ,$$

which is certainly not a finite number. Lemma 31.1 then tells us to not bother looking for a solution of the form

$$y(x) = \sum_{k=0}^{\infty} a_k (x-2)^k \qquad \text{with } a_0 \text{ arbitrary} \qquad .$$

No such solution exists.

Singular and Ordinary Points, and the Radius of Analyticity

Because of the 'singular' behavior just noted, we refer to any point z_0 for which

$$\lim_{z\to z_0} \frac{b(z)}{a(z)}$$

is *not* a well-defined finite number as a *singular point* for the differential equation

$$a(x)y' + b(x)y = 0 \quad .$$

Note that we used "z" in this definition, suggesting that we may be considering points on the complex plane as well. This can certainly be the case when a and b are rational functions. And if a and b are rational functions, then the nonsingular points (i.e., the points that are not singular points) are traditionally referred to as *ordinary points* for the above differential equation.

A related concept is that of the *radius of analyticity* (for the above differential equation) about any given point z_0. This is the distance between z_0 and the singular point z_s closest to z_0, provided the differential equation has at least one singular point. If the equation has no singular points, then we define the equation's *radius of analyticity* (about z_0) to be $+\infty$. For this definition to make sense, of course, we need to be able to view the functions a and b as functions on the complex plane, as well as functions on a line. Again, this is certainly the case when a and b are rational functions.

Validity of the Algebraic Method

We just saw that our algebraic method for finding power series solutions about x_0 will fail if x_0 is a singular point. On the other hand, there is a theorem assuring us that the method will succeed when x_0 is an ordinary point for our differential equation, and even giving us a good idea of the interval over which the general power series solution is valid. Here is that theorem:

Theorem 31.2 (existence of power series solutions)
Let x_0 be an ordinary point on the real line for

$$a(x)y' + b(x)y = 0$$

where a and b are rational functions. Then this differential equation has a general power series solution

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

with a_0 being the arbitrary constant. Moreover, this solution is valid at least on the interval $(x_0 - R, x_0 + R)$ where R it the radius of analyticity about x_0 for the differential equation.

The proof of this theorem requires a good deal more work than did our derivation of the previous lemma. We will save that labor for the next chapter.

Identifying Singular and Ordinary Points

The basic approach to identifying a point z_0 as being either a singular or ordinary point for

$$a(x)y' + b(x)y = 0$$

is to look at the limit

$$\lim_{z \to z_0} \frac{b(z)}{a(z)} \quad .$$

If the limit is a finite number, x_0 is an ordinary point; otherwise x_0 is a singular point. And if you think about how this limit is determined by the values of $a(z)$ and $b(z)$ as $z \to z_0$, you'll derive the shortcuts listed in the next lemma.

Lemma 31.3
Let z_0 be a point in the complex plane, and consider the differential equation

$$a(x)y' + b(x)y = 0$$

where a and b are rational functions. Then

1. *If $a(z_0)$ and $b(z_0)$ are both finite numbers with $a(z_0) \neq 0$, then z_0 is an ordinary point for the differential equation.*
2. *If $a(z_0)$ and $b(z_0)$ both finite numbers with $a(z_0) = 0$ and $b(z_0) \neq 0$, then z_0 is a singular point for the differential equation.*
3. *If $a(z_0)$ is a finite nonzero number, and*
$$\lim_{z \to z_0} |b(z)| = \infty \quad ,$$
then z_0 is a singular point for the differential equation.
4. *If $b(z_0)$ is a finite number, and*
$$\lim_{z \to z_0} |a(z)| = \infty \quad ,$$
then z_0 is an ordinary point for the differential equation.

As actually illustrated in example 31.3, applying the above to the differential equations of interest here, rewritten in the form recommended for the algebraic method, yields the following corollary.

Corollary 31.4
Let $A(x)$ and $B(x)$ be polynomials having no factors in common. Then a point z_0 on the complex plane is a singular point for

$$A(x)y' + B(x)y = 0$$

if and only if $A(z_0) = 0$.

!► ***Example 31.4:*** *To first illustrate the algebraic method, we used*

$$y' + \frac{2}{x-2}y = 0 \quad ,$$

which we rewrote as

$$(x-2)y' + 2y = 0 \quad .$$

Now

$$A(z_s) = z_s - 2 = 0 \quad \Longleftrightarrow \quad z_s = 2 \quad .$$

So this differential equation has just one singular point, $z_s = 2$. Any $x_0 \neq 2$ is then an ordinary point for the differential equation, and the corresponding radius of analyticity is

$$R_{x_0} = \text{distance from } x_0 \text{ to } z_s = |z_s - x_0| = |2 - x_0| \quad .$$

Theorem 31.2 then assures us that, about any $x_0 \neq 2$, the general solution to our differential equation has a power series formula, and its radius of convergence is at least equal to $|2 - x_0|$. In particular, the power series we found,

$$y = a_0 \sum_{k=0}^{\infty} \frac{k+1}{2^k} x^k \quad ,$$

is centered at $x_0 = 0$. So the corresponding radius of analyticity is

$$R = |2 - 0| = 2$$

and our theorems assure us that our series solution is valid at least on the interval $(x_0 - R, x_0 + R) = (-2, 2)$.

In this regard, let us note the following:

1. *If $|x| \geq 2$, then the terms of our power series solution*

$$y = a_0 \sum_{k=0}^{\infty} \frac{k+1}{2^k} x^k = a_0 \sum_{k=0}^{\infty} (k+1) \left(\frac{x}{2}\right)^k \quad ,$$

   clearly increase in magnitude as k increases. Hence, this series diverges whenever $|x| \geq 2$. So, in fact, the radius of convergence is 2, and our power series solution is only valid on $(-2, 2)$.

2. *As we observed on page 31–8, the above power series is the power series about x_0 for*

$$y(x) = \frac{4a_0}{(2 - x)^2} \quad .$$

   But you can easily verify that this simple formula gives us a valid general solution to our differential equation on any interval not containing the singular point $x = 2$, not just $(-2, 2)$.

The last example shows that a power series for a solution may be valid over a smaller interval than the interval of validity for another formula for that solution. Of course, finding that more general formula may not be so easy, especially after we start dealing with higher-order differential equations.

The next example illustrates something slightly different; namely that the radius of convergence for a power series solution can, sometimes, be much larger than the corresponding radius of analyticity for the differential equation.

!►*Example 31.5:* *In example 31.2, we considered the problem of finding the power series solution about $x_0 = 3$ for*

$$\left(x^2 - 6x + 10\right) y' + (12 - 4x)y = 0 \quad .$$

Any singular point z for this differential equation is given by

$$z^2 - 6z + 10 = 0 \quad .$$

Using the quadratic formula, we see that we have two singular points z_+ *and* z_- *given by*

$$z_\pm = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 10}}{2} = 3 \pm 1i \quad .$$

The radius of analyticity about $x_0 = 3$ *for our differential equation is the distance between each of these singular points and* $x_0 = 3$ *,*

$$|z_\pm - x_0| = |[3 \pm 1i] - 3| = |\pm i| = 1 \quad .$$

So the radius of convergence for our series is at least 1 *, which means that our series solution is valid on at least the interval*

$$(x_0 - R, x_0 + R) = (3 - 1, 3 + 1) = (2, 4) \quad .$$

Recall, however, that example 31.2 demonstrated the possibility of a "terminating series", and that our series solution to the above differential equation actually ended up being the polynomial

$$y(x) = a_0 \left[1 + 2x^2 + x^4 \right] \quad ,$$

which is easily verified to be a valid solution on the entire real line $(-\infty, \infty)$ *, not just* $(2, 4)$ *.*

31.4 The Algebraic Method with Second-Order Equations

Extending the algebraic method to deal with second-order differential equations is straightforward. The only real complication (aside from the extra computations required) comes from the fact that our solutions will now involve two arbitrary constants instead of one, and that complication won't be particularly troublesome.

Details of the Method

Our goal, now, is to find a general power series solution to

$$a(x)y'' + b(x)y' + c(x)u = 0$$

assuming $a(x)$, $b(x)$ and $c(x)$ are rational functions. As hinted above, the procedure given here is very similar to that given in the previous section. Because of this, some of the steps will not be given in the same detail as before.

To illustrate the method, we will find a power series solution to

$$y'' - xy = 0 \quad . \tag{31.10}$$

This happens to be *Airy's equation*. It is a famous equation and can*not* be easily solved by any method we've discussed earlier in this text.

Again, we have two preliminary steps:

Pre-step 1: Get the differential equation into the form

$$A(x)y'' + B(x)y' + C(x)y = 0$$

where $A(x)$, $B(x)$ and $C(x)$ are polynomials, preferably with no factors shared by all three.

Our example is already in the desired form.

Pre-step 2: If not already specified, choose a value for x_0 such that that $A(x_0) \neq 0$. If initial conditions are given for $y(x)$ at some point, then use that point for x_0 (provided $A(x_0) \neq 0$). Otherwise, choose x_0 as convenient — which usually means choosing $x_0 = 0$.[6]

For our example, we have no initial values at any point, so we choose x_0 as simply as possible; namely, $x_0 = 0$.

Now for the basic method:

Step 1: Assume

$$y = y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{31.11}$$

with a_0 and a_1 being arbitrary and the other a_k's "to be determined", and then compute/write out the corresponding series for the first two derivatives,

$$y' = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k (x - x_0)^k\right] = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}$$

and

$$y'' = \sum_{k=1}^{\infty} \frac{d}{dx}\left[k a_k (x - x_0)^{k-1}\right] = \sum_{k=2}^{\infty} k(k-1) a_k (x - x_0)^{k-2} \quad .$$

Step 2: Plug these series for y, y', and y'' back into the differential equation and "multiply things out" to get zero equalling the sum of a few power series about x_0.

Step 3: For each series in your last equation, do a change of index so that each series looks like

$$\sum_{n=\text{something}}^{\infty} \left[\text{something not involving } x\right](x - x_0)^n \quad .$$

Step 4: Convert the sum of series in your last equation into one big series. The first few terms may have to be written separately. Simplify what can be simplified.

Since we've already decided $x_0 = 0$ in our example, we let

$$y = y(x) = \sum_{k=0}^{\infty} a_k x^k \quad , \tag{31.12}$$

[6] Again, the requirement that $A(x_0) \neq 0$ is a simplification of requirements we'll develop in the next section. But "$A(x_0) \neq 0$" will suffice for now, especially if A, B and C are polynomials with no factors shared by all three.

and "compute"

$$y' = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^k\right] = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y'' = \sum_{k=1}^{\infty} \frac{d}{dx}\left[k a_k x^{k-1}\right] = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \quad .$$

Plugging these into the given differential equation and carrying out the other steps stated above then yields the following sequence of equalities:

$$\begin{aligned}
0 &= y'' - xy \\
&= \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - x \sum_{k=0}^{\infty} a_k x^k \\
&= \underbrace{\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}}_{n\,=\,k-2} + \underbrace{\sum_{k=0}^{\infty} (-1)a_k x^{k+1}}_{n\,=\,k+1} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} (-1)a_{n-1} x^n \\
&= (0+2)(0+1)a_{0+2}x^0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} (-1)a_{n-1} x^n \\
&= 2a_2 x^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}] x^n \quad .
\end{aligned}$$

Step 5: At this point, you will have an equation of the basic form

$$\sum_{n=0}^{\infty} \left[n^{\text{th}} \text{ formula of the } a_k\text{'s}\right](x - x_0)^n = 0 \quad .$$

Now:

(a) Solve

$$n^{\text{th}} \text{ formula of the } a_k\text{'s} = 0 \qquad \text{for} \quad n = 0,\ 1,\ 2,\ 3,\ 4,\ \ldots \quad .$$

for the a_k with the highest index,

$$a_{\text{highest index}} = \text{formula of } n \text{ and lower-indexed coefficients} \quad .$$

Again, a few of these equations may need to be treated separately, but you will also obtain a relatively simple formula that holds for all indices above some fixed value. This is a *recursion formula* for computing each coefficient from previously computed coefficients.

(b) Using another change of indices, rewrite the recursion formula just derived so that it looks like

$$a_k = \text{formula of } k \text{ and lower-indexed coefficients} \quad .$$

From the previous step in our example, we have

$$2a_2x^0 \ + \ \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - a_{n-1}]x^n \ = \ 0 \quad .$$

So

$$2a_2 \ = \ 0 \quad ,$$

and, for $n = 1,\ 2,\ 3,\ 4,\ \dots,$

$$(n+2)(n+1)a_{n+2} \ - \ a_{n-1} \ = \ 0 \quad .$$

The first tells us that

$$a_2 \ = \ 0 \quad .$$

Solving the second for a_{n+2} *yields the recursion formula*

$$a_{n+2} \ = \ \frac{1}{(n+2)(n+1)}a_{n-1} \qquad \text{for} \quad n = 1,\ 2,\ 3,\ 4,\ \dots \quad .$$

Letting $k = n+2$ *(equivalently,* $n = k-2$ *), this becomes*

$$a_k \ = \ \frac{1}{k(k-1)}a_{k-3} \qquad \text{for} \quad k = 3,\ 4,\ 5,\ 6,\ \dots \quad . \tag{31.13}$$

This is the recursion formula we will use.

Step 6: Use the recursion formula (and any corresponding formulas for the lower-order terms) to find all the a_k's in terms of a_0 and a_1. Look for patterns!

We already saw that

$$a_2 \ = \ 0 \quad .$$

Using this and recursion formula (31.13) with $k = 3,\ 4,\ \dots$ *(and looking for patterns), we see that*

$$a_3 \ = \ \frac{1}{3(3-1)}a_{3-3} \ = \ \frac{1}{3\cdot 2}a_0 \quad ,$$

$$a_4 \ = \ \frac{1}{4(4-1)}a_{4-3} \ = \ \frac{1}{4\cdot 3}a_1 \quad ,$$

$$a_5 \ = \ \frac{1}{5(5-1)}a_{5-3} \ = \ \frac{1}{5\cdot 4}a_2 \ = \ \frac{1}{5\cdot 4}\cdot 0 \ = \ 0 \quad ,$$

$$a_6 \ = \ \frac{1}{6(6-1)}a_{6-3} \ = \ \frac{1}{6\cdot 5}a_3 \ = \ \frac{1}{6\cdot 5}\cdot\frac{1}{3\cdot 2}a_0 \quad ,$$

$$a_7 \ = \ \frac{1}{7(7-1)}a_{7-3} \ = \ \frac{1}{7\cdot 6}a_4 \ = \ \frac{1}{7\cdot 6}\cdot\frac{1}{4\cdot 3}a_1 \quad ,$$

$$a_8 \ = \ \frac{1}{8(8-1)}a_{8-3} \ = \ \frac{1}{8\cdot 7}a_5 \ = \ \frac{1}{8\cdot 7}\cdot\frac{1}{5\cdot 4}\cdot 0 \ = \ 0 \quad ,$$

$$a_9 \ = \ \frac{1}{9(9-1)}a_{9-3} \ = \ \frac{1}{9\cdot 8}a_6 \ = \ \frac{1}{9\cdot 8}\cdot\frac{1}{6\cdot 5}\cdot\frac{1}{3\cdot 2}a_0 \quad ,$$

$$a_{10} = \frac{1}{10(10-1)} a_{10-3} = \frac{1}{10 \cdot 9} a_7 = \frac{1}{10 \cdot 9} \cdot \frac{1}{7 \cdot 6} \cdot \frac{1}{4 \cdot 3} a_1 \quad ,$$

$$\vdots$$

There are three patterns here. The simplest is

$$a_k = 0 \qquad \text{when} \quad k = 2,\ 5,\ 8,\ 11,\ \ldots$$

The other two are more difficult to describe. Look carefully and you'll see that the denominators are basically $k!$ with every third factor removed. If $k = 3$, 6, 9, $\ldots$, then

$$a_k = \frac{1}{(2 \cdot 3)(5 \cdot 6)(8 \cdot 9) \cdots ([k-1] \cdot k)} a_0 \quad .$$

If $k = 4$, 7, 10, $\ldots$, then

$$a_k = \frac{1}{(3 \cdot 4)(6 \cdot 7)(9 \cdot 10) \cdots ([k-1] \cdot k)} a_1 \quad .$$

Let us observe that we can use the change of indices $k = 3n$ and $k = 3n+1$ to rewrite the last two expressions as

$$a_{3n} = \frac{1}{(2 \cdot 3)(5 \cdot 6) \cdots ([3n-1] \cdot 3n)} a_0 \qquad \text{for} \quad n = 1,\ 2,\ 3,\ \ldots$$

and

$$a_{3n+1} = \frac{1}{(3 \cdot 4)(6 \cdot 7) \cdots (3n \cdot [3n+1])} a_1 \qquad \text{for} \quad n = 1,\ 2,\ 3,\ \ldots \quad .$$

Step 7: Using the formulas just derived for the coefficients, write out the resulting series for $y(x)$. Try to simplify it to a linear combination of two power series, $y_1(x)$ and $y_2(x)$, with $y_1(x)$ multiplied by a_0 and $y_2(x)$ multiplied by a_1.

Plugging the formulas just derived for the a_k's into the power series assumed for y, we get

$$\begin{aligned}
y(x) &= \sum_{k=0}^{\infty} a_k x^k \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\
&= \left[a_0 + a_3 x^3 + a_6 x^6 + a_9 x^9 + \cdots\right] \\
&\quad + \left[a_1 + a_4 x^4 + a_7 x^7 + a_{10} x^{10} + \cdots\right] \\
&\quad + \left[a_2 + a_5 x^5 + a_8 x^8 + a_{11} x^{11} + \cdots\right] \\
&= \left[a_0 + a_3 x^3 + a_6 x^6 + \cdots + a_{3n} x^{3n} + \cdots\right] \\
&\quad + \left[a_1 + a_4 x^4 + a_7 x^7 + \cdots + a_{3n+1} x^{3n+1} + \cdots\right] \\
&\quad + \left[a_2 + a_5 x^5 + a_8 x^8 + \cdots + a_{3n+2} x^{3n+2} + \cdots\right]
\end{aligned}$$

$$= \left[a_0 + \frac{1}{3 \cdot 2} a_0 x^3 + \cdots + \frac{1}{(2 \cdot 3)(5 \cdot 6) \cdots ([3n-1] \cdot 3n)} a_0 x^{3n} + \cdots \right]$$

$$+ \left[a_1 x + \frac{1}{4 \cdot 3} a_1 x^4 + \cdots + \frac{1}{(3 \cdot 4)(6 \cdot 7) \cdots (3n \cdot [3n+1])} a_1 x^{3n+1} + \cdots \right]$$

$$+ \left[0 + 0x^5 + 0x^8 + 0x^{11} + \cdots \right]$$

$$= a_0 \left[1 + \frac{1}{3 \cdot 2} x^3 + \cdots + \frac{1}{(2 \cdot 3)(5 \cdot 6) \cdots ([3n-1] \cdot 3n)} x^{3n} + \cdots \right]$$

$$+ a_1 \left[x + \frac{1}{4 \cdot 3} x^4 + \cdots + \frac{1}{(3 \cdot 4)(6 \cdot 7) \cdots (3n \cdot [3n+1])} x^{3n+1} + \cdots \right]$$

So,

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \tag{31.14a}$$

where

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{(2 \cdot 3)(5 \cdot 6) \cdots ([3n-1] \cdot 3n)} x^{3n} \tag{31.14b}$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{1}{(3 \cdot 4)(6 \cdot 7) \cdots (3n \cdot [3n+1])} x^{3n+1} \quad . \tag{31.14c}$$

Last Step: See if you recognize either of the series derived as the series for some well-known function (you probably won't!).

It is unlikely that you have ever seen the above series before. So we cannot rewrite our power series solutions more simply in terms of better-known functions.

Practical Advice on Using the Method

The advice given for using this method with first-order equations certainly applies when using this method for second-order equations. All that can be added is that even greater diligence is needed in the individual computations. Typically, you have to deal with more power series terms when solving second-order differential equations, and that, naturally, provides more opportunities for error. That also leads to a greater probability that you will not succeed in finding "nice" formulas for the coefficients, and may have to simply use the recursion formula to compute as many terms as you think necessary for a reasonably accurate partial sum approximation.

Initial-Value Problems (and Finding Patterns)

Observe that the solution obtained in our example (formula set 31.14) that can be written as

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are power series about $x_0 = 0$ with

$$y_1(x) = 1 + \text{a summation of terms of order } 2 \text{ or more}$$

and

$$y_2(x) = 1 \cdot (x - x_0) + \text{a summation of terms of order 2 or more} \quad .$$

In fact, we can derive this observation more generally after recalling that the general solution to an second-order, homogeneous linear differential equation is given by

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary constants, and y_1 and y_2 form a linearly independent pair of particular solutions to the given differential equation. In particular, we can take y_1 to be the solution satisfying initial conditions

$$y_1(x_0) = 1 \qquad \text{and} \qquad {y_1}'(x_0) = 0 \quad ,$$

while y_2 is the solution satisfying initial conditions

$$y_2(x_0) = 0 \qquad \text{and} \qquad {y_2}'(x_0) = 1 \quad .$$

If our solutions can be written as power series about x_0, then y_1 and y_2 can be written as particular power series

$$y_1(x_0) = \sum_{k=0}^{\infty} \alpha_k (x - x_0)^k \qquad \text{and} \qquad y_2(x_0) = \sum_{k=0}^{\infty} \beta_k (x - x_0)^k$$

where

$$\alpha_0 = y_1(x_0) = 1 \qquad \text{and} \qquad \alpha_1 = {y_1}'(x_0) = 0 \quad ,$$
$$\beta_0 = y_2(x_0) = 0 \qquad \text{and} \qquad \beta_1 = {y_2}'(x_0) = 1 \quad ,$$

and the other α_k's and β_k's are fixed numbers (hopefully given by relatively simple formulas of k). Thus,

$$\begin{aligned} y_1(x) &= \alpha_0 + \alpha_1 (x - x_0) + \alpha_2 (x - x_0)^2 + \alpha_3 (x - x_0)^3 + \cdots \\ &= 1 + 0 \cdot (x - x_0) + \alpha_2 (x - x_0)^2 + \alpha_3 (x - x_0)^3 + \cdots \\ &= 1 + \sum_{k=2}^{\infty} \alpha_k (x - x_0)^k \quad , \end{aligned}$$

while

$$\begin{aligned} y_2(x) &= \beta_0 + \beta_1 (x - x_0) + \beta_2 (x - x_0)^2 + \beta_3 (x - x_0)^3 + \cdots \\ &= 0 + 1 \cdot (x - x_0) + \beta_2 (x - x_0)^2 + \beta_3 (x - x_0)^3 + \cdots \\ &= 1 \cdot (x - x_0) + \sum_{k=2}^{\infty} \beta_k (x - x_0)^k \quad , \end{aligned}$$

verifying that the observation made at the start of this subsection holds in general.

With regard to initial-value problems, we should note that, with these power series for y_1 and y_2,

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

automatically satisfies the initial conditions

$$y(x_0) = a_0 \qquad \text{and} \qquad y'(x_0) = a_1$$

for any choice of constants a_0 and a_1

31.5 Validity of the Algebraic Method for Second-Order Equations

Let's start by defining "ordinary" and "singular" points.

Ordinary and Singular Points, and the Radius of Analyticity

Given a differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{31.15}$$

we classify any point z_0 as a *singular point* if either of the limits

$$\lim_{z \to z_0} \frac{b(z)}{a(z)} \qquad \text{or} \qquad \lim_{z \to z_0} \frac{c(z)}{a(z)}$$

fails to exist as a finite number. Note that (just as with our definition of singular points for first-order differential equations) we used "z" in this definition, indicating that we may be considering points on the complex plane as well. This can certainly be the case when a, b and c are rational functions. And if a, b and c are rational functions, then the nonsingular points (i.e., the points that are not singular points) are traditionally referred to as *ordinary points* for the above differential equation.

The *radius of analyticity* (for the above differential equation) about any given point z_0 is defined just as before: It is the distance between z_0 and the singular point z_s closest to z_0, provided the differential equation has at least one singular point. If the equation has no singular points, then we define the equation's *radius of analyticity* (about z_0) to be $+\infty$. For this definition to make sense, of course, we need to be able to view the functions a, b and c as functions on the complex plane, as well as functions on a line. Again, this is certainly the case when these functions are rational functions.

Nonexistence of Power Series Solutions

The above definitions are inspired by the same sort of computations as led to the analogous definitions for first-order differential equations in section 31.3. I'll leave those computations to you. In particular, rewriting differential equation (31.15) as

$$y''(x) = -\frac{b(x)}{a(x)}y'(x) - \frac{c(x)}{a(x)}y(x) \quad ,$$

and using the relations between the values $y(x_0)$, $y'(x_0)$ and $y''(x_0)$, and the first three coefficients in

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad ,$$

you should be able to prove the second-order analog to lemma 31.1:

Lemma 31.5 (nonexistence of a power series solution)
If x_0 is a singular point for

$$a(x)y'' + b(x)y' + c(x)y = 0 \quad ,$$

then this differential equation does not have a power series solution $y(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ *with* a_0 *and* a_1 *being arbitrary constants.*

?►*Exercise 31.1:* *Verify lemma 31.5.*

(By the way, a differential equation might have a "modified" power series solution about a singular point. We start examining this possibility in chapter 33.)

Validity of the Algebraic Method

Once again, we have a lemma telling us that our algebraic method for finding power series solutions about x_0 will fail if x_0 is a singular point (only now we are considering second-order equations). And, unsurprisingly, we also have a second-order analog of theorem 31.6 assuring us that the method will succeed when x_0 is an ordinary point for our second-order differential equation, and even giving us a good idea of the interval over which the general power series solution is valid. That theorem is:

Theorem 31.6 (existence of power series solutions)
Let x_0 *be an ordinary point on the real line for*

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where a, b *and* c *are rational functions. Then this differential equation has a general power series solution*

$$y(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$$

with a_0 *and* a_1 *being the arbitrary constants. Moreover, this solution is valid at least on the interval* $(x_0 - R, x_0 + R)$ *where* R *it the radius of analyticity about* x_0 *for the differential equation.*

And again, we will wait until the next chapter to prove this theorem (or a slightly more general version of this theorem).

Identifying Singular and Ordinary Points

The basic approach to identifying a point z_0 as being either a singular or ordinary point for

$$a(x)y'' + b(x)y' + c(x)y = 0$$

is to look at the limits

$$\lim_{z \to z_0} \frac{b(z)}{a(z)} \qquad \text{and} \qquad \lim_{z \to z_0} \frac{c(z)}{a(z)} \quad .$$

If the limits are both a finite numbers, x_0 is an ordinary point; otherwise x_0 is a singular point. And if you think about how these limits are determined by the values of $a(z)$ and $b(z)$ as $z \to z_0$, you'll derive the shortcuts listed in the next lemma.

Lemma 31.7 (tests for ordinary/singular points)
Let z_0 be a point on the complex plane, and consider a differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

in which a, b and c are all rational functions. Then:

1. *If $a(z_0)$, $b(z_0)$ and $c(z_0)$ are all finite values with $a(z_0) \neq 0$, then z_0 is an ordinary point for the differential equation.*
2. *If $a(z_0)$, $b(z_0)$ and $c(z_0)$ are all finite values with $a(z_0) = 0$, and either $b(z_0) \neq 0$ or $c(z_0) \neq 0$, then z_0 is a singular point for the differential equation.*
3. *If $a(z_0)$ is a finite value but either*
$$\lim_{z \to z_0} |b(z)| = \infty \qquad \text{or} \qquad \lim_{z \to z_0} |c(z)| = \infty \quad ,$$
then z_0 is a singular point for the differential equation.
4. *If $b(z_0)$ and $c(z_0)$ are finite numbers, and*
$$\lim_{z \to z_0} |a(z)| = \infty \quad ,$$
then z_0 is an ordinary point for the differential equation.

Again, applying the above to the corresponding differential equation rewritten in the form recommended in the first step of our method, we get:

Corollary 31.8
Let A, B, and C be polynomials with no factors shared by all three. Then a point z_0 on the complex plane is a singular point for

$$A(x)y'' + B(x)y' + C(x)y = 0$$

if and only if $A(z_0) = 0$.

!►*Example 31.6:* *The coefficients in Airy's equation*

$$y'' - xy = 0$$

are polynomials with the first coefficient being

$$A(x) = 1 \quad .$$

Since there is no z_s in the complex plane such that $A(z_s) = 0$, Airy's equation has no singular point, and theorem 31.6 assures us that the power series solution we obtained in solving Airy's equation (formula set (31.14) on page 31–24) is valid for all x.

31.6 The Taylor Series Method

The Basic Idea (Expanded)

In this approach to finding power series solutions, we compute the terms in the Taylor series for the solution,

$$y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!}(x - x_0)^k \quad .$$

in a manner reminiscent of the way you probably computed Taylor series in elementary calculus. Unfortunately, this approach often fails to yield a useful general formula for the coefficients of the power series solution (unless the original differential equation is very simple). Consequently, you typically end up with a partial sum of the power series solutions consisting of however many terms you've had the time or desire to compute. But there are two big advantages to this method over the general basic method:

1. The computation of the individual coefficients of the power series solution is a little more direct and may require a little less work than when using the algebraic method, at least for the first few terms (provided you are proficient with product and chain rules of differentiation).

2. The method can be used on a much more general class of differential equations than described so far. In fact, it can be used to formally find the Taylor series solution for any differential equation that can be rewritten in the form

$$y' = F_1(x, y) \qquad \text{or} \qquad y'' = F_2(x, y, y')$$

where F_1 and F_2 are known functions that are sufficiently differentiable with respect to all variables.

With regard to the last comment, observe that

$$a(x)y' + b(x)y = 0 \qquad \text{and} \qquad a(x)y'' + b(x)y' + c(x)y = 0$$

can be rewritten, respectively, as

$$y' = \underbrace{-\frac{b(x)}{a(x)}y}_{F_1(x,y)} \qquad \text{and} \qquad y'' = \underbrace{-\frac{b(x)}{a(x)}y' - \frac{c(x)}{a(x)}y}_{F_2(x,y,y')} \quad .$$

So this method can be used on the same differential equations we used the algebraic method on in the previous sections. Whether you would want to is a different matter.

The Steps in the Taylor Series Method

Here are the steps in our procedure for finding the Taylor series about a point x_0 for the solution to a fairly arbitrary first- or second-order differential equation. As an example, we will find the power series solution to

$$y'' + \cos(x)y = 0 \quad .$$