35

Validating the Method of Frobenius

Let us now focus on verifying the claims made in the big theorems of section 34.1: theorem 34.1 on the indicial equation, and theorem 34.2 on solutions about regular singular points.

We will begin our work in a rather obvious manner — by applying the basic Frobenius method to a generic differential equation with a regular singular point (after rewriting the equation in a "reduced form") and then closely looking at the results of these computations. This, along with a theorem on convergence that we'll discuss, will tell us precisely when the basic method succeeds and why it fails for certain cases. After that, we will derive the alternative solution formulas (formulas (34.3) and (34.4) in theorem 34.2 on page 34–2) and verify that they truly are valid solutions. Dealing with these later cases will be the challenging part.

35.1 Basic Assumptions and Symbology

Throughout this chapter, we are assuming that we have a second-order linear homogeneous differential equation having a point x_0 on the real line as a regular singular point, and having R as the Frobenius radius of convergence about x_0. For simplicity, we will further assume $x_0 = 0$, keeping in mind that corresponding results can be obtained when the regular singular point is nonzero by using the substitution $X = x - x_0$. Also, (after recalling the comments made on page 33–23 about solutions when $x < x_0$), let us agree that we can restrict ourselves to analyzing the possible solutions on the interval $(0, R)$.

As noted in theorem 33.4 on page 33–10, our differential equation can be written as

$$x^2\alpha(x)y'' + x\beta(x)y' + \gamma(x)y = 0 \quad , \tag{35.1a}$$

where α, β and γ are functions analytic at $x_0 = 0$ and with $\alpha(0) \neq 0$. The associated differential equation is then

$$\alpha(x)y'' + \beta(x)y' + \gamma(x)y = 0 \quad . \tag{35.1b}$$

Dividing through by α, we get the corresponding reduced forms for our original differential equation

$$x^2y'' + xP(x)y' + Q(x)y = 0 \quad , \tag{35.2a}$$

and for the associated differential equation

$$y'' + P(x)y' + Q(x)y = 0 \quad . \tag{35.2b}$$

In each of these equations,

$$P(x) = \frac{\beta(x)}{\alpha(x)} \qquad \text{and} \qquad Q(x) = \frac{\gamma(x)}{\alpha(x)} \quad .$$

Do observe that, because of the relation between equations (35.2a) and (35.2b), each equation will have the same nonzero singular points as the other. Hence, R is also the radius of analyticity for equation (35.2b). This means (see lemma 32.7 on page 32–7) that we can express P and Q as power series

$$P(x) = \sum_{k=0}^{\infty} p_k x^k \qquad \text{for} \quad |x| < R \tag{35.3a}$$

and

$$Q(x) = \sum_{k=0}^{\infty} q_k x^k \qquad \text{for} \quad |x| < R \quad . \tag{35.3b}$$

For the rest of this chapter, we will be doing computations involving the above p_k's and q_k's . Don't forget this. And don't forget the relation between P and Q , and the coefficients of the first version of our differential equation. In particular, we might as well note here that

$$p_0 = P(0) = \frac{\beta(0)}{\alpha(0)} \qquad \text{and} \qquad q_0 = Q(0) = \frac{\gamma(0)}{\alpha(0)} \quad .$$

Finally, throughout this chapter, we will let $\mathcal{L}$ be the linear differential operator

$$\mathcal{L}[y] = x^2 y'' + xP(x)y' + Q(x)y \quad ,$$

so that we can write the differential equation we wish to solve, equation (35.2a), in the very abbreviated form

$$\mathcal{L}[y] = 0 \quad .$$

This will make it easier to describe some of our computations.

35.2 The Indicial Equation and Basic Recursion Formula

Basic Derivations

First, let's see what we get from plugging the arbitrary modified power series

$$y(x) = x^r \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^{k+r}$$

into $\mathcal{L}$:

$$\begin{aligned}
\mathcal{L}[y] &= x^2 y'' + xP(x)y' + Q(x)y \\
&= x^2 \sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^{k+r-2} \\
&\quad + x \left(\sum_{k=0}^{\infty} p_k x^k \right) \left(\sum_{k=0}^{\infty} c_k (k+r) x^{k+r-1} \right) + \left(\sum_{k=0}^{\infty} q_k x^k \right) \left(\sum_{k=0}^{\infty} c_k x^{k+r} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} c_k(k+r)(k+r-1)x^{k+r} \\
&\qquad + \sum_{k=0}^{\infty}\sum_{j=0}^{k} c_j p_{k-j}(j+r)x^{k+r} \ + \ \sum_{k=0}^{\infty}\sum_{j=0}^{k} c_j q_{k-j}x^{k+r} \\
&= x^r \sum_{k=0}^{\infty}\left[c_k(k+r)(k+r-1) \ + \ \sum_{j=0}^{k} c_j\left[p_{k-j}(j+r)+q_{k-j}\right]\right]x^k \quad .
\end{aligned}
$$

That is,

$$
\mathcal{L}\left[x^r\sum_{k=0}^{\infty}c_k x^k\right] \ = \ x^r\sum_{k=0}^{\infty}L_k x^k
$$

where

$$
L_k \ = \ c_k(k+r)(k+r-1) \ + \ \sum_{j=0}^{k} c_j\left[p_{k-j}(j+r)+q_{k-j}\right] \quad .
$$

Let's now look at the individual L_k's.

For $k = 0$,

$$
\begin{aligned}
L_0 \ &= \ c_0(0+r)(0+r-1) \ + \ \sum_{j=0}^{0} c_j\left[p_{0-j}(j+r)+q_{0-j}\right] \\
&= \ c_0 r(r-1) \ + \ c_0\left[p_0 r + q_0\right] \\
&= \ c_0\big[r(r-1) + p_0 r + q_0\big] \quad .
\end{aligned}
$$

The expression in the last bracket will arise several more times in our computations. For convenience, we will let I be the corresponding polynomial function

$$
I(\rho) \ = \ \rho(\rho-1) \ + \ p_0\rho + q_0 \quad .
$$

Then

$$
L_0 \ = \ c_0 I(r) \quad .
$$

For $k > 0$,

$$
\begin{aligned}
L_k \ &= \ c_k(k+r)(k+r-1) \ + \ \sum_{j=0}^{k} c_j\left[p_{k-j}(j+r)+q_{k-j}\right] \\
&= \ c_k(k+r)(k+r-1) \ + \ \sum_{j=0}^{k-1} c_j\left[p_{k-j}(j+r)+q_{k-j}\right] \ + \ c_k\left[p_{k-k}(k+r)q_{k-k}\right] \\
&= \ c_k\big[\underbrace{(k+r)(k+r-1)+p_0(k+r)+q_0}_{I(k+r)\,!}\big] \ + \ \sum_{j=0}^{k-1} c_j\left[p_{k-j}(j+r)+q_{k-j}\right] \quad .
\end{aligned}
$$

We'll be repeating the above computations at least two more times in this chapter. To save time, let's summarize what we have.

Lemma 35.1

Let $\mathcal{L}$ *be the differential operator*

$$\mathcal{L}[y] \;=\; x^2 y'' \;+\; xP(x)y' \;+\; Q(x)y$$

where

$$P(x) \;=\; \sum_{k=0}^{\infty} p_k x^k \qquad \text{and} \qquad Q(x) \;=\; \sum_{k=0}^{\infty} q_k x^k \quad .$$

Then, for any modified power series $x^r \sum_k^{\infty} c_k x^k$ *,*

$$\mathcal{L}\left[x^r \sum_{k=0}^{\infty} c_k x^k\right] \;=\; x^r\left[c_0 I(r) \;+\; \sum_{k=1}^{\infty}\left(c_k I(k+r) \;+\; \sum_{j=0}^{k-1} c_j\left[p_{k-j}(j+r) + q_{k-j}\right]\right)x^k\right]$$

where

$$I(\rho) \;=\; \rho(\rho-1) \;+\; p_0\rho + q_0 \quad .$$

Our immediate interest is in finding a modified power series

$$y(x) \;=\; x^r \sum_{k=0}^{\infty} c_k x^k \qquad \text{with} \quad c_0 \neq 0$$

that satisfies our differential equation,

$$\mathcal{L}[y] \;=\; 0 \quad .$$

Applying the above lemma, we see that we must have

$$x^r\left[c_0 I(r) \;+\; \sum_{k=1}^{\infty}\left(c_k I(k+r) \;+\; \sum_{j=0}^{k-1} c_j\left[p_{k-j}(j+r) + q_{k-j}\right]\right)x^k\right] \;=\; 0 \quad ,$$

which means that each term in the above power series must be zero. That is,

$$I(r) \;=\; 0 \tag{35.4a}$$

and

$$c_k I(k+r) \;+\; \sum_{j=0}^{k-1} c_j\left[p_{k-j}(j+r) + q_{k-j}\right] \;=\; 0 \qquad \text{for} \quad k = 1,\,2,\,3,\,\ldots \quad . \tag{35.4b}$$

The Indicial Equation

The Equation and Its Solutions

You probably already recognized equation (35.4a) as the indicial equation from the basic method of Frobenius. In more explicit form, it's the polynomial equation

$$r(r-1) \;+\; p_0 r \;+\; q_0 \;=\; 0 \quad . \tag{35.5a}$$

Equivalently, we can write this equation as

$$r^2 \;+\; (p_0 - 1)r \;+\; q_0 \;=\; 0 \quad , \tag{35.5b}$$

or even

$$(r - r_1)(r - r_2) = 0 \tag{35.5c}$$

or

$$r^2 - (r_1 + r_2)r + r_1 r_2 = 0 \tag{35.5d}$$

where r_1 and r_2 are the solutions to the indicial equation,

$$r_1 = \frac{1 - p_0 + \sqrt{(p_0 - 1)^2 - 4q_0}}{2} \qquad \text{and} \qquad r_2 = \frac{1 - p_0 - \sqrt{(p_0 - 1)^2 - 4q_0}}{2} \quad .$$

This, of course, also means that we can write the formula for I in four different ways:

$$I(\rho) = \rho(\rho - 1) + p_0\rho + q_0 \quad , \tag{35.6a}$$

$$I(\rho) = \rho^2 + (p_0 - 1)\rho + q_0 \quad , \tag{35.6b}$$

$$I(\rho) = (\rho - r_1)(\rho - r_2) \tag{35.6c}$$

and

$$I(\rho) = \rho^2 - (r_1 + r_2)\rho + r_1 r_2 \quad . \tag{35.6d}$$

For the rest of this chapter, we will use whichever of the above formulas for I seems most convenient at the time. Also, r_1 and r_2 will always denote the two values given above. Do note that if both are real, then $r_1 \geq r_2$.

By the way, if you compare the second and last of the above formulas for $I(\rho)$, you'll see that

$$p_0 = 1 - (r_1 + r_2) \qquad \text{and} \qquad q_0 = r_1 r_2 \quad .$$

Later, we may find these observations useful.

Proof of Theorem 34.1

Recall that p_0 and q_0 are related to the coefficients in the equation we first started with,

$$x^2\alpha(x)y'' + x\beta(x)y' + \gamma(x)y = 0 \quad ,$$

via

$$p_0 = P(0) = \frac{\beta_0}{\alpha_0} \qquad \text{and} \qquad q_0 = Q(0) = \frac{\gamma_0}{\alpha_0}$$

where

$$\alpha_0 = \alpha(0) \quad , \quad \beta_0 = \beta(0) \quad \text{and} \quad \gamma_0 = \gamma(0) \quad .$$

Using these relations, we can rewrite the first version of the indicial equation (equation (35.5a)) as

$$r(r - 1) + \frac{\beta_0}{\alpha_0}r + \frac{\gamma_0}{\alpha_0} = 0 \quad ,$$

which, after multiplying through by α_0 is

$$\alpha_0 r(r - 1) + \beta_0 r + \gamma_0 = 0 \quad .$$

This, along with the formulas for r_1 and r_2 , completes the proof of theorem 34.1 on page 34–1.

Recursion Formulas

The Basic Recursion Formula

You probably also recognized that equation (35.4b) is, essentially, a recursion formula for any given value of r. Let us first rewrite it as

$$c_k I(k+r) = -\sum_{j=0}^{k-1} c_j \left[p_{k-j}(j+r) + q_{k-j}\right] \quad \text{for} \quad k = 1, 2, 3, \ldots \quad . \tag{35.7}$$

If

$$I(k+r) \neq 0 \quad \text{for} \quad k = 1, 2, 3, \ldots \quad ,$$

then we can solve the above for c_k, obtaining the generic recursion formula

$$c_k = \frac{-1}{I(k+r)} \sum_{j=0}^{k-1} c_j \left[p_{k-j}(j+r) + q_{k-j}\right] \quad \text{for} \quad k = 1, 2, 3, \ldots \quad . \tag{35.8}$$

It will be worth noting that p_0 and q_0 does not explicitly appear in this recursion formula except in the formula for $I(k+r)$.

More General Recursion Formulas and a Convergence Theorem

Later, we will have to deal with recursion formulas of the form

$$c_k = \frac{1}{I(k+r)} \left(f_k - \sum_{j=0}^{k-1} c_j \left[p_{k-j}(j+r) + q_{k-j}\right] \right)$$

where the f_k's are coefficients of some power series convergent on $(-R, R)$. (Note that this reduces to recursion formula (35.8) if each f_k is 0.) To deal with the convergence of any power series based on any such a recursion formula, we have the following theorem:

Theorem 35.2

Let $R > 0$. Assume

$$\sum_{k=0}^{\infty} p_k x^k \quad , \quad \sum_{k=0}^{\infty} q_k x^k \quad \text{and} \quad \sum_{k=0}^{\infty} f_k x^k$$

are power series convergent for $|x| < R$, and

$$\sum_{k=0}^{\infty} c_k x^k$$

is a power series such that, for some value ω and some integer K_0,

$$c_k = \frac{1}{J(k)} \left(f_k - \sum_{j=0}^{k-1} c_j \left[p_{k-j}(j+\omega) + q_{k-j}\right] \right) \quad \text{for} \quad k \geq K_0$$

where J is some second-degree polynomial function satisfying

$$J(k) \neq 0 \quad \text{for} \quad k = K_0, K_0+1, K_0+2, \ldots \quad .$$

Then $\sum_k^{\infty} c_k x^k$ is also convergent for $|x| < R$.

The proof of this convergence theorem will be given in section 35.6. It is very similar to the convergence proofs developed in chapter 32 for power series solutions.

35.3 The Easily Obtained Series Solutions

Now let r_j be either of the two solutions r_1 and r_2 to the indicial equation,

$$I(r) = 0 \quad .$$

To use recursion formula (35.8) with $r = r_j$, it suffices to have

$$I(k + r_j) \neq 0 \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

But r_1 and r_2 are the only solutions to $I(r) = 0$, so the last line tells us that, to use recursion formula (35.8) with $r = r_j$, it suffices to have

$$k + r_j \neq r_1 \quad \text{and} \quad k + r_j \neq r_2 \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad ;$$

that is, it suffices to have

$$r_1 - r_j \neq k \quad \text{and} \quad r_2 - r_j \neq k \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

As long as this holds, we can start with any nonzero constant c_0 and generate subsequent c_k's via the basic recursion formula (35.8) to create a power series

$$\sum_{k=0}^{\infty} c_k x^k \quad .$$

Moreover, theorem 35.2 assures us that this series is convergent for $|x| < R$. Consequently,

$$y(x) = x^{r_j} \sum_{k=0}^{\infty} c_k x^k$$

is a well-defined function, at least on $(0, R)$ (just what happens at $x = 0$ depends on the x^{r_j} factor in this formula). Plugging this formula back into our differential equation and basically repeating the computations leading to the indicial equation and the recursion formula would then confirm that this y is, indeed, a solution on $(0, R)$ to our differential equation.

Lemma 35.3

For the problem considered in this chapter: If r_j is either of the two solutions r_1 and r_2 to the indicial equation, and

$$r_1 - r_j \neq k \quad \text{and} \quad r_2 - r_j \neq k \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad , \tag{35.9}$$

then a solution on $(0, R)$ to the original differential equation is given by

$$y(x) = x^{r_j} \sum_{k=0}^{\infty} c_k x^k$$

where c_0 is any nonzero constant, and c_1, c_2, c_3, $\ldots$ are given by recursion formula (35.8) with $r = r_j$.

Now let us consider the $r_j = r_1$ and $r_j = r_2$ cases separately, adding the assumption that the coefficients of our original differential equation are all real-valued in some interval about $x_0 = 0$. This means that the coefficients in the indicial equation are all real. Hence, we may assume that either both r_1 and r_2 are real with $r_1 \geq r_2$, or that r_1 and r_2 are complex conjugates of each other.

Solutions Corresponding to r_1

With $r_j = r_1$, condition (35.9) in the above lemma becomes

$$r_1 - r_1 \neq k \quad \text{and} \quad r_2 - r_1 \neq k \qquad \text{for} \quad k = 1,\ 2,\ 3,\ \ldots \quad ,$$

Clearly, the only way this cannot be satisfied is if

$$r_2 - r_1 = K \qquad \text{for some positive integer} \quad K \quad .$$

But, using the formulas for r_1 and r_2 from page 35–5, you can easily verify that

$$r_2 - r_1 = = -\sqrt{(p_0 - 1)^2 - 4q_0} \quad ,$$

which cannot equal some positive integer. Thus, the above lemma assures us that

> *One solution on* $(0, R)$ *to our differential equation is given by*
>
> $$y(x) = x^{r_1} \sum_{k=0}^{\infty} c_k x^k$$
>
> *where* c_0 *is any nonzero constant, and* c_1, c_2, c_3, $\ldots$ *are given by recursion formula (35.8) with* $r = r_1$.

This confirms statement 1 in theorem 34.2.

In particular, for the rest of our discussion, let us let y_1 be the solution

$$y_1(x) = x^{r_1} \sum_{k=0}^{\infty} a_k x^k \tag{35.10}$$

where $a_0 = 1$ and

$$a_k = \frac{-1}{I(k + r_1)} \sum_{j=0}^{k-1} a_j \left[p_{k-j}(j + r_1) + q_{k-j} \right] \qquad \text{for} \quad k = 1,\ 2,\ 3,\ \ldots \quad .$$

On occasion, we may call this our "first" solution.

"Unexceptional" Solutions Corresponding to r_2

With $r_j = r_2$, condition (35.9) in lemma 35.3 becomes

$$r_1 - r_2 \neq k \quad \text{and} \quad r_2 - r_2 \neq k \qquad \text{for} \quad k = 1,\ 2,\ 3,\ \ldots \quad ,$$

which, obviously, is the same as

$$r_1 - r_2 \neq K \qquad \text{for some positive integer} \quad K \quad .$$

Unfortunately, this requirement does not automatically hold. It is certainly possible that

$$r_1 - r_2 = K \qquad \text{for some positive integer} \quad K \quad .$$

This is an "exceptional" case which we will have examine further. For now, the lemma above simply assures us that:

If $r_1 - r_2$ is not a positive integer, then a solution on $(0, R)$ to our differential equation is given by

$$y(x) = x^{r_2} \sum_{k=0}^{\infty} c_k x^k$$

where c_0 is any nonzero constant, and the other c_k's are given by recursion formula (35.8) with $r = r_2$.

Of course, the solutions just described corresponding to r_2 will be the same as those corresponding to r_1 if $r_2 = r_1$ (i.e., $r_1 - r_2 = 0$). This is another exceptional case that we will have to examine later.

For the rest of this chapter, let us say that, if r_1 and r_2 are not equal and do not differ by an integer, then the "second solution" to our differential equation on $(0, R)$ is

$$y_2(x) = x^{r_2} \sum_{k=0}^{\infty} b_k x^k$$

where $b_0 = 1$ and

$$b_k = \frac{-1}{I(k+r_2)} \sum_{j=0}^{k-1} b_j \left[p_{k-j}(j+r_2) + q_{k-j} \right] \qquad \text{for} \quad k = 1,\ 2,\ 3,\ \dots \quad .$$

We should note that, if r_1 and r_2 are two different values not differing by an integer, then the above y_1 and y_2 are clearly not constant multiples of each other (at least, it should be clear once you realize that the first terms of $y_1(x)$ and $y_2(x)$ are, respectively, x^{r_1} and x^{r_2}). Consequently $\{y_1, y_2\}$ is a fundamental set of solutions to our differential equation on $(0, R)$, and

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is a general solution to our differential equation over $(0, R)$. That finishes the proof of theorem 34.2 up through statement 2.

Deriving the "Exceptional" Solutions

In the next two sections, we will derive formulas for the solutions corresponding to $r = r_2$ when r_1 and r_2 are equal or differ by a nonzero integer. In deriving these solutions, we could use the first solution, y_1 , with the reduction of order method from chapter 13. Unfortunately, that gets somewhat messy and does not directly lead to useful recursion formulas. So, instead, we will take somewhat different approaches.

Since the approach we'll take when $r_2 = r_1$ is a bit more elementary (but still tedious) and somewhat less "clever" than the approach we'll take when $r_1 - r_2$ is a positive integer, we will consider the case where $r_2 = r_1$ first.

35.4 Second Solutions When $r_2 = r_1$

Recall that, based on what we learned from studying Euler equations, we suspected that a second solution to our differential equation when $r_2 = r_1$ will be of the form

$$y(x) = \ln|x|\, Y(x) \qquad \text{with} \quad Y(x) = x^{r_1} \sum_{k=0}^{\infty} b_k x^k \quad .$$

Unfortunately, this turns out to not be generally true. But since it seemed so reasonable at the time, let us still try using this, but with an added "error term". That is, let's try something of the form

$$y(x) = \ln|x|\, Y(x) + \epsilon(x) \quad . \tag{35.11}$$

Plugging this into the differential equation:

$$\begin{aligned}
0 &= \mathcal{L}[y] \\
&= x^2 y'' + xPy' + Qy \\
&= x^2 \left[\ln|x|\, Y'' + \frac{2}{x} Y' - \frac{1}{x^2} Y + \epsilon''\right] + xP\left[\ln|x|\, Y' + \frac{1}{x} Y + \epsilon'\right] \\
&\qquad + Q\left[\ln|x|\, Y(x) + \epsilon(x)\right] \\
&= \ln|x| \Big[\underbrace{x^2 Y'' + xPY' + QY}_{\mathcal{L}[Y]}\Big] + 2xY' - Y + PY + \underbrace{x^2\epsilon'' + xP\epsilon' + Q\epsilon}_{\mathcal{L}[\epsilon]} \quad .
\end{aligned}$$

Choosing Y to be our first solution,

$$Y(x) = y_1(x) = x^{r_1} \sum_{k=0}^{\infty} a_k x^k \quad ,$$

causes the natural log term to vanish, leaving us with

$$0 = 2x{y_1}' - y_1 + Py_1 + \mathcal{L}[\epsilon] \quad ,$$

which we can rewrite as

$$\mathcal{L}[\epsilon] = F(x) \tag{35.12a}$$

with

$$F(x) = y_1(x) - 2x{y_1}'(x) - P(x)y_1(x) \quad . \tag{35.12b}$$

It turns out that we will be seeing both the above differential equation and the function F when we deal with the case where r_2 and r_1 differ by a nonzero integer. So, for now, let's expand $F(x)$ using the series formulas for y_1 and P without assuming $r_2 = r_1$:

$$\begin{aligned}
F(x) &= y_1(x) - 2x{y_1}'(x) - P(x)y_1(x) \\
&= x^{r_1} \sum_{n=0}^{\infty} a_n x^n - 2x\left(x^{r_1} \sum_{n=0}^{\infty} a_n(r_1+n)x^{n-1}\right) - \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(x^{r_1} \sum_{m=0}^{\infty} a_m x^m\right)
\end{aligned}$$

$$= x^{r_1}\left[\sum_{n=0}^{\infty} a_n x^n \; - \; \sum_{n=0}^{\infty} a_n(2r_1+2n)x^n \; - \; \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{m=0}^{\infty} a_m x^m\right)\right]$$

$$= x^{r_1}\sum_{n=0}^{\infty}\left(a_n\,[1-2r_1-2n] \; - \; \sum_{j=0}^{n} a_j p_{n-j}\right)x^n \quad .$$

Recalling that $a_0 = 1$ and that, in general $p_0 = 1 - r_1 - r_2$, we see that the first term in the series simplifies somewhat,

$$a_0\,[1-2r_1-2\cdot 0)] \; - \; \sum_{j=0}^{0} a_j p_{0-j} \; = \; a_0[1-2r_1-p_0] \; = \; r_2 \; - \; r_1 \quad .$$

For the other terms, we have

$$\begin{aligned} a_n\,[1-2r_1-2n] \; - \; \sum_{j=0}^{n} a_j p_{n-j} \; &= \; a_n\,[1-2r_1-2n] \; - \; \sum_{j=0}^{n-1} a_j p_{n-j} \; - \; a_n p_0 \\ &= \; a_n\,[1-2r_1-p_0-2n] \; - \; \sum_{j=0}^{n-1} a_j p_{n-j} \\ &= \; a_n\,[r_2-r_1-2n] \; - \; \sum_{j=0}^{n-1} a_j p_{n-j} \quad . \end{aligned}$$

So, in general,

$$F(x) \; = \; x^{r_1}\left[r_2 - r_1 \; + \; \sum_{n=1}^{\infty}\left(a_n\,[r_2-r_1-2n] \; - \; \sum_{j=0}^{n-1} a_j p_{n-j}\right)x^n\right] \quad . \tag{35.13}$$

We should also note that, because of the way F was constructed from power series convergent for $|x| < R$, we automatically have that the power series factor in the above formula for $F(x)$ is convergent for $|x| < R$.

Now, let's again assume $r_2 = r_1$. With this assumption and the change of index $k = n - 1$, the above formula for $F(x)$ reduces further,

$$\begin{aligned} F(x) \; &= \; x^{r_1}\left[0 \; + \; \sum_{n=1}^{\infty}\left(-2na_n \; - \; \sum_{j=0}^{n-1} a_j p_{n-j}\right)x^n\right] \\ &= \; x^{r_1}\sum_{k=0}^{\infty}\left(-2(k+1)a_{k+1} \; - \; \sum_{j=0}^{k} a_j p_{k+1-j}\right)x^{k+1}\Bigg] \quad , \end{aligned}$$

which we can write more succinctly as

$$F(x) \; = \; x^{r_1+1}\sum_{k=0}^{\infty} f_k x^k \tag{35.14a}$$

with

$$f_k \; = \; -2(k+1)a_{k+1} \; - \; \sum_{j=0}^{k} a_j p_{k+1-j} \quad . \tag{35.14b}$$

Let us now consider a modified power series formula for our error term

$$\epsilon(x) = x^{\rho} \sum_{k=0}^{\infty} \epsilon_k x^k \quad .$$

From lemma 35.2, we know that

$$\mathcal{L}[\epsilon(x)] = x^{\rho}\left[\epsilon_0 I(\rho) + \sum_{k=1}^{\infty}\left(\epsilon_k I(k+\rho) + \sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(j+\rho)+q_{k-j}\right]\right)x^k\right]$$

Thus, the differential equation $\mathcal{L}[\epsilon] = F$ becomes

$$x^{\rho}\left[\epsilon_0 I(\rho) + \sum_{k=1}^{\infty}\left(\epsilon_k I(k+\rho) + \sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(j+\rho)+q_{k-j}\right]\right)x^k\right] = x^{r_1+1}\sum_{k=0}^{\infty} f_k x^k$$

which is satisfied when

$$\rho = r_1 + 1 \quad , \tag{35.15a}$$

$$\epsilon_0 I(\rho) = f_0 \tag{35.15b}$$

and, for $k = 1, 2, 3, \ldots,$

$$\epsilon_k I(k+\rho) + \sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(j+r_1+1)+q_{k-j}\right] = f_k \quad . \tag{35.15c}$$

So let $\rho = r_1 + 1$ and observe that because r_1 is a double root of $I(r)$,

$$I(k+\rho) = I(k+r_1+1) = ([r_1+k+1]-r_1)^2 = (k+1)^2 \quad .$$

System (35.15) now reduces to

$$\epsilon_0 = f_0$$

and, when $k = 1, 2, 3, \ldots,$

$$\epsilon_k(k+1)^2 + \sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(j+r_1+1)+q_{k-j}\right] = f_k \quad .$$

That is,

$$\epsilon_0 = f_0 \tag{35.16a}$$

and, for $k = 1, 2, 3, \ldots,$

$$\epsilon_k = \frac{1}{(k+1)^2}\left(f_k - \sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(j+r_1+1)+q_{k-j}\right]\right) \tag{35.16b}$$

where the f_k's are given by formula (35.14b).

Now recall just what we are looking for: We are looking for a function $\epsilon(x)$ such that

$$y_2(x) = y_1(x)\ln|x| + \epsilon(x)$$

is a solution to our original differential equation. We have obtained

$$\epsilon(x) = x^{\rho}\sum_{k=0}^{\infty}\epsilon_k x^k$$

where $\rho = r_1 + 1$ and the ϵ_k's are given by formula set (35.16). Plugging this formula back into the original differential equation and repeating the computations used to derive the above will confirm that y_2 is, indeed, a solution over $(0, R)$, provided the series for ϵ converges. Fortunately, using theorem 35.2 on page 35–6 this convergence is easily confirmed.

Thus, the above y_2 is a solution to our original differential equation on the interval $(0, R)$. Moreover, y_2 is clearly not a constant multiple of y_1. So $\{y_1, y_2\}$ is a fundamental set of solutions,

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is a general solution to our differential equation over $(0, R)$, and we have verified statement 3 in theorem 34.2 (with $b_k = \epsilon_k$).

35.5 Second Solutions When $r_1 - r_2 = K$

Preliminaries

Let's now assume r_1 and r_2 differ by some positive integer K, $r_1 - r_2 = K$. Setting

$$y(x) = x^{r_2} \sum_{k=0}^{\infty} b_k x^k$$

with $b_k = 1$ and using recursion formula (35.8) gives us

$$b_k = \frac{-1}{I(r_2 + k)} \sum_{j=0}^{k-1} b_j \left[p_{k-j}(j + r) + q_{k-j} \right] \qquad \text{for} \quad k = 1, 2, 3, \ldots, K - 1 \quad .$$

Unfortunately, $I(r_2 + K) = I(r_1) = 0$, giving us a "division by zero" when we attempt to compute b_K. This is the complication we will deal with for the rest of this section. It turns out that there are two subcases, depending on whether

$$\Gamma_K = \sum_{j=0}^{K-1} b_j \left[p_{K-j}(j + r_2) + q_{K-j} \right]$$

is zero or not. If it is zero, we get lucky.

The Case Where We Get Lucky

Recall that we actually derived our recursion formula from the requirement that, for

$$y(x) = x^{r_2} \sum_{k=0}^{\infty} b_k x^k$$

to be a solution to our differential equation, it suffices to have

$$b_k I(r_2 + k) = -\sum_{j=0}^{k-1} b_j \left[p_{k-j}(j + r) + q_{k-j} \right] \qquad \text{for} \quad k = 1, 2, 3, \ldots, \quad . \tag{35.17}$$

As noted above, we can use this to find b_k for $k < K$. For $k = K$, we have $I(r_2 + K) = I(r_1) = 0$ and the above equation becomes

$$b_K \cdot 0 = \Gamma_K$$

If we are lucky, then $\Gamma_K = 0$ and the above equation is trivially true for any value of b_K. So b_K is arbitrary if $\Gamma_K = 0$. Pick any value you wish (say, $b_K = 0$) and use equation (35.17) to compute the rest of the b_k's for

$$y_2(x) = x^{r_2} \sum_{k=0}^{\infty} b_k x^k \quad .$$

Convergence theorem 35.2 on page 35–6 now applies and assures us that the above power series converges for $|x| < R$. And then, again, the very computations leading to the indicial equation and recursion formulas verify that this $y(x)$ is a solution to our differential equation. Moreover, the leading term is x^{r_2}. Consequently, $y_1(x)$ and $y_2(x)$ are not constant multiples of each other. Hence, $\{y_1, y_2\}$ is a fundamental set of solutions, and

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is a general solution to our differential equation over $(0, R)$.

If you go back and check, you will see that the above y_2 is the solution claimed to exist in statement 4 of theorem 34.2 when $\mu = 0$. Hence we've confirmed that part of the claim.

?►**Exercise 35.1:** *Show that, if we took $b_0 = 0$ and $b_K = 1$ in the above (instead of $b_0 = 1$ and $b_K = 0$), we would have obtained the first solution, $y_1(x)$. (Thus, if $r_1 - r_2 = K$ and $\Gamma_k = 0$, the Frobenius method will generate the complete general solution when using $r = r_2$.)*

The Other Case

Let us now assume

$$\Gamma_K = \sum_{j=0}^{K-1} b_j \left[p_{K-j}(j + r_2) + q_{K-j} \right] \neq 0 \quad .$$

Ultimately, we want to confirm that formula (34.4) in theorem 34.2 does describe a solution to our differential equation. Before doing that, however, let us see how anyone could have come up with formula (34.4) in the first place.

Deriving a Solution as the Limit of Other Second Solutions

Suppose we have two differential equations that are very similar to each other. Does it not seem reasonable to expect one solution of one of these equations to also be very similar to some solution to the other differential equation? I hope your answer is *yes*, because this what we will use to derive the second solution to our differential equation,

$$x^2 y'' + xPy' + Qy = 0 \quad . \tag{35.18}$$

Remember: This has the corresponding indicial equation

$$I(r) = 0 \quad \text{with} \quad I(\rho) = (\rho - r_2)(\rho - r_1) \quad .$$

Also remember that we are assuming $r_1 = r_2 + K$ for some positive integer K, and that Γ_K (as defined above) is nonzero.

Now let r be any real value close to r_2 (say, $|r - r_2| < 1$) and consider

$$x^2 y'' + x P_r y' + Q_r y = 0$$

where P_r and Q_r differ from P and Q only in having the first coefficients in their power series about 0 adjusted so that the corresponding indicial equation is

$$I_r(r) = 0 \quad \text{with} \quad I_r(\rho) = (\rho - r)(\rho - r_1) \quad .$$

If $r = r_2$, this is our original equation. If $r \neq r_2$, this is our "approximating differential equation". From our discussion in section 35.3 on the "easily obtained solutions", we know that, when $r \neq r_2$, a second solution to this equation is given by

$$y(x, r) = x^r \sum_{k=0}^{\infty} b_k(r) x^k$$

with $b_0(r) = 1$ and

$$b_k(r) = -\frac{1}{I_r(r+k)} \sum_{j=0}^{k-1} b_j(r)[P_{k-j}(j+r) + Q_{k-j}] \quad \text{for} \quad k = 1, 2, \dots \quad .$$

This, presumably, will approximate some second solution $y(x, r_1)$ to equation (35.18),

$$y(x, r_2) \approx y(x, r) \quad .$$

Presumably, also, this approximation improves as $r \to r_1$. So, let us go further and seek the $y(x, r_2)$ given by

$$y(x, r_2) = \lim_{r \to r_2} y(x, r)$$

Before going further, let us observe that

$$\begin{aligned} I_r(r+k) &= (r + k - r)(r + k - r_1) \\ &= k(k + r - [r_2 + K]) = k(k - K + r - r_2) \quad . \end{aligned}$$

Thus,

$$b_k(r) = \frac{-1}{k(k - K + r - r_2)} \sum_{j=0}^{k-1} b_j(r)[P_{k-j}(j+r) + Q_{k-j}] \quad \text{for} \quad k = 1, 2, \dots \quad .$$

In particular,

$$b_K(r) = \frac{-1}{K(r - r_2)} \sum_{j=0}^{K-1} b_j(r)[P_{K-j}(j+r) + Q_{K-j}]$$

So, while we have

$$b_k(r_2) = \lim_{r \to r_2} b_k(r) \quad \text{when} \quad k < K$$

being well-defined finite values, we also have

$$\lim_{r\to r_2} |b_K(r)| = \infty \quad ,$$

suggesting that

$$\lim_{r\to r_2} |b_k(r)| = \infty \qquad \text{for} \quad k > K$$

since the recursion formula for these $b_k(r)$'s all contain $b_K(r)$.

It must be noted, however, that we are assuming $\lim_{r\to r_2} y(x,r)$ exists despite the fact that individual terms in $y(x,r)$ behave badly as $r \to r_2$. Let's hold to this hope. Assuming this,

$$\begin{aligned}\lim_{r\to r_2} x^r \sum_{k=K}^{\infty} b_k(r)x^k &= \lim_{r\to r_2}\left[x^r\sum_{k=0}^{\infty} b_k(r)x^k - x^r\sum_{k=0}^{K-1} b_k(r)x^k\right]\\ &= \lim_{r\to r_2}\left[y(x,r) - x^r\sum_{k=0}^{K-1} b_k(r)x^k\right] = y(x,r_2) - x^{r_2}\sum_{k=0}^{K-1} b_k(r_2)x^k \quad ,\end{aligned}$$

which is finite for each x in the interval of convergence. Consequently,

$$\lim_{r\to r_2} (r - r_2)x^r \sum_{k=K}^{\infty} b_k(r)x^k = 0 \quad .$$

which we will rewrite as

$$\lim_{r\to r_2} x^r \sum_{k=K}^{\infty} \beta_k(r)x^k = 0 \quad . \tag{35.19}$$

by letting

$$\beta_k(r) = (r - r_2)b_k(r) \qquad \text{for} \quad r \neq r_2 \quad .$$

Now, let's start computing $y(x,r_2)$ as a limit using a simple, cheap trick:

$$\begin{aligned} y(x,r_2) &= \lim_{r\to r_2} y(x,r)\\ &= \lim_{r\to r_2} x^r \sum_{k=0}^{\infty} b_k(r)x^k\\ &= \lim_{r\to r_2} x^r \sum_{k=0}^{K-1} b_k(r)x^k + \lim_{r\to r_2} x^r \sum_{k=K}^{\infty} b_k(r)x^k\\ &= x^{r_2}\sum_{k=0}^{K-1} b_k(r_2)x^k + \lim_{r\to r_2}\frac{r-r_2}{r-r_2}x^r\sum_{k=K}^{\infty} b_k(r)x^k\\ &= x^{r_2}\sum_{k=0}^{K-1} b_k(r_2)x^k + \lim_{r\to r_2}\frac{x^r\sum_{k=K}^{\infty}\beta_k(r)x^k}{r-r_2} \quad .\end{aligned}$$

Using L'Hôpital's rule, we see that

$$\lim_{r\to r_2}\frac{x^r\sum_{k=K}^{\infty}\beta_k(r)x^k}{r-r_2} = \lim_{r\to r_2}\frac{\dfrac{\partial}{\partial r}\left[x^r\sum_{k=K}^{\infty}\beta_k(r)x^k\right]}{\dfrac{\partial}{\partial r}[r-r_2]}$$

$$= \lim_{r \to r_2} \frac{x^r \ln|x| \sum_{k=K}^{\infty} \beta_k(r) x^k \ + \ x^r \sum_{k=K}^{\infty} {\beta_k}'(r) x^k}{1}$$

$$= x^{r_2} \ln|x| \sum_{k=K}^{\infty} \beta_k(r_2) x^k \ + \ x^{r_2} \sum_{k=K}^{\infty} {\beta_k}'(r_2) x^k$$

Combining the last two results gives

$$y(x, r_2) \ = \ x^{r_2} \ln|x| \sum_{k=K}^{\infty} \beta_k(r_2) x^k \ + \ x^{r_2} \sum_{k=0}^{\infty} \begin{Bmatrix} b_k(r_2) & \text{if} \quad k < K \\ {\beta_k}'(r_2) & \text{if} \quad K \le k \end{Bmatrix} x^k \quad .$$

This is not a very "pretty" expression. To simplify it, let

$$\epsilon_k \ = \ \begin{cases} b_k(r_2) & \text{if} \quad k < K \\ {\beta_k}'(r_2) & \text{if} \quad K \le k \end{cases} \quad ,$$

and observe that, letting $\alpha_k = \beta_{k+K}(r_2)$,

$$x^{r_2} \sum_{k=K}^{\infty} \beta_k(r_2) x^k \ = \ x^{r_1 - K} \left[\alpha_0 x^K \ + \ \alpha_1 x^{K+1} \ + \ \alpha_2 x^{K+2} \ + \ \cdots \right] \ = \ x^{r_1} \sum_{k=0}^{\infty} \alpha_k x^k \quad .$$

Then

$$y(x, r_2) \ = \ \ln|x| \, Y(x) \ + \ \epsilon(x) \tag{35.20a}$$

where

$$Y(x) \ = \ x^{r_1} \sum_{k=0}^{\infty} \alpha_k x^k \qquad \text{and} \qquad \epsilon(x) \ = \ x^{r_2} \sum_{k=0}^{\infty} \epsilon_k x^k \quad . \tag{35.20b}$$

Admittedly, part of the derivation of formula (35.20) was based on "hope" and assumptions that seemed reasonable but were not rigorously justified. So we are not yet certain this formula does yield the desired solution. Moreover, the methods given in this derivation for computing the α_k's and ϵ_k's certainly appear to be rather difficult to carry out in practice. These are valid concerns that we will deal with by now ignoring just how we derived this formula. Instead, we will see about validating this formula and obtaining more usable recursion formulas for the α_k's and ϵ_k's via methods that, by now, should be familiar to the reader.

Verifying Our Solution

Notice how similar formula (35.20a) for $y(x, r_2)$ is to formula (35.11) on page 35–10 from which we derived the second solution $y(x)$ when $r_1 - r_2 = 0$ in section 35.4. Let us be inspired by the work done in that section (and reuse as much of that work as possible) and try to find a solution of the form

$$y(x) \ = \ \ln|x| \, Y(x) \ + \ \epsilon(x)$$

where

$$\epsilon(x) \ = \ x^{r_2} \sum_{k=0}^{\infty} \epsilon_k x^k \quad .$$

Glancing back at the work near the beginning of section 35.4, it should be clear that

$$y(x) \ = \ \ln|x| \, Y(x) \ + \ \epsilon(x)$$

will satisfy our differential equation $\mathcal{L}[y] = 0$ if

$$Y(x) = y_1(x) \qquad \text{and} \qquad \mathcal{L}[\epsilon] = F(x)$$

where, taking into account that $I(r_2) = 0$ and $r_1 - r_2 = K$,

$$\begin{aligned}\mathcal{L}[\epsilon] &= x^{r_2}\left[\epsilon_0 I(r_2) + \sum_{k=1}^{\infty}\left(\epsilon_k I(r_2+k) + \sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(r_2+j) + q_{k-j}\right]\right)x^k\right] \\ &= x^{r_2}\sum_{k=1}^{\infty}\left(\epsilon_k I(r_2+k) + \sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(r_2+j) + q_{k-j}\right]\right)x^k\end{aligned}$$

and

$$\begin{aligned}F(x) &= x^{r_1}\left[r_2 - r_1 + \sum_{n=1}^{\infty}\left(a_n\left[r_2 - r_1 - 2n\right] - \sum_{j=0}^{n-1} a_j p_{n-j}\right)x^n\right] \\ &= x^{r_2+K}\left[-K + \sum_{n=1}^{\infty}\left(a_n\left[-K - 2n\right] - \sum_{j=0}^{n-1} a_j p_{n-j}\right)x^n\right] \\ &= x^{r_2}\left[-Kx^K + \sum_{n=1}^{\infty}\left(a_n\left[-K - 2n\right] - \sum_{j=0}^{n-1} a_j p_{n-j}\right)x^{K+n}\right] \quad .\end{aligned}$$

Using $k = n + K$, we can rewrite our last formula as

$$F(x) = x^{r_2}\left[-Kx^K + \sum_{k=K+1}^{\infty} f_k x^k\right]$$

with

$$f_k = a_{k-K}\left[K - 2k\right] - \sum_{j=0}^{k-K-1} a_j p_{k-K-j} \quad .$$

As in the previous section, we know the power series in the formula for $F(x)$ converges for $|x| < R$ because of the way it was constructed from power series already known to be convergent for these values of x .

So the differential equation, $\mathcal{L}[\epsilon] = F$, expands to

$$x^{r_2}\sum_{k=1}^{\infty}\left(\epsilon_k I(r_2+k) + \sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(r_2+j) + q_{k-j}\right]\right)x^k = x^{r_2}\left[-Kx^K + \sum_{k=K+1}^{\infty} f_k x^k\right] \quad ,$$

which means that we are seeking ϵ_k's satisfying the system

$$\epsilon_k I(r_2+k) + \sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(r_2+j) + q_{k-j}\right] = \begin{cases} 0 & \text{if} \quad 1 \le k < K \\ -K & \text{if} \quad k = K \\ f_k & \text{if} \quad k > K \end{cases} \quad . \tag{35.21}$$

Solving for ϵ_k in the first few equations of this set yields

$$\epsilon_k = \frac{-1}{I(r_2+k)}\sum_{j=0}^{k-1}\epsilon_j\left[p_{k-j}(r_2+j) + q_{k-j}\right] \qquad \text{for} \quad k = 1, 2, \ldots, K-1 \quad .$$

To simplify matters, let's recall that, at the start of this section, we had already obtained a set $\{b_0, b_1, \ldots, b_{K-1}\}$ satisfying $b_0 = 1$ and

$$b_k = \frac{-1}{I(r_2+k)} \sum_{j=0}^{k-1} b_j \left[p_{k-j}(r_2+j) + q_{k-j} \right] \qquad \text{for} \quad k = 1, 2, \ldots, K-1 \quad .$$

It is then easily verified that, whatever value we have for ϵ_0,

$$\epsilon_k = \epsilon_0 b_k \qquad \text{for} \quad k = 1, 2, \ldots, K-1 \quad .$$

Now, also recall that

$$\Gamma_K = \sum_{j=0}^{K-1} b_j \left[p_{K-j}(r_2+j) + q_{K-j} \right] \neq 0 \quad .$$

and take a look at the K^{th} equation in system (35.21):

$$\epsilon_K I(r_2+K) + \sum_{j=0}^{K-1} \epsilon_j \left[p_{K-j}(r_2+j) + q_{K-j} \right] = -K$$

$$\hookrightarrow \qquad \epsilon_K I(r_1) + \sum_{j=0}^{K-1} \epsilon_0 b_j \left[p_{K-j}(r_2+j) + q_{K-j} \right] = -K$$

$$\hookrightarrow \qquad \epsilon_K \cdot 0 + \epsilon_0 \Gamma_K = -K \quad .$$

So ϵ_K can be any value, while

$$\epsilon_0 = -\frac{K}{\Gamma_K} \quad ,$$

and

$$\epsilon_k = \epsilon_0 \cdot b_k = -\frac{K b_k}{\Gamma_K} \qquad \text{for} \quad k = 1, 2, \ldots, K-1 \quad .$$

For the remaining ϵ_k's, we simply solve each of the remaining equations in system (35.21) for ϵ_k (using whatever value of ϵ_K we choose), obtaining

$$\epsilon_k = \frac{1}{I(r_2+k)} \left[f_k - \sum_{j=0}^{k-1} \epsilon_j \left[p_{k-j}(r_2+j) + q_{k-j} \right] \right] \qquad \text{for} \quad k > K \quad .$$

Theorem 35.2 tells us that the resulting $\sum_{k=0}^{\infty} \epsilon_k x^k$ converges for $|x| < R$, and that, along with all the computations above, tells us that

$$y(x) = y_1(x) \ln|x| + x^{r_2} \sum_{k=0}^{\infty} \epsilon_k x^k$$

is a solution to our original differential equation on $(0, R)$. Clearly, it is not a constant multiple of y_1, and so $\{y_1, \mu y\}$ is a fundamental set of solutions for any nonzero constant μ. In particular, the solution mentioned in theorem 34.2 is the one with

$$\mu = \frac{1}{\epsilon_0} = -\frac{\Gamma_K}{K} \quad .$$

And that, except for verifying convergence theorem 35.2, confirms statement 4 of theorem 34.2, and completes the proof of theorem 34.2, itself.

35.6 Convergence of the Solution Series

Finally, let's verify theorem 35.2 on page 35–6 on the convergence of our series.

Assumptions and Claim

We are assuming that ω is some constant,

$$\sum_{k=0}^{\infty} f_k x^k \quad , \quad \sum_{k=0}^{\infty} p_k x^k \quad \text{and} \quad \sum_{k=0}^{\infty} q_k x^k$$

are power series convergent for $|x| < R$, J is a second-degree polynomial function, and K_0 is some nonnegative integer such that

$$J(k) \neq 0 \quad \text{for} \quad k = K_0,\ K_0+1,\ K_0+2,\ \dots \quad .$$

We are also assuming that we have a power series

$$\sum_{k=0}^{\infty} c_k x^k$$

whose coefficients satisfy

$$c_k = \frac{1}{J(k)}\left[f_k - \sum_{j=0}^{k-1} c_j \left[p_{k-j}(j+\omega) + q_{k-j} \right] \right] \quad \text{for} \quad k \geq K_0 \quad .$$

The claim of the theorem is that $\sum_k^{\infty} c_k x^k$ converges for all x satisfying $|x| < R$. This, of course, can be verified by showing $\sum_k^{\infty} |c_k|\, |x|^k$ converges for each x in $(-R, R)$.

The Proof

The proof is very similar to (and a bit simpler than) the proofs of convergence in chapter 32.

We start by letting x be any single value in $(-R, R)$. We then can (and do) choose X to be some value with $|x| < X < R$. Also, by the convergence of the series, we can (and do) choose M to be a positive value such that, for $k = 0,\ 1,\ 2,\ \dots$,

$$\left| f_k X^k \right| < M \quad , \quad \left| p_k X^k \right| < M \quad \text{and} \quad \left| q_k X^k \right| < M \quad .$$

Now consider the power series $\sum_{k=0}^{\infty} C_k x^k$ with

$$C_k = |c_k| \quad \text{for} \quad k < K_0$$

and

$$C_k = \left| \frac{1}{J(k)} \right| \left[M X^{-k} + \sum_{j=0}^{k-1} C_j \left[M X^{-[k-j]}(j+|\omega|) + M X^{-[k-j]} \right] \right] \quad \text{for} \quad k \geq K_0 \quad .$$

Comparing the recursion formulas for c_k and C_k, it is obvious that

$$|c_k|\, |x|^k \leq C_k\, |x|^k \quad \text{for} \quad k = 0,\ 1,\ 2,\ \dots \quad .$$

Consequently, the convergence of $\sum_k^\infty c_k x^k$ can be confirmed by showing $\sum_k^\infty C_k |x|^k$ converges, and (by the limit ratio test) that can be shown by verifying that

$$\lim_{k\to\infty} \left| \frac{C_{k+1}x^{k+1}}{C_k x^k} \right| \leq 1 \quad .$$

Fortunately, for $k > K_0$,

$$\begin{aligned}
C_{k+1} &= \left|\frac{1}{J(k+1)}\right| \left[MX^{-(k+1)} + \sum_{j=0}^{k} C_j \left[MX^{-[k+1-j]}(j+|\omega|) + MX^{-[k+1-j]} \right] \right] \\
&= \left|\frac{X^{-1}}{J(k+1)}\right| \left[\left(MX^{-k} + \sum_{j=0}^{k-1} C_j \left[MX^{-[k-j]}(j+|\omega|) + MX^{-[k-j]} \right] \right) \right. \\
&\qquad\qquad \left. + C_k \left[MX^{-[k-k]}(k+|\omega|) + MX^{-[k-k]} \right] \right] \\
&= \left|\frac{X^{-1}}{J(k+1)}\right| (|J(k)|\, C_k + C_k[M(k+\omega+1)]) \\
&= \left| \frac{|J(k)| + M(k+\omega+1)}{J(k+1)} \right| \cdot \frac{C_k}{X} \quad .
\end{aligned}$$

Thus,

$$\left| \frac{C_{k+1}x^{k+1}}{C_k x^k} \right| = \frac{C_{k+1}}{C_k} |x| = \left| \frac{|J(k)| + M(k+\omega+1)}{J(k+1)} \right| \cdot \frac{|x|}{X} \quad .$$

Since J is a second-degree polynomial, you can easily verify that

$$\lim_{k\to\infty} \left| \frac{|J(k)| + M(k+\omega+1)}{J(k+1)} \right| = 1 \quad .$$

Hence, since $|x| < X$,

$$\lim_{k\to\infty} \left| \frac{C_{k+1}x^{k+1}}{C_k x^k} \right| = \lim_{k\to\infty} \left| \frac{|J(k)| + M(k+\omega+1)}{J(k+1)} \right| \cdot \frac{|x|}{X} = 1 \cdot \frac{|x|}{X} < 1 \quad ,$$

which is all we needed to verify the claimed convergence. ∎