28

Piecewise-Defined Functions and Periodic Functions

At the start of our study of the Laplace transform, it was claimed that the Laplace transform is "particularly useful when dealing with nonhomogeneous equations in which the forcing functions are not continuous". Thus far, however, we've done precious little with any discontinuous functions other than step functions. Let us now rectify the situation by looking at the sort of discontinuous functions (and, more generally, "piecewise-defined" functions) that often arise in applications, and develop tools and skills for dealing with these functions.

We will also take a brief look at transforms of periodic functions other than sines and cosines. As you will see, many of these functions are, themselves, piecewise defined. And finally, we will use some of the material we've recently developed to re-examine the issue of resonance in mass/spring systems.

28.1 Piecewise-Defined Functions

Piecewise-Defined Functions, Defined

When we talk about a "discontinuous function f" in the context of Laplace transforms, we usually mean f is a piecewise continuous function that is not continuous on the interval $(0, \infty)$. Such a function will have jump discontinuities at isolated points in this interval. Computationally, however, the real issue is often not so much whether there is a nonzero jump in the graph of f at a point t_0, but whether the formula for computing $f(t)$ is the same on either side of t_0. So we really should be looking at the more general class of "piecewise-defined" functions that, at worst, have jump discontinuities.

Just what is a *piecewise-defined* function? It is any function given by different formulas on different intervals. For example,

$$f(t) = \begin{cases} 0 & \text{if} \quad t < 1 \\ 1 & \text{if} \quad 1 < t < 2 \\ 0 & \text{if} \quad 2 < t \end{cases} \qquad \text{and} \qquad g(t) = \begin{cases} 0 & \text{if} \quad t \le 1 \\ t - 1 & \text{if} \quad 1 < t < 2 \\ 1 & \text{if} \quad 2 \le t \end{cases}$$

are two relatively simple piecewise-defined functions. The first (sketched in figure 28.1a) is discontinuous because it has nontrivial jumps at $t = 1$ and $t = 2$. However, the second

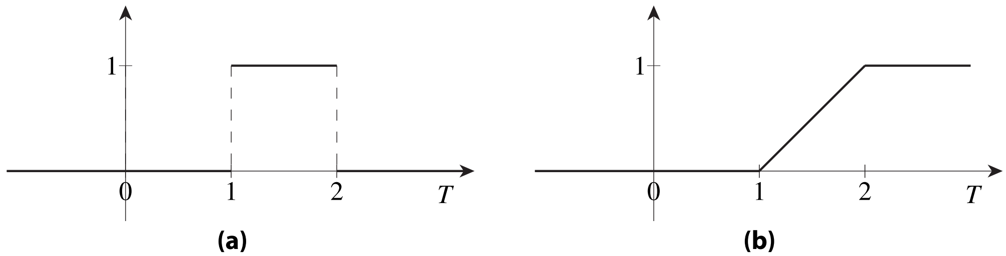

Figure 28.1: The graphs of two piecewise-defined functions.

function (sketched in figure 28.1b) is continuous because $t - 1$ goes from 0 to 1 as t goes from 1 to 2. There are no jumps in the graph of g.

By the way, we may occasionally refer to the sort of lists used above to define $f(t)$ and $g(t)$ as *conditional sets of formulas* or *sets of conditional formulas* for f and g, simply because these are sets of formulas with conditions stating when each formula is to be used.

Do note that, in the above formula set for f, we did not specify the values of $f(t)$ when $t = 1$ or $t = 2$. This was because f has jump discontinuities at these points and, as we agreed in chapter 24 (see page 496), we are not concerned with the precise value of a function at its discontinuities. On the other hand, using the formula set given above for g, you can easily verify that

$$\lim_{t \to 1^-} g(t) = 0 = \lim_{t \to 2^+} g(t) \qquad \text{and} \qquad \lim_{t \to 1^-} g(t) = 1 = \lim_{t \to 2^+} g(t) \quad ;$$

so there is not a true jump in g at these points. That is why we went ahead and specified that

$$g(1) = 0 \qquad \text{and} \qquad g(2) = 1 \quad .$$

In the future, let us agree that, even if the value of a particular function f or g is not explicitly specified at a particular point t_0, as long as the left- and right-hand limits of the function at t_0 are defined and equal, then the function is defined at t_0 and is equal to those limits. That is, we'll assume

$$f(t_0) = \lim_{t \to t_0^-} f(t) = \lim_{t \to t_0^+} f(t)$$

whenever

$$\lim_{t \to t_0^-} f(t) = \lim_{t \to t_0^+} f(t) \quad .$$

This will simplify notation a little, and may keep us from worrying about issues of continuity when those issues are not important.

Step Functions, Again

Most people would probably consider the step functions to be the simplest piecewise-defined functions. These include the basic step function,

$$\text{step}(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad 0 \le t \end{cases}$$

(sketched in figure 28.2a), as well as the step function at a point α,

$$\text{step}_\alpha(t) = \text{step}(t - \alpha) = \begin{cases} 0 & \text{if} \quad t < \alpha \\ 1 & \text{if} \quad \alpha < t \end{cases}$$

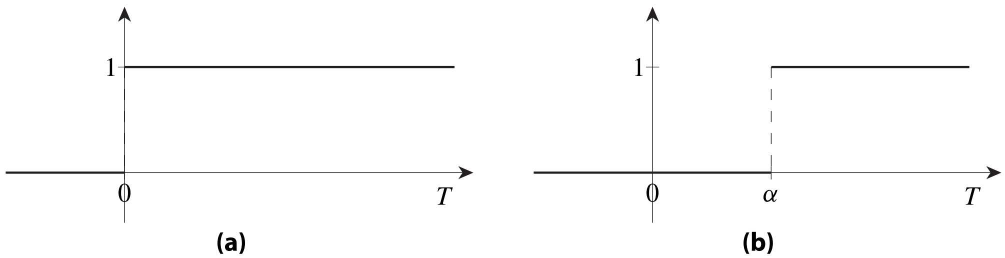

Figure 28.2: The graphs of **(a)** the basic step function $\text{step}(t)$ and **(b)** a shifted step function $\text{step}_\alpha(t)$ with $\alpha > 0$.

(sketched in figure 28.2b).

We will be dealing with other piecewise-defined functions, but, even with these other functions, we will find step functions useful. Step functions can be used as 'switches' — turning on and off the different formulas in our piecewise-defined functions. In this regard, let us quickly observe what we get when we multiply the step function by any function/formula $g(t)$:

$$g(t)\,\text{step}_\alpha(t) = \begin{Bmatrix} g(t)\cdot 0 & \text{if} & t < \alpha \\ g(t)\cdot 1 & \text{if} & \alpha < t \end{Bmatrix} = \begin{cases} 0 & \text{if} \quad t < \alpha \\ g(t) & \text{if} \quad \alpha < t \end{cases} .$$

Here, the step function at α 'switches on' $g(t)$ at $t = \alpha$. For example,

$$t^2\,\text{step}_3(t) = \begin{cases} 0 & \text{if} \quad t < 3 \\ t^2 & \text{if} \quad 3 < t \end{cases}$$

and

$$\sin(t-4)\,\text{step}_4(t) = \begin{cases} 0 & \text{if} \quad t < 4 \\ \sin(t-4) & \text{if} \quad 4 < t \end{cases} .$$

This fact will be especially useful when applying Laplace transforms in problems involving piecewise-defined functions, and we will find ourselves especially interested in cases where the formula being multiplied by $\text{step}_\alpha(t)$ describes a function that is also translated by α (as in $\sin(t-4)\,\text{step}_4(t)$).

The Laplace transform of $\text{step}_\alpha(t)$ was computed in chapter 24. If you don't recall how to compute this transform, it would be worth your while to go back to review that discussion. It is also worthwhile for us to look at a differential equation involving a step function.

!▶*Example 28.1:* *Consider finding the solution to*

$$y'' + y = \text{step}_3 \qquad \textit{with} \quad y(0) = 0 \quad \textit{and} \quad y'(0) = 0 \quad .$$

Taking the Laplace transform of both sides:

$$\mathcal{L}[y'' + y]\big|_s = \mathcal{L}[\text{step}_3]\big|_s$$

$$\hookrightarrow \qquad \mathcal{L}[y'']\big|_s + \mathcal{L}[y]|_s = \frac{1}{s}e^{-3s}$$

$$\hookrightarrow \qquad \left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{1}{s}e^{-3s}$$

$$\hookrightarrow \qquad \left[s^2 + 1\right] Y(s) = \frac{1}{s} e^{-3s}$$

$$\hookrightarrow \qquad Y(s) = \frac{1}{s(s^2+1)} e^{-3s} \quad .$$

Thus,

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)} e^{-3s}\right]\Big|_t \quad .$$

Here, we have the inverse transform of an exponential multiplied by a function whose inverse transform can easily be computed using, say, partial fraction. This would be a good point to pause and discuss, in general, what can be done in such situations.[1]

28.2 The "Translation Along the T-Axis" Identity

The Identity

As illustrated in the above example, we may often find ourselves with

$$\mathcal{L}^{-1}\left[e^{-\alpha s} F(s)\right]\Big|_t$$

where α is some positive number and $F(s)$ is some function whose inverse Laplace transform, $f = \mathcal{L}^{-1}[F]$, is either known or can be found with relative ease. Remember, this means

$$F(s) = \mathcal{L}[f(t)]|_s = \int_0^\infty f(t) e^{-st}\, dt \quad .$$

Consequently,

$$e^{-\alpha s} F(s) = e^{-\alpha s} \int_0^\infty f(t) e^{-st}\, dt = \int_0^\infty f(t) e^{-\alpha s} e^{-st}\, dt = \int_0^\infty f(t) e^{-s(t+\alpha)}\, dt \quad .$$

Using the change of variables $\tau = t + \alpha$ (thus, $t = \tau - \alpha$), and being careful with the limits of integration, we see that

$$e^{-\alpha s} F(s) = \cdots = \int_{t=0}^\infty f(t) e^{-s(t+\alpha)}\, dt = \int_{\tau=\alpha}^\infty f(\tau - \alpha) e^{-s\tau}\, d\tau \quad . \tag{28.1}$$

This last integral is almost, but not quite, the integral for the Laplace transform of $f(\tau - \alpha)$ (using τ instead of t as the symbol for the variable of integration). And the reason it is not is that this integral's limits start at α instead of 0. But that is where the limits would start if the function being transformed were 0 for $\tau < \alpha$. This, along with observations made a page or so ago, suggests viewing this integral as the transform of

$$f(t-\alpha)\,\text{step}_\alpha(t) = \begin{cases} 0 & \text{if } t < \alpha \\ f(t-\alpha) & \text{if } \alpha \le t \end{cases} \quad .$$

[1] The observant reader will note that y can be found directly using convolution. However, beginners may find the computation of the needed convolution, $\sin(t) * \text{step}_3(t)$, a little tricky. The approach being developed here reduces the need for such convolutions, and can be applied when convolution cannot be used. Still, convolutions with piecewise-defined functions can be useful, and will be discussed in section 28.4.

After all,

$$\begin{aligned}
\int_{\tau=\alpha}^{\infty} f(\tau-\alpha)e^{-s\tau}\,d\tau &= \int_{t=\alpha}^{\infty} f(t-\alpha)e^{-st}\,dt \\
&= \int_{t=0}^{\alpha} f(t-\alpha)\cdot 0\cdot e^{-st}\,dt \;+\; \int_{t=\alpha}^{\infty} f(t-\alpha)\cdot 1\cdot e^{-st}\,dt \\
&= \int_{t=0}^{\alpha} f(t-\alpha)\,\text{step}_{\alpha}(t)e^{-st}\,dt \;+\; \int_{t=\alpha}^{\infty} f(t-\alpha)\,\text{step}_{\alpha}(t)e^{-st}\,dt \\
&= \int_{t=0}^{\infty} f(t-\alpha)\,\text{step}_{\alpha}(t)e^{-st}\,dt \\
&= \mathcal{L}\big[f(t-\alpha)\,\text{step}_{\alpha}\big]\big|_s \quad .
\end{aligned}$$

Combining the above computations with equation set (28.1) then gives us

$$e^{-\alpha s}F(s) \;=\; \cdots \;=\; \int_{\tau=\alpha}^{\infty} f(\tau-\alpha)e^{-s\tau}\,d\tau \;=\; \cdots \;=\; \mathcal{L}\big[f(t-\alpha)\,\text{step}_{\alpha}(t)\big]\big|_s \quad .$$

Cutting out the middle, we get our second translation identity:

Theorem 28.1 (translation along the T–axis)
Let

$$F(s) \;=\; \mathcal{L}[f(t)]|_s$$

where f is any Laplace transformable function. Then, for any positive constant α,

$$\mathcal{L}\big[f(t-\alpha)\,\text{step}_{\alpha}(t)\big]\big|_s \;=\; e^{-\alpha s}F(s) \quad . \tag{28.2a}$$

Equivalently,

$$\mathcal{L}^{-1}\big[e^{-\alpha s}F(s)\big]\big|_t \;=\; f(t-\alpha)\,\text{step}_{\alpha}(t) \quad . \tag{28.2b}$$

Computing Inverse Transforms
The Basic Computations

Computing inverse transforms using the translation along the T–axis identity is usually straightforward.

!►*Example 28.2:* *Consider finding the inverse Laplace transform of*

$$\frac{e^{-2s}}{s^2+1} \quad .$$

Applying the identity, we have

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2+1}\right]\Bigg|_t \;=\; \mathcal{L}^{-1}\Big[e^{-2s}\underbrace{\frac{1}{s^2+1}}_{F(s)}\Big]\Big|_t \;=\; \mathcal{L}^{-1}\big[e^{-2s}F(s)\big]\big|_t \;=\; f(t-2)\,\text{step}_2(t) \quad .$$

Here the inverse transform of F is easily read off the tables:

$$f(t) \;=\; \mathcal{L}^{-1}[F(s)]|_t \;=\; \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]\Bigg|_t \;=\; \sin(t) \quad .$$

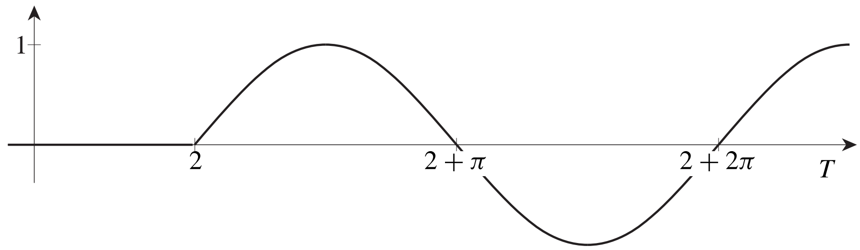

Figure 28.3: The graph of $\sin(t-2)\,\mathrm{step}(t-2)$.

So, for any X ,

$$f(X) = \sin(X) \quad .$$

Using this with $X = t - 2$ in the above inverse transform computation then yields

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2+1}\right]\bigg|_t = f(t-2)\,\mathrm{step}_2(t) = \sin(t-2)\,\mathrm{step}_2(t) \quad .$$

Keep in mind that

$$\sin(t-2)\,\mathrm{step}_2(t) = \begin{cases} 0 & \text{if } t < 2 \\ \sin(t-2) & \text{if } 2 < t \end{cases} \quad .$$

The graph of this function is sketched in figure 28.3.

Observe that, as illustrated in figure 28.3, the graph of

$$\mathcal{L}^{-1}\left[e^{-\alpha s}F(s)\right]\big|_t = f(t-\alpha)\,\mathrm{step}_\alpha(t)$$

is always zero for $t < \alpha$, and is the graph shifted by α of $f(t)$ on $[0,\infty)$ on $\alpha \leq t$. Remembering this can simplify graphing these types of functions.

Describing Piecewise-Defined Functions Arising From Inverse Transforms

Let us start with a simple, but illustrative, example.

!►*Example 28.3:* *Consider computing the inverse Laplace transform of*

$$F(s) = \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} \quad .$$

Going to the tables, we see that

$$G(s) = \frac{1}{s^2} \quad \Longrightarrow \quad g(t) = t \quad .$$

Using this, along with linearity and the second translation identity, we get

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)]|_t &= \mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s}\right]\bigg|_t \\ &= \mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-1s}\right]\bigg|_t - \mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-2s}\right]\bigg|_t \\ &= (t-1)\,\mathrm{step}_1(t) - (t-2)\,\mathrm{step}_2(t) \quad . \end{aligned}$$

Note that the step functions tell us that 'significant changes' occur in $f(t)$ at the points $t = 1$ and $t = 2$.

While the above is a valid answer, it is not a particularly convenient answer. It would be much easier to graph and see what f really is if we go further and completely compute $f(t)$ on the intervals having $t = 1$ and $t = 2$ as endpoints:

For $t < 1$, then

$$f(t) \;=\; (t-1)\underbrace{\text{step}_1(t)}_{0} - (t-2)\underbrace{\text{step}_2(t)}_{0} \;=\; 0 \;-\; 0 \;=\; 0 \quad .$$

For $1 < t < 2$, then

$$f(t) \;=\; (t-1)\underbrace{\text{step}_1(t)}_{1} - (t-2)\underbrace{\text{step}_2(t)}_{0} \;=\; (t-1) \;-\; 0 \;=\; t-1 \quad .$$

For $2 < t$, then

$$f(t) \;=\; (t-1)\underbrace{\text{step}_1(t)}_{1} - (t-2)\underbrace{\text{step}_2(t)}_{1} \;=\; (t-1) \;-\; (t-2) \;=\; 1 \quad .$$

Thus,

$$f(t) \;=\; \begin{cases} 0 & \textit{if} \quad t < 1 \\ t-1 & \textit{if} \quad 1 < t < 2 \\ 1 & \textit{if} \quad 2 < t \end{cases} \quad .$$

(This is the function sketched in figure 28.1b on page 555.)

As just illustrated, piecewise-defined functions naturally arise when computing inverse Laplace transforms using the second translation identity. Typically, use of this identity leads to an expression of the form

$$f(t) \;=\; g_0(t) \;+\; g_1(t)\,\text{step}_{\alpha_1}(t) \;+\; g_2(t)\,\text{step}_{\alpha_2}(t) \;+\; g_3(t)\,\text{step}_{\alpha_3}(t) \;+\; \cdots \tag{28.3}$$

where f is the function of interest, the $g_k(t)$'s are various formulas, and the α_k's are positive constants. This expression is a valid formula for f, and the step functions tell us that 'significant changes' occur in $f(t)$ at the points $t = \alpha_1$, $t = \alpha_2$, $t = \alpha_3$, $\ldots$. Still, to get a better picture of the function $f(t)$, we will want to obtain the formulas for $f(t)$ over each of the intervals bounded by the α_k's. Assuming we were reasonably intelligent and indexed the α_k's so that

$$0 \;<\; \alpha_1 \;<\; \alpha_2 \;<\; \alpha_3 \;<\; \cdots \quad ,$$

we would have

For $t < \alpha_1$,

$$\begin{aligned} f(t) \;&=\; g_0(t) \;+\; g_1(t)\underbrace{\text{step}_{\alpha_1}(t)}_{0} + g_2(t)\underbrace{\text{step}_{\alpha_2}(t)}_{0} + g_3(t)\underbrace{\text{step}_{\alpha_3}(t)}_{0} \;+\; \cdots \\ &=\; g_0(t) \;+\; 0 \;+\; 0 \;+\; 0 \;+\; \cdots \;=\; g_0(t) \quad . \end{aligned}$$

For $\alpha_1 < t < \alpha_2$,

$$f(t) = g_0(t) + g_1(t)\underbrace{\text{step}_{\alpha_1}(t)}_{1} + g_2(t)\underbrace{\text{step}_{\alpha_2}(t)}_{0} + g_3(t)\underbrace{\text{step}_{\alpha_3}(t)}_{0} + \cdots$$

$$= g_0(t) + g_1(t) + 0 + 0 + \cdots = g_0(t) + g_1(t) \quad .$$

For $\alpha_2 < t < \alpha_3$,

$$f(t) = g_0(t) + g_1(t)\underbrace{\text{step}_{\alpha_1}(t)}_{1} + g_2(t)\underbrace{\text{step}_{\alpha_2}(t)}_{1} + g_3(t)\underbrace{\text{step}_{\alpha_3}(t)}_{0} + \cdots$$

$$= g_0(t) + g_1(t) + g_2(t) + 0 + \cdots = g_0(t) + g_1(t) + g_2(t) \quad .$$

And so on.

Thus, the function f described by formula (28.3), above, is also given by the conditional set of formulas

$$f(t) = \begin{cases} f_0(t) & \text{if} \quad t < \alpha_1 \\ f_1(t) & \text{if} \quad \alpha_1 < t < \alpha_2 \\ f_2(t) & \text{if} \quad \alpha_2 < t < \alpha_3 \\ \quad\vdots & \end{cases}$$

where

$$f_0(t) = g_0(t) \quad ,$$

$$f_1(t) = g_0(t) + g_1(t) \quad ,$$

$$f_2(t) = g_0(t) + g_1(t) + g_2(t) \quad ,$$

$$\vdots$$

$$.$$

Computing Transforms with the Identity

The translation along the T–axis identity is also helpful in computing the transforms of piecewise-defined functions. Here, though, the computations typically require little more care. We'll deal with fairly simple cases here, and develop this topic further in the next section.

!►*Example 28.4:* *Consider finding* $\mathcal{L}[g(t)]|_s$ *where*

$$g(t) = \begin{cases} 0 & \text{if} \quad t < 3 \\ t^2 & \text{if} \quad 3 < t \end{cases} \quad .$$

Remember, this function can also be written as

$$g(t) = t^2 \,\text{step}_3(t) \quad .$$

Plugging this into the transform and applying our new translation identity gives

$$\mathcal{L}[g(t)]|_s = \mathcal{L}\big[t^2 \,\text{step}_3(t)\big]\big|_s = \mathcal{L}\big[f(t-3)\,\text{step}_3(t)\big]\big|_s = e^{-3s}F(s)$$

where

$$f(t-3) = t^2 \quad .$$

But we need the formula for $f(t)$ *, not* $f(t-3)$ *, to compute* $F(s)$ *. To find that that formula, let* $X = t-3$ *(hence,* $t = X+3$ *) in the formula for* $f(t-3)$ *. This gives*

$$f(X) = (X+3)^2 \quad .$$

Thus,

$$f(t) = (t+3)^2 = t^2 + 6t + 9 \quad ,$$

and

$$\begin{aligned} F(s) &= \mathcal{L}[f(t)]\big|_s = \mathcal{L}\big[t^2+6t+9\big]\big|_s \\ &= \mathcal{L}\big[t^2\big]\big|_s + 6\mathcal{L}[t]\big|_s + 9\mathcal{L}[1]\big|_s = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \quad . \end{aligned}$$

Plugging this back into the above formula for $\mathcal{L}[g(t)]\big|_s$ *gives us*

$$\mathcal{L}[g(t)]\big|_s = e^{-3s}F(s) = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right] \quad .$$

28.3 Rectangle Functions and Transforms of More Complicated Piecewise-Defined Functions

Rectangle Functions

"Rectangle functions" are slight generalizations of step functions. Given any interval (α, β) , the *rectangle function on* (α, β) , denoted $\mathrm{rect}_{(\alpha,\beta)}$, is the function given by

$$\mathrm{rect}_{(\alpha,\beta)}(t) = \begin{cases} 0 & \text{if} \quad t < \alpha \\ 1 & \text{if} \quad \alpha < t < \beta \\ 0 & \text{if} \quad \beta < t \end{cases} \quad .$$

The graph of $\mathrm{rect}_{(\alpha,\beta)}$ with $-\infty < \alpha < \beta < \infty$ has been sketched in figure 28.4. You can see why it is called a rectangle function — it's graph looks rather "rectangular", at least when α and β are finite. If $\alpha = -\infty$ or $\beta = \infty$, the corresponding rectangle functions simplify to

$$\mathrm{rect}_{(-\infty,\beta)}(t) = \begin{cases} 1 & \text{if} \quad t < \beta \\ 0 & \text{if} \quad \beta < t \end{cases}$$

and

$$\mathrm{rect}_{(\alpha,\infty)}(t) = \begin{cases} 0 & \text{if} \quad t < \alpha \\ 1 & \text{if} \quad \alpha < t \end{cases} \quad .$$

And if both $a = -\infty$ and $b = \infty$, then we have

$$\mathrm{rect}_{(-\infty,\infty)}(t) = 1 \qquad \text{for all} \quad t \quad .$$

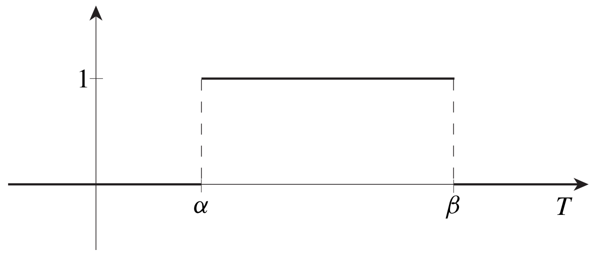

Figure 28.4: Graph of the rectangle function $\text{rect}_{(\alpha,\beta)}(t)$ with $-\infty < \alpha < \beta < \infty$.

All of these rectangle functions can be written as simple linear combinations of 1 and step functions at α and/or β, with, again, the step functions acting as 'switches' — switching the rectangle function 'on' (from 0 to 1 at α), and switching it 'off' (from 1 back to 0 at β). In particular, we clearly have

$$\text{rect}_{(-\infty,\infty)}(t) = 1 \qquad \text{and} \qquad \text{rect}_{(\alpha,\infty)}(t) = \text{step}_\alpha(t) \quad .$$

Somewhat more importantly (for us), we should observe that, for $-\infty < \alpha < \beta < \infty$,

$$1 - \text{step}_\beta(t) = \begin{Bmatrix} 1-0 & \text{if} \quad t < \beta \\ 1-1 & \text{if} \quad \beta < t \end{Bmatrix} = \begin{Bmatrix} 1 & \text{if} \quad t < \beta \\ 0 & \text{if} \quad \beta < t \end{Bmatrix} = \text{rect}_{(-\infty,\beta)}(t) \quad ,$$

and

$$\begin{aligned} \text{step}_\alpha(t) - \text{step}_\beta(t) &= \begin{Bmatrix} 0-0 & \text{if} \quad t < \alpha \\ 1-0 & \text{if} \quad \alpha < t < \beta \\ 1-1 & \text{if} \quad \beta < t \end{Bmatrix} \\ &= \begin{Bmatrix} 0 & \text{if} \quad t < a \\ 1 & \text{if} \quad \alpha < t < \beta \\ 0 & \text{if} \quad \beta < t \end{Bmatrix} = \text{rect}_{(\alpha,\beta)}(t) \quad . \end{aligned}$$

In summary, for $-\infty < \alpha < \beta < \infty$,

$$\text{rect}_{(\alpha,\beta)}(t) = \text{step}_\alpha(t) - \text{step}_\beta(t) \quad , \tag{28.4a}$$

$$\text{rect}_{(-\infty,\beta)}(t) = 1 - \text{step}_\beta(t) \tag{28.4b}$$

and

$$\text{rect}_{(\alpha,\infty)}(t) = \text{step}_\alpha(t) \quad . \tag{28.4c}$$

These formulas allow us to quickly compute the Laplace transforms of rectangle functions using the known transforms of 1 and the step functions.

!►*Example 28.5:*

$$\begin{aligned} \mathcal{L}\left[\text{rect}_{(3,4)}(t)\right]\Big|_s &= \mathcal{L}\left[\text{rect}_{(3,4)}(t)\right]\Big|_s \\ &= \mathcal{L}\left[\text{step}_3(t) - \text{step}_4(t)\right]\Big|_s \\ &= \mathcal{L}\left[\text{step}_3(t)\right]\Big|_s - \mathcal{L}\left[\text{step}_4(t)\right]\Big|_s = \frac{1}{s}e^{-3s} - \frac{1}{s}e^{-4s} \quad . \end{aligned}$$

Transforming More General Piecewise-Defined Functions

To help us deal with more general piecewise-defined functions, let us make the simple observations that

$$g(t)\,\mathrm{rect}_{(a,b)}(t) \;=\; \begin{Bmatrix} g(t)\cdot 0 & \text{if} & t < a \\ g(t)\cdot 1 & \text{if} & a < t < b \\ g(t)\cdot 0 & \text{if} & b < t \end{Bmatrix} \;=\; \begin{cases} 0 & \text{if} \quad t < a \\ g(t) & \text{if} \quad a < t < b \\ 0 & \text{if} \quad b < t \end{cases} ,$$

and

$$g(t)\,\mathrm{rect}_{(-\infty,b)}(t) \;=\; \begin{Bmatrix} g(t)\cdot 1 & \text{if} & t < b \\ g(t)\cdot 0 & \text{if} & b < t \end{Bmatrix} \;=\; \begin{cases} g(t) & \text{if} \quad t < b \\ 0 & \text{if} \quad b < t \end{cases} .$$

So functions of the form

$$f(t) \;=\; \begin{cases} 0 & \text{if} \quad t < a \\ g(t) & \text{if} \quad a < t < b \\ 0 & \text{if} \quad b < t \end{cases} \qquad \text{and} \qquad h(t) \;=\; \begin{cases} g(t) & \text{if} \quad t < b \\ 0 & \text{if} \quad b < t \end{cases}$$

can be rewritten, respectively, as

$$f(t) \;=\; g(t)\,\mathrm{rect}_{(a,b)}(t) \qquad \text{and} \qquad h(t) \;=\; g(t)\,\mathrm{rect}_{(-\infty,b)}(t) \quad .$$

More generally, it should now be clear that anything of the form

$$f(t) \;=\; \begin{cases} g_0(t) & \text{if} \quad t < \alpha_1 \\ g_1(t) & \text{if} \quad \alpha_1 < t < \alpha_2 \\ g_2(t) & \text{if} \quad \alpha_2 < t < \alpha_3 \\ \quad\vdots & \end{cases} \tag{28.5a}$$

can be rewritten as

$$f(t) \;=\; g_0(t)\,\mathrm{rect}_{(-\infty,\alpha_1)}(t) \;+\; g_1(t)\,\mathrm{rect}_{(\alpha_1,\alpha_2)}(t) \;+\; g_2(t)\,\mathrm{rect}_{(\alpha_2,\alpha_3)}(t) \;+\; \cdots \quad . \tag{28.5b}$$

The second form (with the rectangle functions) is a bit more concise than the "conditional set of formulas" used in form (28.5a), and is generally preferred by typesetters. Of course, there is a more important advantage of form (28.5b): Assuming f is piecewise continuous and of exponential order, its Laplace transform can now be taken by expressing the rectangle functions in formula (28.5b) as the linear combinations of 1 and step functions given in equation set (28.4), and then using linearity and what we learned in the previous section about taking transforms of functions multiplied by step functions.

!►*Example 28.6:* *Consider finding* $F(s) = \mathcal{L}[f(t)]|_s$ *when*

$$f(t) \;=\; \begin{cases} t^2 & \textit{if} \quad t < 3 \\ 0 & \textit{if} \quad 3 < t \end{cases} .$$

From the above, we see that

$$\begin{aligned} f(t) \;&=\; t^2\,\mathrm{rect}_{(-\infty,3)}(t) \\ &=\; t^2\left[1 - \mathrm{step}_3(t)\right] \;=\; t^2 \;-\; t^2\,\mathrm{step}_3(t) \quad . \end{aligned}$$

So
$$F(s) \;=\; \mathcal{L}[f(t)]\big|_s \;=\; \mathcal{L}\big[t^2 - t^2\,\text{step}_3(t)\big]\big|_s \;=\; \mathcal{L}\big[t^2\big]\big|_s \;-\; \mathcal{L}\big[t^2\,\text{step}_3(t)\big]\big|_s \quad .$$

The Laplace transform of t^2 *is in the tables, while the transform of* $t^2\,\text{step}_3(t)$ *just happened to have been computed in example 28.4 a few pages ago. Using these transforms, the above formula for* F *becomes*
$$F(s) \;=\; \frac{2}{s^3} \;-\; e^{-3s}\left[\frac{2}{s^3} \;+\; \frac{6}{s^2} \;+\; \frac{9}{s}\right] \quad .$$

!►Example 28.7: *Consider finding* $F(s) = \mathcal{L}[f(t)]|_s$ *when*
$$f(t) \;=\; \begin{cases} 0 & \text{if } \; t < 2 \\ e^{3t} & \text{if } \; 2 < t < 4 \\ 0 & \text{if } \; 4 < t \end{cases} \quad .$$

From the above, we see that
$$\begin{aligned} f(t) \;&=\; e^{3t}\,\text{rect}_{(2,4)}(t) \\ &=\; e^{3t}\big[\text{step}_2(t) - \text{step}_4(t)\big] \;=\; e^{3t}\,\text{step}_2(t) \;-\; e^{3t}\,\text{step}_4(t) \quad . \end{aligned}$$

Thus,
$$\mathcal{L}[f(t)]\big|_s \;=\; \mathcal{L}\big[e^{3t}\,\text{step}_2(t)\big]\big|_s \;-\; \mathcal{L}\big[e^{3t}\,\text{step}_4(t)\big]\big|_s \quad . \tag{28.6}$$

Both of the transforms on the right side of our last equation are easily computed via the translation identity developed in this chapter. For the first, we have
$$\mathcal{L}\big[e^{3t}\,\text{step}_2(t)\big]\big|_s \;=\; \mathcal{L}\big[g(t-2)\,\text{step}_2(t)\big]\big|_s \;=\; e^{-2s}G(s)$$

where
$$g(t-2) \;=\; e^{3t} \quad .$$

Letting $X = t - 2$ *(so* $t = X + 2$ *), the last expression becomes*
$$g(X) \;=\; e^{3(X+2)} \;=\; e^{3X+6} \;=\; e^6 e^{3X} \quad .$$

So
$$g(t) \;=\; e^6 e^{3t}$$

and
$$G(s) \;=\; \mathcal{L}[g(t)]\big|_s \;=\; \mathcal{L}\big[e^6 e^{3t}\big]\big|_s \;=\; e^6\mathcal{L}\big[e^{3t}\big]\big|_s \;=\; e^6\frac{1}{s-3} \quad .$$

This, along with the first equation in this paragraph, gives us
$$\mathcal{L}\big[e^{3t}\,\text{step}_2(t)\big]\big|_s \;=\; e^{-2s}G(s) \;=\; e^{-2s}e^6\frac{1}{s-3} \;=\; \frac{e^{-2(s-3)}}{s-3} \quad .$$

The transform of $e^{3t}\,\text{step}_4(t)$ *can be computed in the same manner, yielding*
$$\mathcal{L}\big[e^{3t}\,\text{step}_4(t)\big]\big|_s \;=\; \frac{e^{-4(s-3)}}{s-3} \quad .$$

(The details of this computation are left to you.)

Finally, combining the formulas just obtained for the transforms of $e^{3t}\,\mathrm{step}_2(t)$ *and* $e^{3t}\,\mathrm{step}_4(t)$ *with equation (28.6), we have*

$$\begin{aligned}\mathcal{L}[f(t)]|_s &= \mathcal{L}\big[e^{3t}\,\mathrm{step}_2(t)\big]\big|_s - \mathcal{L}\big[e^{3t}\,\mathrm{step}_4(t)\big]\big|_s \\ &= \frac{e^{-2(s-3)}}{s-3} - \frac{e^{-4(s-3)}}{s-3} \quad .\end{aligned}$$

!►Example 28.8: *Let's find the Laplace transform* $F(s)$ *of*

$$f(t) = \begin{cases} 2 & \text{if } t < 1 \\ e^{3t} & \text{if } 1 < t < 3 \\ t^2 & \text{if } 3 < t \end{cases} \quad .$$

To apply the Laplace transform, we first convert the above to an equivalent expression involving step functions:

$$\begin{aligned} f(t) &= 2\,\mathrm{rect}_{(-\infty,1)}(t) + e^{3t}\,\mathrm{rect}_{(1,3)}(t) + t^2\,\mathrm{rect}_{(3,\infty)}(t) \\ &= 2\big[1 - \mathrm{step}_1(t)\big] + e^{3t}\big[\mathrm{step}_1(t) - \mathrm{step}_3(t)\big] + t^2\,\mathrm{step}_3(t) \\ &= 2 - 2\,\mathrm{step}_1(t) + e^{3t}\,\mathrm{step}_1(t) - e^{3t}\,\mathrm{step}_3(t) + t^2\,\mathrm{step}_3(t) \quad .\end{aligned}$$

Using the tables and methods already discussed earlier in this chapter (as in examples 28.7 and 28.4), we discover that

$$\mathcal{L}[2]|_s = \frac{2}{s} \quad , \quad \mathcal{L}\big[2\,\mathrm{step}_1(t)\big]\big|_s = \frac{2e^{-s}}{s} \quad ,$$

$$\mathcal{L}\big[e^{3t}\,\mathrm{step}_1(t)\big]\big|_s = \frac{e^{-(s-3)}}{s-3} \quad , \quad \mathcal{L}\big[e^{3t}\,\mathrm{step}_3(t)\big]\big|_s = \frac{e^{-3(s-3)}}{s-3}$$

and

$$\mathcal{L}\big[t^2\,\mathrm{step}_3(t)\big]\big|_s = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right] \quad .$$

Combining the above and using the linearity of the Laplace transform, we obtain

$$\begin{aligned} F(s) &= \mathcal{L}[f(t)]|_s \\ &= \mathcal{L}\big[2 - 2\,\mathrm{step}_1(t) + e^{3t}\,\mathrm{step}_1(t) - e^{3t}\,\mathrm{step}_3(t) + t^2\,\mathrm{step}_3(t)\big]\big|_s \\ &= \frac{2}{s} - \frac{2e^{-s}}{s} + \frac{e^{-(s-3)}}{s-3} - \frac{e^{-3(s-3)}}{s-3} + e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right] \quad .\end{aligned}$$

28.4 Convolution with Piecewise-Defined Functions

Take another look at example 28.1 on page 557. As noted in the footnote, we could have by-passed much of the Laplace transform computation by simply observing that

$$y(t) = \sin(t) * \text{step}_3(t)$$

and computing that convolution. But in the footnote, it was claimed that computing such convolutions can be "a little tricky". Well, to be honest, it's not all that tricky. It's more an issue of careful bookkeeping.

When computing a convolution $h * f$ in which f is piecewise defined, you need to realize that the resulting convolution will also be piecewise defined, with (as you will see in the examples) the formula for $h * f$ changing at the same points where the formula for f changes. Hence, you should compute $h * f$ separately over the different intervals bounded by these points. Moreover, in computing the corresponding integrals, you will also need to account for the piecewise-defined nature of f, and break up the integral appropriately. To simplify all this, it is strongly recommended that you compute the convolution $h * f$ using the integral formula

$$h * f(t) = \int_0^t h(t - x) f(x)\, dx$$

(and not with the integrand $h(x) f(t - x)$).

One or two examples should clarify matters.

!►Example 28.9: *Let's compute* $\sin(t) * \text{step}_3(t)$. *Since* step_3 *is piecewise defined, we will, as suggested, use the integral formula*

$$\sin(t) * \text{step}_3(t) = \int_0^t \sin(t - x)\, \text{step}_3(x)\, dx$$

First, we compute the integral assuming $t < 3$. *This one is easy:*

$$\sin(t) * \text{step}_3(t) = \int_0^t \sin(t - x) \underbrace{\text{step}_3(x)}_{\substack{=0 \\ \text{since } x<t<3}} dx = \int_0^t \sin(t - x) \cdot 0\, dx = 0 \quad .$$

So,

$$\sin(t) * \text{step}_3(t) = 0 \qquad \text{if} \quad t < 3 \quad . \tag{28.7}$$

On the other hand, if $3 < t$, *then the interval of integration includes* $x = 3$, *the point at which the value of* $\text{step}_3(x)$ *radically changes from* 0 *to 1. Thus, we must break up our integral at the point* $x = 3$ *in computing* $h * f$:

$$\begin{aligned}
\sin(t) * \text{step}_3(t) &= \int_0^t \sin(t - x)\, \text{step}_3(x)\, dx \\
&= \int_0^3 \sin(t - x) \underbrace{\text{step}_3(x)}_{\substack{=0 \\ \text{since } x<3}} dx + \int_3^t \sin(t - x) \underbrace{\text{step}_3(x)}_{\substack{=1 \\ \text{since } 3<x}} dx \\
&= \int_0^3 \sin(t - x) \cdot 0\, dx + \int_3^t \sin(t - x) \cdot 1\, dx
\end{aligned}$$

$$= 0 + \cos(t-t) - \cos(t-3)$$
$$= 1 - \cos(t-3) \quad .$$

Thus,

$$\sin(t) * \text{step}_3(t) = 1 - \cos(t-3) \qquad \text{if} \quad 3 < t \quad . \tag{28.8}$$

Combining our two results (formulas (28.7) and (28.7)), we have the complete set of conditional formulas for our convolution,

$$\sin(t) * \text{step}_3(t) = \begin{cases} 0 & \text{if} \quad t < 3 \\ 1 - \cos(t-3) & \text{if} \quad 3 < t \end{cases} \quad .$$

Glance back at the above example and observe that, immediately after the computation of $\sin(t) * \text{step}_3(t)$ for each different case ($t < 3$ and $3 < t$), the resulting formula for the convolution was rewritten along with the values assumed for t (formulas (28.7) and (28.7), respectively). Do the same in your own computations! Always rewrite any derived formula for your convolution along with the values assumed for t. And write this someplace safe where you can easily find it. This is part of the bookkeeping, and helps ensure that you do not lose parts of your work when you compose the full set of conditional formulas for the convolution.

One more example should be quite enough.

!►*Example 28.10:* *Let's compute* $e^{-3t} * f(t)$ *where*

$$f(t) = \begin{cases} t & \text{if} \quad t < 2 \\ 2 & \text{if} \quad 2 < t < 4 \\ 0 & \text{if} \quad 4 < t \end{cases} \quad .$$

For this convolution, we can do a little "pre-computing" to simplify later steps:

$$e^{-3t} * f(t) = \int_0^t e^{-3(t-x)} f(x)\,dx$$
$$= \int_0^t e^{-3t+3x} f(x)\,dx = e^{-3t} \int_0^t e^{3x} f(x)\,dx \quad .$$

Now, if $t < 2$,

$$e^{-3t} * f(t) = e^{-3t} \int_0^t e^{3x} \underbrace{f(x)}_{\substack{=x \\ \text{since } x < t < 2}} dx = e^{-3t} \int_0^t e^{3x} x\,dx \quad .$$

This integral is easily computed using integration by parts, yielding

$$e^{-3t} * f(t) = e^{-3t} \left[\frac{t}{3} e^{3t} - \frac{1}{9} e^{3t} + \frac{1}{9} \right] = \frac{1}{9} \left[3t - 1 + e^{-3t} \right] \quad .$$

Thus,

$$e^{-3t} * f(t) = \frac{1}{9} \left[3t - 1 + e^{-3t} \right] \quad . \tag{28.9}$$

On the other hand, when $2 < t < 4$,

$$
\begin{aligned}
e^{-3t} * f(t) &= e^{-3t} \int_0^t e^{3x} f(x)\,dx \\
&= e^{-3t} \left[\int_0^2 e^{3x} \underbrace{f(x)}_{\substack{=x \\ \text{since } x<2}} dx \;+\; \int_2^t e^{3x} \underbrace{f(x)}_{\substack{=2 \\ \text{since } 2<x<t<4}} dx \right] \\
&= e^{-3t} \left[\int_0^2 e^{3x} x\,dx \;+\; \int_2^t e^{3x} \cdot 2\,dx \right] \\
&= \cdots \\
&= e^{-3t} \left[\frac{1}{9} \left(5e^6 + 1\right) \;+\; \frac{2}{3} \left(e^{3t} - e^6\right) \right] \\
&= \cdots \\
&= \frac{2}{3} \;+\; \frac{1}{9} \left[1 - e^6\right] e^{-3t} \quad .
\end{aligned}
$$

Thus,

$$
e^{-3t} * f(t) \;=\; \frac{2}{3} \;+\; \frac{1}{9} \left[1 - e^6\right] e^{-3t} \qquad \text{for} \quad 2 < t < 4 \quad . \tag{28.10}
$$

Finally, when $6 < t$,

$$
\begin{aligned}
e^{-3t} * f(t) &= e^{-3t} \int_0^t e^{3x} f(x)\,dx \\
&= e^{-3t} \left[\int_0^2 e^{3x} \underbrace{f(x)}_{\substack{=x \\ \text{since } x<2}} dx \;+\; \int_2^4 e^{3x} \underbrace{f(x)}_{\substack{=2 \\ \text{since } 2<x<4}} dx \;+\; \int_4^t e^{3x} \underbrace{f(x)}_{\substack{=0 \\ \text{since } 4<x}} dx \right] \\
&= e^{-3t} \left[\int_0^2 e^{3x} x\,dx \;+\; \int_2^4 e^{3x} \cdot 2\,dx \;+\; \int_4^t e^{3x} \cdot 0\,dx \right] \\
&= \cdots \\
&= e^{-3t} \left[\frac{1}{9} \left(5e^6 + 1\right) \;+\; \frac{2}{3} \left(e^{12} - e^6\right) \;+\; 0 \right] \\
&= \cdots \\
&= \frac{1}{9} \left[6e^{12} + 1 - e^6\right] e^{-3t} \quad .
\end{aligned}
$$

Thus,

$$
e^{-3t} * f(t) \;=\; \frac{1}{9} \left[6e^{12} + 1 - e^6\right] e^{-3t} \qquad \text{for} \quad 4 < t \quad . \tag{28.11}
$$

Putting it all together, equations (28.9), (28.10) and (28.11) give us

$$
e^{-3t} * f(t) \;=\; \begin{cases}
\dfrac{1}{9} \left[3t - 1 + e^{-3t}\right] & \text{if} \quad t < 2 \\
\dfrac{2}{3} \;+\; \dfrac{1}{9} \left[1 - e^6\right] e^{-3t} & \text{if} \quad 2 < t < 4 \\
\dfrac{1}{9} \left[6e^{12} + 1 - e^6\right] e^{-3t} & \text{if} \quad 4 < t
\end{cases} \quad .
$$

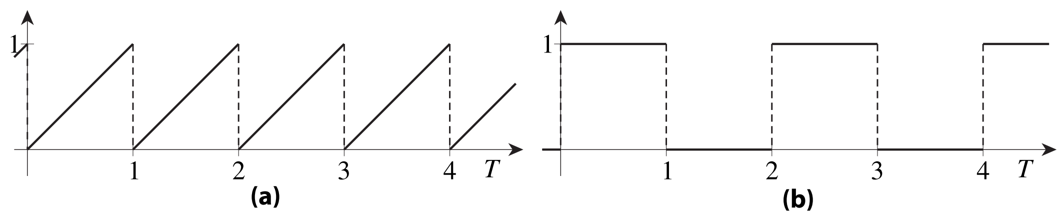

Figure 28.5: Two periodic functions: **(a)** a basic saw function, and **(b)** a basic square wave function.

28.5 Periodic Functions

Basics

Often, a function of interest f is *periodic with period* p for some positive value p. This means that the graph of the function remains unchanged when shifted to the left or right by p. This is equivalent to saying

$$f(t+p) = f(t) \qquad \text{for all} \quad t \quad . \tag{28.12}$$

You are well-acquainted with several periodic functions — the trigonometric functions, for example. In particular, the basic sine and cosine functions

$$\sin(t) \qquad \text{and} \qquad \cos(t)$$

are periodic with period $p = 2\pi$. But other periodic functions, such as the "saw" function sketched in figure 28.5a and the "square-wave" function sketched in figure 28.5b can arise in applications.

Strictly speaking, a truly periodic function is defined on the entire real line, $(-\infty, \infty)$. For our purposes, though, it will suffice to have f "periodic on $(0, \infty)$" with period p. This simply means that f is that part of a periodic function along the positive T–axis. What $f(t)$ is for $t < 0$ is irrelevant. Accordingly, for functions *periodic on* $(0, \infty)$, we modify requirement (28.12) to

$$f(t+p) = f(t) \qquad \text{for all} \quad t > 0 \quad . \tag{28.13}$$

In what follows, however, it will usually be irrelevant as to whether a given function is truly periodic or merely periodic on $(0, \infty)$, In either case, we will refer to the function as "periodic", and specify whether it is defined on all of $(-\infty, \infty)$ or just $(0, -\infty)$ only if necessary.

A convenient way to describe a periodic function f with period p is by

$$f(t) = \begin{cases} f_0(t) & \text{if} \quad 0 < t < p \\ f(t+p) & \text{in general} \end{cases} \quad .$$

The $f_0(t)$ is the formula for f over the *base period interval* $(0, p)$. The second line is simply telling us that the function is periodic and that equation (28.12) or (28.13) holds and can be used to compute the function at points outside of the base period interval. (The value of $f(t)$ at $t = 0$ and integral multiples of p are determined — or ignored — following the conventions for piecewise-defined functions discussed in section 28.1.)

!►Example 28.11: *Let* $\mathrm{saw}(t)$ *denote the basic saw function sketched in figure 28.5a. It clearly has period* $p = 1$ *, has jump discontinuities at integer values of* t *, and is given on* $(0,\infty)$ *by*

$$\mathrm{saw}(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ \mathrm{saw}(t+1) & \text{in general} \end{cases} .$$

In this case, the formula for computing $\mathrm{saw}(\tau)$ *when* $0 < \tau < 1$ *is*

$$\mathrm{saw}_0(\tau) = \tau \quad .$$

So, for example, $\mathrm{saw}(3/4) = 3/4$ *.*

On the other hand, to compute $\mathrm{saw}(\tau)$ *when* $\tau > 1$ *(and not an integer), we must use*

$$\mathrm{saw}(t+1) = \mathrm{saw}(t)$$

repeatedly until we finally reach in a value t *in the base period interval* $(0, 1)$ *. For example,*

$$\begin{aligned} \mathrm{saw}\left(\frac{8}{3}\right) &= \mathrm{saw}\left(\frac{5}{3}+1\right) = \mathrm{saw}\left(\frac{5}{3}\right) \\ &= \mathrm{saw}\left(\frac{2}{3}+1\right) = \mathrm{saw}\left(\frac{2}{3}\right) = \frac{2}{3} \quad . \end{aligned}$$

Often, the formula for the function over the base period interval is, itself, piecewise defined.

!►Example 28.12: *Let* $\mathrm{sqwave}(t)$ *denote the square-wave function in figure 28.5b. This function has period* $p = 2$ *, and, over its base period interval* $(0, 2)$ *, is given by*

$$\mathrm{sqwave}(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t < 2 \end{cases} .$$

So,

$$\mathrm{sqwave}(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t < 2 \\ \mathrm{sqwave}(t-2) & \text{in general} \end{cases} .$$

Before discussing Laplace transforms of periodic functions, let's make a couple of observations concerning a function f which is periodic with period p over $(0,\infty)$. We won't prove them. Instead, you should think about why these statements are "obviously true".

1. If f is piecewise continuous over $(0, p)$, then f is piecewise continuous over $(0,\infty)$.

2. If f is piecewise continuous over $(0, p)$, then f is of exponential order $s_0 = 0$.

Transforms of Periodic Functions

Suppose we want to find the Laplace transform

$$F(s) = \mathcal{L}[f(t)]|_s = \int_0^\infty f(t)e^{-st}\,dt$$

when f is piecewise continuous and periodic with period p. Because $f(t)$ satisfies

$$f(t) = f(t+p) \qquad \text{for} \quad t > 0 \quad ,$$

we should expect to (possibly) simplify our computations by partitioning the integral of the transform into integrals over subintervals of length p,

$$\begin{aligned} F(s) &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^p f(t)e^{-st}\,dt + \int_p^{2p} f(t)e^{-st}\,dt + \int_{2p}^{3p} f(t)e^{-st}\,dt \\ &\quad + \int_{3p}^{4p} f(t)e^{-st}\,dt + \int_{4p}^{5p} f(t)e^{-st}\,dt + \cdots \quad . \end{aligned}$$

For brevity, let's rewrite this as

$$F(s) = \sum_{k=0}^{\infty} \int_{kp}^{(k+1)p} f(t)e^{-st}\,dt \quad . \tag{28.14}$$

Now consider using the substitution $\tau = t - kp$ in the k^{th} term of this summation. Then $t = \tau + kp$,

$$e^{-st} = e^{-s(\tau+kp)} = e^{-kps}e^{-s\tau} \quad ,$$

and, by the periodicity of f,

$$\begin{aligned} f(\tau+p) &= f(\tau) \\ f(\tau+2p) &= f([\tau+p]+p) = f(\tau+p) = f(\tau) \\ f(\tau+3p) &= f([\tau+2p]+p) = f(\tau+2p) = f(\tau) \\ &\vdots \\ f(\tau+kp) &= \cdots = f(\tau) \quad . \end{aligned}$$

So,

$$\begin{aligned} \int_{t=kp}^{(k+1)p} f(t)e^{-st}\,dt &= \int_{\tau=kp-kp}^{(k+1)p-kp} f(\tau+kp)e^{-s(\tau+kp)}\,dt \\ &= \int_0^p f(\tau)e^{-kps}e^{-s\tau}\,dt = e^{-kps}\int_0^p f(\tau)e^{-s\tau}\,dt \quad . \end{aligned}$$

Note that the last integral does not depend on k. Consequently, combining the last result with equation (28.14), we have

$$F(s) = \sum_{k=0}^{\infty} e^{-kps}\int_0^p f(\tau)e^{-s\tau}\,dt = \left[\sum_{k=0}^{\infty} e^{-kps}\right]\int_0^p f(\tau)e^{-s\tau}\,d\tau \quad .$$

Here we have an incredible stroke of luck, at least if you recall what a geometric series is and how to compute its sum. Assuming you do recall this, we have

$$\sum_{k=0}^{\infty} e^{-kps} = \sum_{k=0}^{\infty} \left[e^{-ps}\right]^k = \frac{1}{1-e^{-ps}} \quad . \tag{28.15}$$

We also have this if you do not recall about geometric series, but will you certainly want to go to the addendum on page 576 to see how we get this equation.

Whether or not you recall about geometric series, equation (28.15) combined with the last formula for F (along with the observations made earlier regarding piecewise continuity and periodic functions) gives us the following theorem.

Theorem 28.2
Let f be a piecewise continuous and periodic function with period p. Then its Laplace transform F is given by

$$F(s) = \frac{F_0(s)}{1 - e^{-ps}} \qquad \text{for} \quad s > 0$$

where

$$F_0(s) = \int_0^p f(t)e^{-st}\,dt \quad .$$

There are at least two alternative ways of describing F_0 in the above theorem. First of all, if f is given by

$$f(t) = \begin{cases} f_0(t) & \text{if} \quad 0 < t < p \\ f(t+p) & \text{in general} \end{cases} \quad ,$$

then, of course,

$$F_0(s) = \int_0^p f_0(t)e^{-st}\,dt \quad .$$

Also, using the fact that

$$\int_0^p f_0(t)e^{-st}\,dt = \int_0^\infty f_0(t)\,\text{rect}_{(0,p)}(t)\,e^{-st}\,dt \quad ,$$

we see that

$$F_0(s) = \mathcal{L}\big[f_0(t)\,\text{rect}_{(0,p)}(t)\big]\big|_s$$

or, equivalently, that

$$F_0(s) = \mathcal{L}\big[f(t)\,\text{rect}_{(0,p)}(t)\big]\big|_s \quad .$$

Whether any of alternative descriptions of $F_0(s)$ is useful may depend on what transforms you have already computed.

!► Example 28.13: *Let's find the Laplace transform of the saw function from example 28.11 and sketched in figure 28.5a,*

$$\text{saw}(t) = \begin{cases} t & \textit{if} \quad 0 < t < 1 \\ \text{saw}(t+1) & \textit{in general} \end{cases} \quad .$$

Here, $p = 1$, and the last theorem tells us that

$$\mathcal{L}[\text{saw}(t)]|_s = \frac{F_0(s)}{1 - e^{-1\cdot s}} = \frac{F_0(s)}{1 - e^{-s}} \qquad \text{for} \quad s > 0$$

where (using each of the formulas discussed for F_0)

$$F_0(s) = \int_0^1 \operatorname{saw}(t) e^{-st}\, dt \tag{28.16a}$$

$$= \int_0^1 t e^{-st}\, dt \tag{28.16b}$$

$$= \mathcal{L}\big[t \operatorname{rect}_{(0,1)}(t)\big]\big|_s \quad . \tag{28.16c}$$

Had the author been sufficiently clever, $\mathcal{L}\big[t \operatorname{rect}_{(0,1)}(t)\big]$ would have already been computed in a previous example, and we could write out the final result using formula (28.16c). But he wasn't, so let's just compute $F_0(s)$ using formula (28.16b) and integration by parts:

$$\begin{aligned} F_0(s) &= \int_0^1 t e^{-st}\, dt \\ &= -\frac{t}{s} e^{-st}\Big|_{t=0}^{1} - \int_0^1 \left(-\frac{1}{s}\right) e^{-st}\, dt \\ &= -\frac{1}{s} e^{-s\cdot 1} + 0 - \frac{1}{s^2}\left[e^{-s\cdot 1} - e^{-s\cdot 0}\right] = \frac{1}{s^2}\left[1 - e^{-s} - s e^{-s}\right] \quad . \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}[\operatorname{saw}(t)]|_s &= \frac{F_0(s)}{1 - e^{-s}} \\ &= \frac{1}{s^2} \cdot \frac{1 - e^{-s} - s e^{-s}}{1 - e^{-s}} \\ &= \frac{1}{s^2}\left[1 - \frac{s e^{-s}}{1 - e^{-s}}\right] = \frac{1}{s^2} - \frac{1}{s} \cdot \frac{e^{-s}}{1 - e^{-s}} \quad . \end{aligned}$$

This is our transform. If you wish, you can apply a little algebra and 'simplify' it to

$$\mathcal{L}[\operatorname{saw}(t)]|_s = \frac{1}{s^2} - \frac{1}{s} \cdot \frac{1}{e^s - 1} \quad ,$$

though you may prefer to keep the formula with $1 - e^{-ps}$ in the denominator to remind you that this transform came from a periodic function with period p .

Just for fun, let's go even further using the fact that

$$\frac{e^{-s}}{1 - e^{-s}} = \frac{e^{-s}}{1 - e^{-s}} \cdot \frac{2e^{s/2}}{2e^{s/2}} = \frac{1}{2} \cdot \frac{2e^{-s/2}}{e^{s/2} - e^{-s/2}} = \frac{1}{2} \cdot \frac{e^{-s/2}}{\sinh(s/2)} \quad .$$

Thus, the above formula for the Laplace transform of the saw function can also be written as

$$\mathcal{L}[\operatorname{saw}(t)]|_s = \frac{1}{s^2} - \frac{1}{2s} \cdot \frac{e^{-s/2}}{\sinh(s/2)} \quad .$$

This is significant only in that it demonstrates why hyperbolic trigonometric functions are sometimes found in tables of transforms.

Addendum: Verifying Equation (28.15)

Equation (28.15) gives a formula for adding up

$$\sum_{k=0}^{\infty} e^{-kps}$$

assuming p and s are positive values. To derive that formula, we start with the N^{th} partial sum of the series,

$$S_N = \sum_{k=0}^{N} e^{-kps} = \underbrace{e^{-0ps}}_{=1} + e^{-1ps} + e^{-2ps} + e^{-3ps} + \cdots + e^{-Nps} \quad .$$

Multiplying this by e^{-ps}, we get

$$\begin{aligned} e^{-ps} S_N &= e^{-ps}\left[1 + e^{-1ps} + e^{-2ps} + e^{-3ps} + \cdots + e^{-Nps}\right] \\ &= e^{-1ps} + e^{-2ps} + e^{-3ps} + \cdots + e^{-Nps} + e^{-(N+1)ps} \quad . \end{aligned}$$

The similarity between S_N and $e^{-ps} S_N$ naturally leads us to compute their difference,

$$\begin{aligned} \left[1 - e^{-ps}\right] S_N &= S_N - e^{-ps} S_N \\ &= \left\{1 + e^{-1ps} + e^{-2ps} + e^{-3ps} + \cdots + e^{-Nps}\right\} \\ &\quad - \left\{e^{-1ps} + e^{-2ps} + e^{-3ps} + \cdots + e^{-Nps} + e^{-(N+1)ps}\right\} \\ &= 1 - e^{-(N+1)ps} \quad . \end{aligned}$$

Dividing through by $1 - e^{-ps}$ then yields

$$S_N = \frac{1 - e^{-(N+1)ps}}{1 - e^{-ps}} \quad .$$

The above formula for S_N holds for any choice of p and s, but if $p > 0$ and $s > 0$, as assumed here, then

$$\lim_{N \to \infty} e^{-(N+1)ps} = 0 \quad ,$$

and thus,

$$\begin{aligned} \sum_{k=0}^{\infty} e^{-kps} &= \lim_{N\to\infty} \sum_{k=0}^{N} e^{-kps} = \lim_{N\to\infty} S_N \\ &= \lim_{N\to\infty} \frac{1 - e^{-(N+1)ps}}{1 - e^{-ps}} = \frac{1 - 0}{1 - e^{-ps}} = \frac{1}{1 - e^{-ps}} \quad , \end{aligned}$$

confirming equation (28.15).

Table 28.1: Commonly Used Identities (Version 2)

In the following, $F(s) = \mathcal{L}[f(t)]|_s$.

$h(t)$	$H(s) = \mathcal{L}[h(t)]\|_s$	Restrictions
$f(t)$	$\displaystyle\int_0^\infty f(t)e^{-st}\,dt$	
$e^{\alpha t} f(t)$	$F(s-\alpha)$	α is real
$f(t-\alpha)\,\text{step}_\alpha(t)$	$F(s)\,e^{-\alpha s}$	$\alpha > 0$
$\dfrac{df}{dt}$	$sF(s) - f(0)$	
$\dfrac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0) - f'(0)$	
$\dfrac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0)$ $- s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$	$n = 1, 2, 3, \ldots$
$t\,f(t)$	$-\dfrac{dF}{ds}$	
$t^n\,f(t)$	$(-1)^n \dfrac{d^n F}{ds^n}$	$n = 1, 2, 3, \ldots$
$\displaystyle\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	
$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(\sigma)\,d\sigma$	
$f * g(t)$	$F(s)G(s)$	
f is periodic with period p	$\dfrac{\int_0^p f(t)e^{-st}\,dt}{1 - e^{-ps}}$	

28.6 An Expanded Table of Identities

For reference, let us write out a new table of Laplace transform identities containing the identities listed in our first table of Laplace transform identities, table 25.1 on page 514, along with some of the more important identities derived after making that table. Our new table is table 28.1.

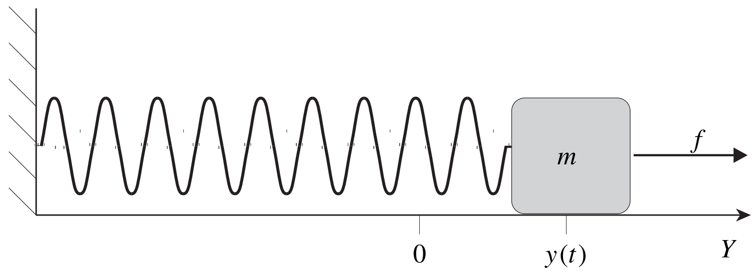

Figure 28.6: A mass/spring system with mass m and an outside force f acting on the mass.

28.7 Duhamel's Principle and Resonance

The Problem

Now is a good time to re-examine some of those "forced" mass/spring systems originally discussed in chapters 17 and 22, and diagramed in figure 28.6. Recall that this system is modeled by

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = f$$

where $y = y(t)$ is the position of the mass at time t (with $y = 0$ being the "equilibrium" position of the mass when $f = 0$), m is the mass of the object attached to the spring, κ is the spring constant, γ is the damping constant, and $f = f(t)$ is the sum of all forces acting on the spring other than the damping friction and the spring's reaction to being stretched and compressed (f was called F_{other} in chapter 17 and F in chapter 22). Remember, also, that m and κ are positive constants.

Our main interest will be in the phenomenon of resonance in an undamped system. Accordingly, we will assume $\gamma = 0$, and restrict our attention to solving

$$m\frac{d^2y}{dt^2} + \kappa y = f \quad . \tag{28.17}$$

Ultimately, we will further restrict our attention to cases in which f is periodic. But let's wait on that, and derive some basic formulas without assuming this periodicity.

Solutions Using Arbitrary f

The General Solution

As you know quite well by now, the general solution to our differential equation, equation (28.17), is

$$y(t) = y_p(t) + y_h(t)$$

where y_h is the general solution to the corresponding homogeneous differential equation, and y_p is any particular solution to the given nonhomogeneous differential equation.

The formula for y_h is already known. In chapter 17, we found that

$$y_h(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) \qquad \text{where} \quad \omega_0 = \sqrt{\frac{\kappa}{m}} \quad .$$

Recall that ω_0 is the natural angular frequency of the mass/spring system, and is related to the system's natural frequency ν_0 and natural period p_0 via

$$\nu_0 = \frac{\omega_0}{2\pi} \qquad \text{and} \qquad p_0 = \frac{1}{\nu_0} = \frac{2\pi}{\omega_0} \quad .$$

For future use, note that y_h is a periodic function with period p_0 ; hence

$$y_h(t + p_0) - y_h(t) = 0 \qquad \text{for all} \quad t \quad .$$

That leaves finding a particular solution y_p . Let's take this to be the solution to the initial-value problem

$$m\frac{d^2y}{dt^2} + \kappa y = f \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0 \quad .$$

This is easily found by either applying the Laplace transform and using the convolution identity in taking the inverse transform, or by appealing directly to Duhamel's principle. Either way, we get

$$y_p(t) = h * f(t) = \int_0^t h(t - x) f(x)\, dx$$

where

$$h(\tau) = \mathcal{L}^{-1}\left[\frac{1}{ms^2 + \kappa}\right]\Bigg|_\tau = \frac{1}{m}\mathcal{L}^{-1}\left[\frac{1}{s^2 + \kappa/m}\right]\Bigg|_\tau \quad .$$

Since $\omega_0 = \sqrt{\kappa/m}$,

$$h(\tau) = \frac{1}{m}\mathcal{L}^{-1}\left[\frac{1}{s^2 + (\omega_0)^2}\right]\Bigg|_\tau = \frac{1}{\omega_0 m}\sin(\omega_0\tau) \quad .$$

Thus, the above integral formula for y_p can be written as

$$y_p(t) = \frac{1}{\omega_0 m}\int_0^t \sin(\omega_0[t - x])\, f(x)\, dx \quad . \tag{28.18}$$

The Difference Formula and First Theorem

For our studies, we will want to see how any solution y varies "over a cycle" (i.e., as t increases by p_0). This variance in y over a cycle is given by the difference $y(t + p_0) - y(t)$, and will be especially meaningful when the forcing function is periodic with period p_0 .

For now, let's consider the difference $y(t+p_0)-y(t)$ assuming $y = y_p+y_h$ is any solution to our differential equation. Of course, the y_h term is irrelevant because of its periodicity,

$$\begin{aligned} y(t + p_0) - y(t) &= \big[y_p(t + p_0) + y_h(t + p_0)\big] - \big[y_p(t) + y_h(t)\big] \\ &= y_p(t + p_0) - y_p(t) + \underbrace{y_h(t + p_0) - y_h(t)}_{0} \quad . \end{aligned}$$

Now, using formula (28.18) for y_p , we see that

$$\begin{aligned} y_p(t + p_0) &= \frac{1}{\omega_0 m}\int_0^{t+p_0} \sin(\omega_0[(t + p_0) - x])\, f(x)\, dx \\ &= \frac{1}{\omega_0 m}\int_0^{t+p_0} \sin(\omega_0[t - x] + \underbrace{\omega_0 p_0}_{2\pi})\, f(x)\, dx \end{aligned}$$

$$= \frac{1}{\omega_0 m} \int_0^{t+p_0} \sin(\omega_0[t-x])\, f(x)\, dx$$

$$= \frac{1}{\omega_0 m} \int_0^{t} \sin(\omega_0[t-x])\, f(x)\, dx \;+\; \frac{1}{\omega_0 m} \int_t^{t+p_0} \sin(\omega_0[t-x])\, f(x)\, dx \quad .$$

But the first integral in the last line is simply the integral formula for $y_p(t)$ given in equation (28.18). So the above reduces to

$$y(t+p_0) \;=\; y(t) \;+\; \frac{1}{\omega_0 m} \int_t^{t+p_0} \sin(\omega_0[t-x])\, f(x)\, dx \quad . \tag{28.19}$$

To further "reduce" our difference formula, let us use a well-known trigonometric identity:

$$\begin{aligned}
&\int_t^{t+p_0} \sin(\omega_0[t-x])\, f(x)\, dx \\
&\qquad = \int_t^{t+p_0} \sin(\omega_0 t - \omega_0 x)\, f(x)\, dx \\
&\qquad = \int_t^{t+p_0} [\sin(\omega_0 t)\cos(\omega_0 x) \;-\; \cos(\omega_0 t)\sin(\omega_0 x)]\, f(x)\, dx \\
&\qquad = \sin(\omega_0 t) \int_t^{t+p_0} \cos(\omega_0 x)\, f(x)\, dx \;-\; \cos(\omega_0 t) \int_t^{t+p_0} \sin(\omega_0 x)\, f(x)\, dx \quad .
\end{aligned}$$

Combining this result with the last equation for $y(t+p_0)$ and recalling the previous results derived in this section then yields:

Theorem 28.3

Let m and κ be positive constants, and let f be any piecewise continuous function of exponential order. Then, the general solution to

$$m\frac{d^2y}{dt^2} \;+\; \kappa y \;=\; f$$

is

$$y(t) \;=\; y_p(t) \;+\; c_1\cos(\omega_0 t) \;+\; c_2\sin(\omega_0 t)$$

where

$$\omega_0 \;=\; \sqrt{\frac{\kappa}{m}} \qquad \text{and} \qquad y_p(t) \;=\; \frac{1}{\omega_0 m} \int_0^{t} \sin(\omega_0[t-x])\, f(x)\, dx \quad .$$

Moreover,

$$y(t+p_0) \;-\; y(t) \;=\; \frac{1}{\omega_0 m}\, [\mathcal{I}_S(t)\sin(\omega_0 t) \;+\; \mathcal{I}_C(t)\cos(\omega t)] \qquad \text{for} \quad t \geq 0$$

where

$$\mathcal{I}_S(t) \;=\; \int_t^{t+p_0} \cos(\omega_0 x)\, f(x)\, dx \qquad \text{and} \qquad \mathcal{I}_C(t) \;=\; -\int_t^{t+p_0} \sin(\omega_0 x)\, f(x)\, dx \quad .$$

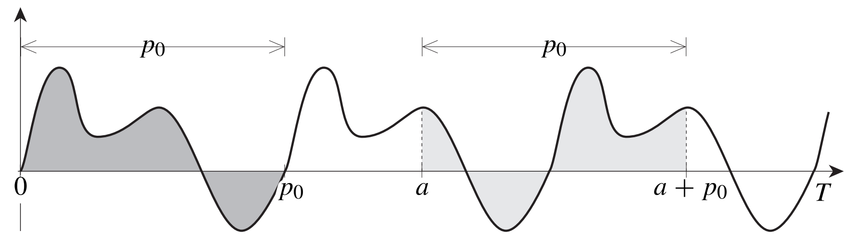

Figure 28.7: Illustration for lemma 28.4.

Resonance from Periodic Forcing Functions
A Useful Fact

Take a look at figure 28.7. It shows the graph of some periodic function g with period p_0, and with two regions of width p_0 "greyed in" in two shades of grey. The darker grey region is between the graph and the T–axis with $0 < t < p_0$. The lighter grey region is between the graph and the T–axis with $a < t < a + p_0$ for some real number a. Note the similarity in the shapes of the regions. In particular, note how the pieces of the lighter grey region can be rearranged to perfectly match the darker grey region. Consequently, the areas in each of these two regions, both above and below the T–axis, are the same. Add to this the relationship between "integrals" and "area", and you get the useful fact stated in the next lemma.

Lemma 28.4
Let g be a periodic, piecewise continuous function with period p_0. Then, for any t,

$$\int_t^{t+p_0} g(x)\,dx = \int_0^{p_0} g(x)\,dx \quad .$$

If you wish, you can rigorously prove this lemma using some basic theory from elementary calculus.

?►*Exercise 28.1:* *Prove lemma 28.4. A good start would be to show that*

$$\frac{d}{dt}\int_t^{t+p_0} g(x)\,dx = 0 \quad .$$

Resonance

Now consider the formulas for $\mathcal{I}_S(t)$ and $\mathcal{I}_C(t)$ from theorem 28.3,

$$\mathcal{I}_S(t) = \int_t^{t+p_0} \cos(\omega_0 x)\, f(x)\,dx \qquad \text{and} \qquad \mathcal{I}_C(t) = -\int_t^{t+p_0} \sin(\omega_0 x)\, f(x)\,dx \quad .$$

If f is also periodic with period p_0, then the products in these integrals are also periodic, each with period p_0. Lemma 28.4 then tells us that

$$\mathcal{I}_S(t) = \int_t^{t+p_0} \cos(\omega_0 x)\, f(x)\,dx = \int_0^{p_0} \cos(\omega_0 x)\, f(x)\,dx$$

and

$$\mathcal{I}_C(t) \;=\; -\int_t^{t+p_0} \sin(\omega_0 x)\, f(x)\, dx \;=\; -\int_0^{p_0} \sin(\omega_0 x)\, f(x)\, dx \quad .$$

Thus, if f is periodic with period p_0, the difference formula in theorem 28.3 reduces to

$$y(t+p_0) \;-\; y(t) \;=\; \frac{1}{\omega_0 m}\left[\mathcal{I}_S \sin(\omega_0 t) \;+\; \mathcal{I}_C \cos(\omega t)\right]$$

where $\mathcal{I}_S$ and $\mathcal{I}_C$ are the *constants*

$$\mathcal{I}_S \;=\; \int_0^{p_0} \cos(\omega_0 x)\, f(x)\, dx \qquad \text{and} \qquad \mathcal{I}_C \;=\; -\int_0^{p_0} \sin(\omega_0 x)\, f(x)\, dx \quad .$$

Using a little more trigonometry (see the derivation of formula (17.8b) on page 363), we can reduce this to the even more convenient form given in the the next theorem.

Theorem 28.5 (resonance in undamped systems)
Let m and p be positive constants, and f a periodic piecewise continuous function. Assume further that f has period p_0, the natural period of the mass/spring system modeled by

$$m\frac{d^2y}{dt^2} \;+\; \kappa y \;=\; f \quad .$$

That is,

$$\textit{period of } f \;=\; p_0 \;=\; \frac{2\pi}{\omega_0} \qquad \textit{with} \qquad \omega_0 \;=\; \sqrt{\frac{\kappa}{m}} \quad .$$

Also let

$$\mathcal{I}_S \;=\; \int_0^{p_0} \cos(\omega_0 x)\, f(x)\, dx \qquad \textit{and} \qquad \mathcal{I}_C \;=\; -\int_0^{p_0} \sin(\omega_0 x)\, f(x)\, dx \quad .$$

Then, for any solution y to the above differential equation, and any $t > 0$,

$$y(t+p_0) \;-\; y(t) \;=\; A\cos(\omega_0 t - \phi) \tag{28.20}$$

where

$$A \;=\; \frac{1}{\omega_0 m}\sqrt{(\mathcal{I}_S)^2 \;+\; (\mathcal{I}_C)^2}$$

and with ϕ being the constant satisfying $0 \le \phi < 2\pi$,

$$\cos(\phi) \;=\; \frac{\mathcal{I}_C}{\sqrt{(\mathcal{I}_S)^2 \;+\; (\mathcal{I}_C)^2}} \qquad \textit{and} \qquad \sin(\phi) \;=\; \frac{\mathcal{I}_S}{\sqrt{(\mathcal{I}_S)^2 \;+\; (\mathcal{I}_C)^2}} \quad .$$

To see what all this implies, assume f, y, etc. are as in the theorem, and look at what the difference formula tells us about $y(t_n)$ when τ is any fixed value in $[0, p_0)$, and

$$t_n \;=\; \tau \;+\; np_0 \qquad \text{for} \quad n = 1, 2, 3, \ldots \quad .$$

The value of $y(\tau)$ can be computed using the integral formula for y_p in theorem 28.3. To compute each $y(t_n)$, however, it is easier to use this computed value for $y(\tau)$ along with difference formula (28.20) and the fact that, for any integer k,

$$\cos(\omega_0 t_k - \phi) \;=\; \cos(\omega_0[\tau + kp_0] - \phi) \;=\; \cos(\omega_0 \tau - \phi + k\underbrace{\omega_0 p_0}_{2\pi}) \;=\; \cos(\omega_0 \tau - \phi) \quad .$$

Doing so, we get

$$y(t_1) \;=\; y(\tau + p_0) \;=\; y(\tau) \;+\; A\cos(\omega_0\tau - \phi) \quad ,$$

$$\begin{aligned} y(t_2) \;=\; y(\tau + 2p_0) \;&=\; y(\tau + p_0) \;+\; A\cos(\omega_0[\tau + p_0] - \phi) \\ &=\; [y(\tau) + A\cos(\omega_0 t - \phi)] \;+\; A\cos(\omega_0\tau - \phi) \\ &=\; y(\tau) \;+\; 2A\cos(\omega_0\tau - \phi) \quad , \end{aligned}$$

$$\begin{aligned} y(t_3) \;=\; y(\tau + 3p_0) \;&=\; y(\tau + 2p_0) \;+\; A\cos(\omega_0[\tau + 2p_0] - \phi) \\ &=\; [y(\tau) + 2A\cos(\omega_0 t - \phi)] \;+\; A\cos(\omega_0\tau - \phi) \\ &=\; y(\tau) \;+\; 3A\cos(\omega_0\tau - \phi) \quad , \end{aligned}$$

and so on. In general,

$$y(t_n) \;=\; y(\tau) \;+\; nA\cos(\omega_0\tau - \phi) \quad . \tag{28.21}$$

Clearly, if $A \neq 0$ and $\omega_0\tau - \phi$ is neither $\pi/2$ or $3\pi/2$, then

$$y(t_n) \;\to\; \pm\infty \quad \text{as} \quad n \to \infty \quad .$$

This is clearly "runaway resonance".

Thus, it is the A in difference formula (28.20) that determines if we have "runaway resonance". If $A \neq 0$, the solution contains an oscillating term with a steadily increasing amplitude. On the other hand, if $A = 0$, then the solution y is periodic, and does not "blow up".

By the way, for graphing purposes it may be convenient to use the periodicity of the cosine term and rewrite equation (28.21) as

$$y(t_n) \;=\; y(\tau) \;+\; nA\cos(\omega_0 t_n - \phi) \quad .$$

Replacing t_n with t, and recalling what n and τ represent, we see that this is the same as saying

$$y(t) \;=\; y(\tau) \;+\; nA\cos(\omega_0 t - \phi) \tag{28.22}$$

where n is the largest integer such that $np_0 \leq t$ and $\tau \;=\; t - np_0$.

!►Example 28.14: *Let us use the theorems in this section to analyze the response of an undamped mass/spring system with natural period $p_0 = 1$ to a force f given by the basic saw function sketched in figure 28.8a,*

$$f(t) \;=\; \text{saw}(t) \;=\; \begin{cases} t & \text{if} \quad 0 < t < 1 \\ \text{saw}(t-1) & \text{if} \quad 1 < t \end{cases} \quad .$$

The corresponding natural angular frequency is

$$\omega_0 \;=\; \frac{2\pi}{p_0} \;=\; 2\pi \quad .$$

The actual values of the mass m and spring constant κ are irrelevant provided they satisfy

$$2\pi \;=\; \omega_0 \;=\; \sqrt{\frac{\kappa}{m}} \quad .$$

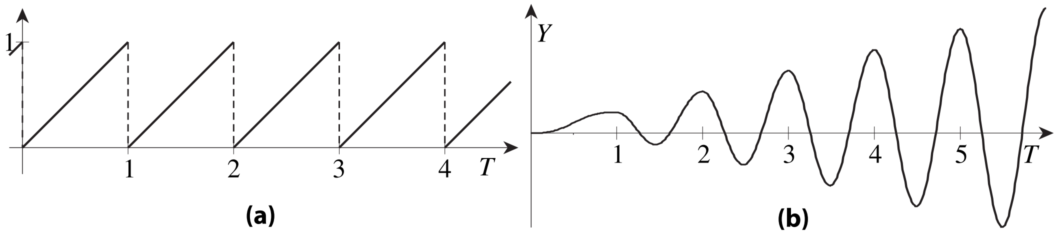

Figure 28.8: **(a)** A basic saw function, and **(b)** the corresponding response of an undamped mass/spring system with natural period 1 over 6 cycles.

Also, since the solution to the corresponding homogeneous differential equation was pretty much irrelevant in the discussion leading to our last theorem, let's assume our solution satisfies

$$y(0) = 0 \quad \textit{and} \quad y'(0) = 0 \quad ,$$

so that the solution formula described in theorem 28.3 becomes

$$y(t) = y_p(t) = \frac{1}{2\pi m}\int_0^t \sin(\omega_0[t-x])\,\mathrm{saw}(x)\,dx \quad .$$

In particular, if $0 \le x \le t < 1$ *, then* $\mathrm{saw}(x) = x$ *and we can complete our computations of* $y(t)$ *using integration by parts:*

$$\begin{aligned} y(t) &= \frac{1}{2\pi m}\int_0^t \sin(2\pi[t-x])\,x\,dx \\ &= \frac{1}{2\pi m}\left[\frac{x}{2\pi}\cos(2\pi[t-x])\bigg|_{x=0}^{t} - \int_0^t \frac{1}{2\pi}\cos(2\pi[t-x])\,dx\right] \\ &= \frac{1}{2\pi m}\left[\frac{t}{2\pi}\cos(2\pi[t-t]) - \frac{0}{2\pi}\cos(2\pi[t-0])\right. \\ &\qquad\qquad \left. + \frac{1}{(2\pi)^2}\sin(2\pi[t-t]) - \frac{1}{(2\pi)^2}\sin(2\pi[t-0])\right] \quad . \end{aligned}$$

This simplifies to

$$y(t) = \frac{1}{8\pi^3 m}\left[2\pi t - \sin(2\pi t)\right] \qquad \textit{when} \quad 0 \le t < 1 \quad . \tag{28.23}$$

In a similar manner, we find that

$$\mathcal{I}_S = \int_0^{p_0} \cos(\omega_0 x)\,f(x)\,dx = \int_0^1 \cos(2\pi x)\,x\,dx = \cdots = 0$$

and

$$\mathcal{I}_C = -\int_0^{p_0} \sin(\omega_0 x)\,f(x)\,dx = -\int_0^1 \sin(2\pi x)\,x\,dx = \cdots = \frac{1}{2\pi} \quad .$$

Thus,

$$A = \frac{1}{2\pi m}\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2} = \frac{1}{4\pi^2 m} \quad .$$

Since $A \neq 0$*, we have resonance. There is an oscillatory term whose amplitude steadily increases as* t *increases.*

To actually graph our solution, we still need to find the phase, ϕ, which (according to our last theorem) is the value in $[0, 2\pi)$ such that

$$\cos(\phi) = \frac{\mathcal{I}_C}{\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2}} = 1 \qquad \text{and} \qquad \sin(\phi) = \frac{\mathcal{I}_S}{\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2}} = 0 \quad .$$

Clearly $\phi = 0$.

So let $t \geq 0$. Then, employing formula (28.22) (derived just before this example),

$$\begin{aligned} y(t) &= y(\tau) + nA\cos(\omega_0 t - \phi) \\ &= \frac{1}{8\pi^3 m}[2\pi\tau - \sin(2\pi\tau)] + \frac{n}{4\pi^2 m}\cos(2\pi t) \end{aligned}$$

where (since $p_0 = 1$) n is the largest integer with $n \leq t$ and $\tau = t - n$. This is the function graphed in figure 28.8b.

Additional Exercises

28.2. *Using the first translation identity or one of the differentiation identities, compute each of the following:*

a. $\mathcal{L}\left[e^{4t}\,\text{step}_6(t)\right]\Big|_s$ **b.** $\mathcal{L}\left[t\,\text{step}_6(t)\right]\Big|_s$

28.3. *Compute (using the translation along the T–axis identity) and then graph the inverse transforms of the following functions:*

a. $\dfrac{e^{-4s}}{s^3}$ **b.** $\dfrac{e^{-3s}}{s+2}$ **c.** $\sqrt{\pi}\,s^{-3/2}e^{-s}$

d. $\dfrac{\pi}{s^2+\pi^2}e^{-2s}$ **e.** $\dfrac{e^{-4s}}{(s-5)^3}$ **f.** $\dfrac{(s+2)e^{-5s}}{(s+2)^2+16}$

28.4. *Finish solving the differential equation in example 28.1.*

28.5. *Compute and then graph the inverse transforms of the following functions (express your answers as conditional sets of formulas):*

a. $\dfrac{1-e^{-s}}{s^2}$ **b.** $\dfrac{e^{-s}+e^{-3s}}{s}$ **c.** $\dfrac{2}{s^3} - \dfrac{2+4s}{s^3}e^{-2s}$

d. $\dfrac{\pi\left(1+e^{-s}\right)}{s^2+\pi^2}$ **e.** $\dfrac{(s+4)e^{-12}-8e^{-3s}}{s^2-16}$ **f.** $\dfrac{e^{-2s}-2e^{-4s}+e^{-6s}}{s^2}$

28.6. *Find and graph the solution to each of the following initial-value problems:*

a. $y' = \text{step}_3(t)$ *with* $y(0) = 0$

b. $y' = \text{step}_3(t)$ *with* $y(0) = 4$

c. $y'' = \text{step}_2(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

d. $y'' = \text{step}_2(t)$ *with* $y(0) = 4$ *and* $y'(0) = 6$

e. $y'' + 9y = \text{step}_{10}(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

28.7. *Compute the Laplace transforms of the following functions using the translation along the T–axis identity. (Trigonometric identities may also be useful for some of these.)*

a. $f(t) = \begin{cases} 0 & \text{if } t < 6 \\ e^{4t} & \text{if } 6 < t \end{cases}$

b. $g(t) = \begin{cases} 0 & \text{if } t < 4 \\ \dfrac{1}{\sqrt{t-4}} & \text{if } 4 < t \end{cases}$

c. $t \operatorname{step}_6(t)$

d. $te^{3t} \operatorname{step}_2(t)$

e. $t^2 \operatorname{step}_6(t)$

f. $\sin(2(t-1)) \operatorname{step}_1(t)$

g. $\sin(2t) \operatorname{step}_{\pi/2}(t)$

h. $\sin(2t) \operatorname{step}_{\pi/4}(t)$

i. $\sin(2t) \operatorname{step}_{\pi/6}(t)$

28.8. *For each of the following choices of f :*

i. *Graph the given function over the positive T–axis.*

ii. *Rewrite the function in terms of appropriate rectangle functions, and then rewrite that in terms of appropriate step functions.*

iii. *Then find the Laplace transform $F(s) = \mathcal{L}[f(t)]|_s$.*

a. $f(t) = \begin{cases} e^{-4t} & \text{if } t < 6 \\ 0 & \text{if } 6 < t \end{cases}$

b. $f(t) = \begin{cases} 2t - t^2 & \text{if } t < 2 \\ 0 & \text{if } 2 < t \end{cases}$

c. $f(t) = \begin{cases} 2 & \text{if } t < 3 \\ 2e^{-4(t-3)} & \text{if } 3 < t \end{cases}$

d. $f(t) = \begin{cases} \sin(\pi t) & \text{if } t < 1 \\ 0 & \text{if } 1 < t \end{cases}$

e. $f(t) = \begin{cases} t^2 & \text{if } t < 3 \\ 9 & \text{if } 3 < t \end{cases}$

f. $f(t) = \begin{cases} 0 & \text{if } t < 2 \\ 3 & \text{if } 2 < t < 4 \\ 0 & \text{if } 4 < t \end{cases}$

g. $f(t) = \begin{cases} 1 & \text{if } t < 1 \\ 2 & \text{if } 2 < t < 3 \\ 4 & \text{if } 3 < t \end{cases}$

h. $f(t) = \begin{cases} 0 & \text{if } t < 1 \\ \sin(\pi t) & \text{if } 1 < t < 2 \\ 0 & \text{if } 2 < t \end{cases}$

i. $f(t) = \begin{cases} 0 & \text{if } t \le 1 \\ (t-1)^2 & \text{if } 1 < t < 3 \\ 4 & \text{if } 3 \le t \end{cases}$

j. $f(t) = \begin{cases} t & \text{if } t \le 2 \\ 4-t & \text{if } 2 < t < 4 \\ 0 & \text{if } 3 \le t \end{cases}$

28.9. *The infinite stair function,* $\operatorname{stair}(t)$, *can be described in terms of rectangle functions by*

$$\begin{aligned} \operatorname{stair}(t) &= \sum_{n=0}^{\infty} (n+1) \operatorname{rect}_{(n,n+1)}(t) \\ &= 1 \operatorname{rect}_{(0,1)}(t) + 2 \operatorname{rect}_{(1,2)}(t) + 3 \operatorname{rect}_{(2,3)}(t) + 4 \operatorname{rect}_{(3,4)}(t) + \cdots \quad . \end{aligned}$$

Using this:

a. *Sketch the graph of* $\text{stair}(t)$ *over the positive* T*–axis, and rewrite the formula for* $\text{stair}(t)$ *in terms of step functions.*

b. *Assuming the linearity of the Laplace transform holds for infinite sums as well as finite sums, find an infinite sum formula for* $\mathcal{L}[\text{stair}(t)]|_s$.

c. *Recall the formula for the sum of a geometric series,*

$$\sum_{n=0}^{\infty} X^n = \frac{1}{1-X} \qquad \text{when} \quad |X| < 1 \quad .$$

Using this, simplify the infinite sum formula for $\mathcal{L}[\text{stair}(t)]|_s$ *which you (we hope) obtained in the previous part of this exercise.*

28.10. *Find and graph the solution to each of the following initial-value problems:*

a. $y' = \text{rect}_{(1,3)}(t) \qquad \text{with} \quad y(0) = 0$

b. $y'' = \text{rect}_{(1,3)}(t) \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0$

c. $y'' + 9y = \text{rect}_{(1,3)}(t) \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0$

28.11. *Compute each of the following convolutions:*

a. $t^2 * \text{step}_3(t)$

b. $1 * \text{step}_4(t)$

c. $\cos(t) * \text{rect}_{(0,\pi)}(t)$

d. $2e^{-2t} * \text{rect}_{(1,3)}(t)$

e. $e^{-2t} * \left[e^{5t}\,\text{rect}_{(1,3)}(t)\right]$

f. $\sin(t) * \left[\sin(t)\,\text{rect}_{(2\pi,3\pi)}(t)\right]$

g. $t * f(t) \qquad \text{where} \quad f(t) = \begin{cases} \sqrt{t} & \text{if} \quad 0 \le t < 4 \\ 2 & \text{if} \quad 4 < t \end{cases}$

h. $\sin(t) * f(t) \qquad \text{where} \quad f(t) = \begin{cases} 1 & \text{if} \quad t < 2\pi \\ \cos(t) & \text{if} \quad 2\pi < t < 3\pi \\ -1 & \text{if} \quad 3\pi < t \end{cases}$

28.12. *Each function listed below is at least periodic on* $(0, \infty)$. *Sketch graph of each, and then find its Laplace transform using the methods developed in section 28.5.*

a. $f(t) = \begin{cases} e^{-2t} & \text{if} \quad 0 < t < 3 \\ f(t-3) & \text{if} \quad t > 3 \end{cases}$

b. $f(t) = \text{sqwave}(t)$ *(from example 28.12)*

c. $f(t) = \begin{cases} 1 & \text{if} \quad 0 < t < 1 \\ -1 & \text{if} \quad 1 < t < 2 \\ f(t-2) & \text{if} \quad t > 2 \end{cases}$

d. $f(t) = \begin{cases} 2t - t^2 & \text{if} \quad t < 2 \\ f(t-2) & \text{if} \quad 2 < t \end{cases}$ *(see exercise 28.8 a)*

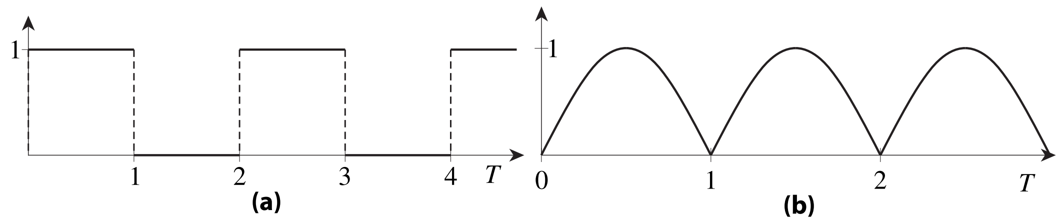

Figure 28.9: **(a)** The square wave function for exercise 28.13 a, and **(b)** the rectified sine function for exercise 28.13 b.

e. $f(t) = \begin{cases} t & \text{if } 0 < t < 2 \\ 4 - t & \text{if } 2 < t < 4 \\ f(t-4) & \text{if } t > 4 \end{cases}$ *(see exercise 28.8 i)*

f. $f(t) = |\sin(t)|$

28.13. *In each of the following exercises, you are given the natural period* p_0 *and a forcing function* f *for an undamped mass/spring system modeled by*

$$m\frac{d^2y}{dt^2} + \kappa y = f \quad .$$

Analyze the corresponding resonance occurring in each system. In particular, let y *be any solution to the modeling differential equation and:*

i. *Compute the difference* $y(t + p_0) - y(t)$.

ii. *Compute the formula for* $y(t)$ *assuming* $y(0) = 0$ *and* $y'(0) = 0$. *(Express your answer using* τ *and* n *where* n *is the largest integer such that* $np_0 \leq t$ *and* $\tau = t - np_0$ *.)*

iii. *Using the formula just computed for part ii along with your favorite computer math package, sketch the graph of* y *over several cycles. (For convenience, assume* m *is a unit mass.)*

a. $p_0 = 2$ *and* f *is basic squarewave sketched in figure 28.9a. That is,*

$$f(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t < 2 \\ f(t+2) & \text{in general} \end{cases} \quad .$$

b. $p_0 = 1$ *and* $f(t) = |\sin(\pi t)|$ *, the "rectified sine function" sketched in figure 28.9b.*

c. $p_0 = 1$ *and* $f(t) = \sin(4\pi t)$.