24

The Laplace Transform (Intro)

The Laplace transform is a mathematical tool based on integration that has a number of applications. It particular, it can simplify the solving of many differential equations. We will find it particularly useful when dealing with nonhomogeneous equations in which the forcing functions are not continuous. This makes it a valuable tool for engineers and scientists dealing with "real-world" applications.

By the way, the Laplace transform is just one of many "integral transforms" in general use. Conceptually and computationally, it is probably the simplest. If you understand the Laplace transform, then you will find it much easier to pick up the other transforms as needed.

24.1 Basic Definition and Examples

Definition, Notation and Other Basics

Let f be a 'suitable' function (more on that later). The *Laplace transform of* f, denoted by either F or $\mathcal{L}[f]$, is the function given by

$$F(s) = \mathcal{L}[f]|_s = \int_0^\infty f(t)e^{-st}\,dt \quad . \tag{24.1}$$

!►*Example 24.1:* *For our first example, let us use*

$$f(t) = \begin{cases} 1 & \text{if } t \le 2 \\ 0 & \text{if } 2 < t \end{cases} \quad .$$

This is the relatively simple discontinuous function graphed in figure 24.1a. To compute the Laplace transform of this function, we need to break the integral into two parts:

$$\begin{aligned} F(s) = \mathcal{L}[f]|_s &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^2 \underbrace{f(t)}_{1} e^{-st}\,dt + \int_2^\infty \underbrace{f(t)}_{0} e^{-st}\,dt \\ &= \int_0^2 e^{-st}\,dt + \int_2^\infty 0\,dt = \int_0^2 e^{-st}\,dt \quad . \end{aligned}$$

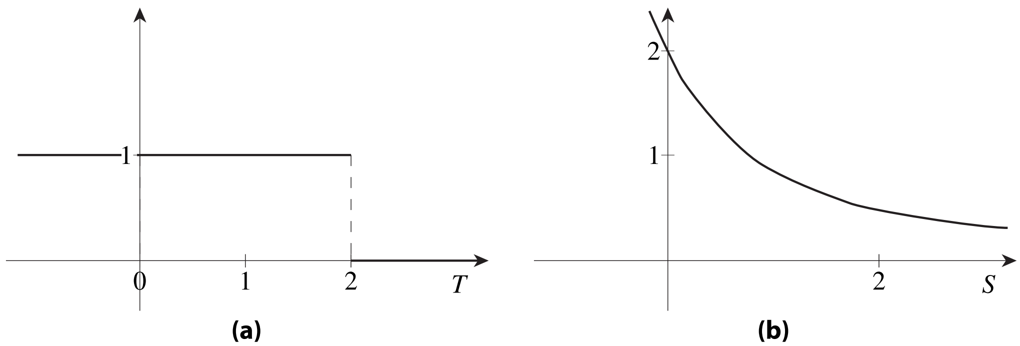

Figure 24.1: The graph of **(a)** the discontinuous function $f(t)$ from example 24.1 and **(b)** its Laplace transform $F(s)$.

So, if $s \neq 0$,

$$F(s) = \int_0^2 e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_{t=0}^{2} = -\frac{1}{s}\left[e^{-s\cdot 2} - e^{-s\cdot 0}\right] = \frac{1}{s}\left[1 - e^{-2s}\right] \quad .$$

And if $s = 0$,

$$F(s) = F(0) = \int_0^2 e^{-0\cdot t}\,dt = \int_0^2 1\,dt = 2 \quad .$$

This is the function sketched in figure 24.1b. (Using L'Hôpital's rule, you can easily show that $F(s) \to F(0)$ as $s \to 0$. So, despite our need to compute $F(s)$ separately when $s = 0$, F is a continuous function.)

As the example just illustrated, we really are 'transforming' the function $f(t)$ into another function $F(s)$. This process of transforming $f(t)$ to $F(s)$ is also called the *Laplace transform* and, unsurprisingly, is denoted by $\mathcal{L}$. Thus, when we say "the Laplace transform", we can be referring to either the transformed function $F(s)$ or to the process of computing $F(s)$ from $f(t)$.

Some other quick notes:

1. There are standard notational conventions that simplify bookkeeping. The functions 'to be transformed' are (almost) always denoted by lower case Roman letters — f, g, h, etc. — and t is (almost) always used as the variable in the formulas for these functions (because, in applications, these are typically functions of time). The corresponding 'transformed functions' are (almost) always denoted by the corresponding upper case Roman letters — F, G, H, ETC. — and s is (almost) always used as the variable in the formulas for these functions.

   Thus, if we happen to refer to functions $f(t)$ and $F(s)$, it is a good bet that $F = \mathcal{L}[f]$.

2. Observe that, in the integral for the Laplace transform, we are integrating the inputted function $f(t)$ multiplied by the exponential e^{-st} over the positive T–axis. Because of the sign in the exponential, this exponential is a *rapidly decreasing* function of t when $s > 0$ and is a *rapidly increasing* function of t when $s < 0$. This will help determine both the sort of functions that are 'suitable' for the Laplace transform, and the domains of the transformed functions.

3. It is also worth noting that, because the lower limit in the integral for the Laplace transform is $t = 0$, the formula for $f(t)$ when $t < 0$ is completely irrelevant. In fact, $f(t)$ need not even be defined for $t < 0$. For this reason, some authors explicitly limit the values for t to being nonnegative. We won't do this explicitly, but do keep in mind that the Laplace transform of a function $f(t)$ is only based on the values/formula for $f(t)$ with $t \geq 0$. This will become a little more relevant when we discuss inverting the Laplace transform (in chapter 26).

4. As indicated by our discussion so far, we are treating the s in

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

as a real variable, that is, we are assuming s denotes a relatively arbitrary real value. Be aware, however, that in more advanced developments, s is often treated as a complex variable, $s = \sigma + i\xi$. This allows the use of results from the theory of analytic complex functions. But we won't need that theory (a theory which few readers of this text are likely to have yet seen). So, in this text (with one very brief exception in chapter 26), s will always be assumed to be real.

Transforms of Some Common Functions

Before we can make much use of the Laplace transform, we need to build a repertoire of common functions whose transforms we know. It would also be a good idea to compute a number of transforms simply to get a better grasp of this whole 'Laplace transform' idea.

So let's get started.

!►*Example 24.2 (transforms of favorite constants):* *Let f be the zero function, that is,*

$$f(t) = 0 \qquad \text{for all} \quad t \quad .$$

Then its Laplace transform is

$$F(s) = \mathcal{L}[0]|_s = \int_0^\infty 0 \cdot e^{-st}\,dt = \int_0^\infty 0\,dt = 0 \quad . \tag{24.2}$$

Now let h be the unit constant function, that is,

$$h(t) = 1 \qquad \text{for all} \quad t \quad .$$

Then

$$H(s) = \mathcal{L}[1]|_s = \int_0^\infty 1 \cdot e^{-st}\,dt = \int_0^\infty e^{-st}\,dt \quad .$$

What comes out of this integral depends strongly on whether s is positive or not. If $s < 0$, then $0 < -s = |s|$ and

$$\begin{aligned}\int_0^\infty e^{-st}\,dt &= \int_0^\infty e^{|s|t}\,dt = \frac{1}{|s|}e^{|s|t}\bigg|_{t=0}^{\infty} \\ &= \lim_{t\to\infty}\frac{1}{|s|}e^{|s|t} - \frac{1}{|s|}e^{|s|\cdot 0} = \infty - \frac{1}{|s|} = \infty \quad .\end{aligned}$$

If $s = 0$ *, then*

$$\int_0^\infty e^{-st}\,dt \;=\; \int_0^\infty e^{0\cdot t}\,dt \;=\; \int_0^\infty 1\,dt \;=\; t\Big|_{t=0}^{\infty} \;=\; \infty \quad .$$

Finally, if $s > 0$ *, then*

$$\int_0^\infty e^{-st}\,dt \;=\; \frac{1}{-s}e^{-st}\bigg|_{t=0}^{\infty} \;=\; \lim_{t\to\infty}\frac{1}{-s}e^{-st} \;-\; \frac{1}{-s}e^{-s\cdot 0} \;=\; 0 \;+\; \frac{1}{s} \;=\; \frac{1}{s} \quad .$$

So,

$$\mathcal{L}[1]|_s \;=\; \int_0^\infty 1\cdot e^{-st}\,dt \;=\; \begin{cases} \dfrac{1}{s} & \text{if} \quad 0 < s \\ \infty & \text{if} \quad s \le 0 \end{cases} \quad . \tag{24.3}$$

As illustrated in the last example, a Laplace transform $F(s)$ is often a well-defined (finite) function of s only when s is greater that some fixed number s_0 ($s_0 = 0$ in the example). This is a result of the fact that the larger s is, the faster e^{-st} goes to zero as $t \to \infty$ (provided $s > 0$). In practice, we will only give the formulas for the transforms over the intervals where these formulas are well-defined and finite. Thus, in place of equation (24.3), we will write

$$\mathcal{L}[1]|_s \;=\; \frac{1}{s} \qquad \text{for} \quad s > 0 \quad . \tag{24.4}$$

As we compute Laplace transforms, we will note such restrictions on the values of s . To be honest, however, these restrictions will usually not be that important in practice. What will be important is that there is some finite value s_0 such that our formulas are valid whenever $s > s_0$.

Keeping this in mind, let's go back to computing transforms.

!►Example 24.3 (transforms of some powers of t): *We want to find*

$$\mathcal{L}\big[t^n\big]\big|_s \;=\; \int_0^\infty t^n e^{-st}\,dt \;=\; \int_0^\infty t^n e^{-st}\,dt \qquad \text{for} \quad n \;=\; 1,\,2,\,3,\,\dots \quad .$$

With a little thought, you will realize this integral will not be finite if $s \le 0$ *. So we will assume* $s > 0$ *in these computations. This, of course, means that*

$$\lim_{t\to\infty} e^{-st} \;=\; 0 \quad .$$

It also means that, using L'Hôpital's rule, you can easily verify that

$$\lim_{t\to\infty} t^n e^{-st} \;=\; 0 \qquad \text{for} \quad n \ge 0 \quad .$$

Keeping the above in mind, consider the case where $n = 1$ *,*

$$\mathcal{L}[t]|_s \;=\; \int_0^\infty t e^{-st}\,dt \quad .$$

This integral just cries out to be integrated "by parts":

$$\begin{aligned} \mathcal{L}[t]|_s \;&=\; \int_0^\infty \underbrace{t}_{u}\,\underbrace{e^{-st}\,dt}_{dv} \\ &=\; uv\Big|_{t=0}^{\infty} \;-\; \int_0^\infty v\,du \end{aligned}$$

$$= t\left(\frac{1}{-s}\right)e^{-st}\bigg|_{t=0}^{\infty} - \int_0^\infty \left(\frac{1}{-s}\right)e^{-st}\,dt$$

$$= -\frac{1}{s}\Big[\underbrace{\lim_{t\to\infty} te^{-st}}_{0} - \underbrace{0\cdot e^{-s\cdot 0}}_{0} - \int_0^\infty e^{-st}\,dt\Big] = \frac{1}{s}\int_0^\infty e^{-st}\,dt \quad .$$

Admittedly, this last integral is easy to compute, but why bother since we computed it in the previous example! In fact, it is worth noting that combining the last computations with the computations for $\mathcal{L}[1]$ *yields*

$$\mathcal{L}[t]|_s = \frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s}\int_0^\infty 1\cdot e^{-st}\,dt = \frac{1}{s}\mathcal{L}[1]|_s = \frac{1}{s}\left[\frac{1}{s}\right] \quad .$$

So,

$$\mathcal{L}[t]|_s = \frac{1}{s^2} \qquad \text{for} \quad s > 0 \quad . \tag{24.5}$$

Now consider the case where $n = 2$ *. Again, we start with an integration by parts:*

$$\begin{aligned}
\mathcal{L}\big[t^2\big]\big|_s &= \int_0^\infty t^2 e^{-st}\,dt \\
&= \int_0^\infty \underbrace{t^2}_{u}\underbrace{e^{-st}\,dt}_{dv} \\
&= uv\big|_{t=0}^{\infty} - \int_0^\infty v\,du \\
&= t^2\left(\frac{1}{-s}\right)e^{-st}\bigg|_{t=0}^{\infty} - \int_0^\infty \left(\frac{1}{-s}\right)e^{-st}2t\,dt \\
&= \frac{1}{-s}\Big[\underbrace{\lim_{t\to\infty} t^2e^{-st}}_{0} - \underbrace{0^2e^{-s\cdot 0}}_{0} - 2\int_0^\infty te^{-st}\,dt\Big] = \frac{2}{s}\int_0^\infty te^{-st}\,dt \quad .
\end{aligned}$$

But remember,

$$\int_0^\infty te^{-st}\,dt = \mathcal{L}[t]|_s \quad .$$

Combining the above computations with this (and referring back to equation (24.5)), we end up with

$$\mathcal{L}\big[t^2\big]\big|_s = \frac{2}{s}\int_0^\infty te^{-st}\,dt = \frac{2}{s}\mathcal{L}[t]|_s = \frac{2}{s}\left[\frac{1}{s^2}\right] = \frac{2}{s^3} \quad . \tag{24.6}$$

Clearly, a pattern is emerging. I'll leave the computation of $\mathcal{L}\big[t^3\big]$ *to you.*

?►Exercise 24.1: *Assuming* $s > 0$ *, verify (using integration by parts) that*

$$\mathcal{L}\big[t^3\big]\big|_s = \frac{3}{s}\mathcal{L}\big[t^2\big]\big|_s \quad ,$$

and from that and the formula for $\mathcal{L}\big[t^2\big]$ *computed above, conclude that*

$$\mathcal{L}\big[t^3\big]\big|_s = \frac{3\cdot 2}{s^4} = \frac{3!}{s^4} \quad .$$

?►Exercise 24.2: *More generally, use integration by parts to show that, whenever $s > 0$ and n is a positive integer,*

$$\mathcal{L}\big[t^n\big]\big|_s = \frac{n}{s}\mathcal{L}\big[t^{n-1}\big]\big|_s \quad .$$

Using the results from the last two exercises, we have, for $s > 0$,

$$\mathcal{L}\big[t^4\big]\big|_s = \frac{4}{s}\mathcal{L}\big[t^3\big]\big|_s = \frac{4}{s}\cdot\frac{3\cdot 2}{s^4} = \frac{4\cdot 3\cdot 2}{s^5} = \frac{4!}{s^5} \quad ,$$

$$\mathcal{L}\big[t^5\big]\big|_s = \frac{5}{s}\mathcal{L}\big[t^4\big]\big|_s = \frac{5}{s}\cdot\frac{4\cdot 3\cdot 2}{s^5} = \frac{5\cdot 4\cdot 3\cdot 2}{s^6} = \frac{5!}{s^6} \quad ,$$

$$\vdots$$

In general, for $s > 0$ and $n = 1, 2, 3, \ldots$,

$$\mathcal{L}\big[t^n\big]\big|_s = \frac{n!}{s^{n+1}} \quad . \tag{24.7}$$

(If you check, you'll see that it even holds for $n = 0$.)

It turns out that a formula very similar to (24.7) also holds when n is not an integer. Of course, there is then the issue of just what $n!$ means if n is not an integer! Since the discussion of that issue may distract our attention away from one the main issues at hand — that of getting a basic understanding of what the Laplace transform is by computing transforms of simple functions — let us hold off on that discussion for a few pages.

Instead, let's compute the transforms of some exponentials:

!►Example 24.4 (transform of a real exponential): *Consider computing the Laplace transform of e^{3t},*

$$\mathcal{L}\big[e^{3t}\big]\big|_s = \int_0^\infty e^{3t}e^{-st}\,dt = \int_0^\infty e^{3t-st}\,dt = \int_0^\infty e^{-(s-3)t}\,dt \quad .$$

If $s-3$ is not positive, then $e^{-(s-3)t}$ is not a decreasing function of t, and, hence, the above integral will not be finite. So we must require $s-3$ to be positive (that is, $s > 3$). Assuming this, we can continue our computations

$$\begin{aligned}\mathcal{L}\big[e^{3t}\big]\big|_s &= \int_0^\infty e^{-(s-3)t}\,dt \\ &= \frac{-1}{s-3}\,e^{-(s-3)t}\Big|_{t=0}^{\infty} \\ &= \frac{-1}{s-3}\left[\lim_{t\to\infty} e^{-(s-3)t} - e^{-(s-3)0}\right] = \frac{-1}{s-3}[0 - 1] \quad .\end{aligned}$$

So

$$\mathcal{L}\big[e^{3t}\big]\big|_s = \frac{1}{s-3} \qquad \text{for} \quad 3 < s \quad .$$

Replacing 3 with any other real number is trivial.

?►Exercise 24.3 (transforms of real exponentials): *Let α be any real number and show that*

$$\mathcal{L}\big[e^{\alpha t}\big]\big|_s = \frac{1}{s-\alpha} \qquad \text{for} \quad \alpha < s \quad . \tag{24.8}$$

Complex exponentials are also easily done:

!▶Example 24.5 (transform of a complex exponential): *Computing the Laplace transform of e^{i3t} leads to*

$$\begin{aligned}\mathcal{L}\left[e^{i3t}\right]\Big|_s &= \int_0^\infty e^{i3t}e^{-st}\,dt \\ &= \int_0^\infty e^{-(s-i3)t}\,dt \\ &= \frac{-1}{s-i3}\,e^{-(s-i3)t}\Big|_{t=0}^{\infty} = \frac{-1}{s-i3}\left[\lim_{t\to\infty} e^{-(s-i3)t} - e^{-(s-i3)0}\right] \quad .\end{aligned}$$

Now,

$$e^{-(s-i3)0} = e^0 = 1$$

and

$$\lim_{t\to\infty} e^{-(s-i3)t} = \lim_{t\to\infty} e^{-st+i3t} = \lim_{t\to\infty} e^{-st}\left[\cos(3t) + i\sin(3t)\right] \quad .$$

Since sines and cosines oscillate between 1 and -1 as $t \to \infty$, the last limit does not exist unless

$$\lim_{t\to\infty} e^{-st} = 0 \quad ,$$

and this occurs if and only if $s > 0$. In this case,

$$\lim_{t\to\infty} e^{-(s-i3)t} = \lim_{t\to\infty} e^{-st}\left[\cos(3t) + i\sin(3t)\right] = 0 \quad .$$

Thus, when $s > 0$,

$$\mathcal{L}\left[e^{i3t}\right]\Big|_s = \frac{-1}{s-i3}\left[\lim_{t\to\infty} e^{-(s-i3)t} - e^{-(s-i3)0}\right] = \frac{-1}{s-i3}[0 - 1] = \frac{1}{s-i3} \quad .$$

Again, replacing 3 with any real number is trivial.

?▶Exercise 24.4 (transforms of complex exponentials): *Let α be any real number and show that*

$$\mathcal{L}\left[e^{i\alpha t}\right]\Big|_s = \frac{1}{s-i\alpha} \qquad \text{for} \quad 0 < s \quad . \tag{24.9}$$

24.2 Linearity and Some More Basic Transforms

Suppose we have already computed the Laplace transforms of two functions $f(t)$ and $g(t)$, and, thus, already know the formulas for

$$F(s) = \mathcal{L}[f]|_s \qquad \text{and} \qquad G(s) = \mathcal{L}[g]|_s \quad .$$

Now look at what happens if we compute the transform of any linear combination of f and g: Letting α and β be any two constants, we have

$$\begin{aligned}\mathcal{L}[\alpha f(t) + \beta g(t)]|_s &= \int_0^\infty [\alpha f(t) + \beta g(t)] e^{-st}\,dt \\ &= \int_0^\infty \left[\alpha f(t)e^{-st} \;+\; \beta g(t)e^{-st}\right] dt \\ &= \alpha \int_0^\infty f(t)e^{-st}\,dt \;+\; \beta \int_0^\infty g(t)e^{-st}\,dt \\ &= \alpha\mathcal{L}[f(t)]|_s \;+\; \beta\mathcal{L}[g(t)]|_s \;=\; \alpha F(s) \;+\; \beta G(s) \quad .\end{aligned}$$

Thus, the Laplace transform is a *linear transform*; that is, for any two constants α and β, and any two Laplace transformable functions f and g,

$$\mathcal{L}[\alpha f(t) + \beta g(t)] \;=\; \alpha\mathcal{L}[f] \;+\; \beta\mathcal{L}[g] \quad .$$

This fact will simplify many of our computations, and is important enough to enshrine as a theorem. While we are at it, let's note that the above computations can be done with more functions than two, and that we, perhaps, should have noted the values of s for which the integrals are finite. Taking all that into account, we can prove:

Theorem 24.1 (linearity of the Laplace transform)
The Laplace transform transform is linear. That is,

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)] \;=\; c_1\mathcal{L}[f_1(t)] \;+\; c_2\mathcal{L}[f_2(t)] \;+\; \cdots \;+\; c_n\mathcal{L}[f_n(t)]$$

where each c_k is a constant and each f_k is a "Laplace transformable" function.

Moreover, if, for each f_k we have a value s_k such that

$$F_k(s) \;=\; \mathcal{L}[f_k(t)]|_s \qquad \text{for} \quad s_k < s \quad ,$$

then, letting s_{max} be the largest of these s_k's,

$$\begin{aligned}&\mathcal{L}[c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)]|_s \\ &\qquad = c_1 F_1(s) \;+\; c_2 F_2(s) \;+\; \cdots \;+\; c_n F_n(s) \qquad \text{for} \quad s_{max} < s \quad .\end{aligned}$$

!► ***Example 24.6 (transform of the sine function):*** *Let us consider finding the Laplace transform of $\sin(\omega t)$ for any real value ω. There are several ways to compute this, but the easiest starts with using Euler's formula for the sine function along with the linearity of the Laplace transform:*

$$\begin{aligned}\mathcal{L}[\sin(\omega t)]|_s &= \mathcal{L}\left[\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right]\bigg|_s \\ &= \frac{1}{2i}\mathcal{L}\left[e^{i\omega t} - e^{-i\omega t}\right]\Big|_s \;=\; \frac{1}{2i}\left[\mathcal{L}\left[e^{i\omega t}\right]\Big|_s \;-\; \mathcal{L}\left[e^{-i\omega t}\right]\Big|_s\right] \quad .\end{aligned}$$

From example 24.5 and exercise 24.4, we know

$$\mathcal{L}\left[e^{i\omega t}\right]\Big|_s \;=\; \frac{1}{s - i\omega} \qquad \text{for} \quad s > 0 \quad .$$

Thus, also,

$$\mathcal{L}\left[e^{-i\omega t}\right]\Big|_s = \mathcal{L}\left[e^{i(-\omega)t}\right]\Big|_s = \frac{1}{s - i(-\omega)} = \frac{1}{s + i\omega} \qquad \text{for} \quad s > 0 \quad .$$

Plugging these into the computations for $\mathcal{L}[\sin(\omega t)]$ *(and doing a little algebra) yields, for* $s > 0$ *,*

$$\begin{aligned}
\mathcal{L}[\sin(\omega t)]|_s &= \frac{1}{2i}\left[\mathcal{L}\left[e^{i\omega t}\right]\Big|_s - \mathcal{L}\left[e^{-i\omega t}\right]\Big|_s\right] \\
&= \frac{1}{2i}\left[\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right] \\
&= \frac{1}{2i}\left[\frac{s + i\omega}{(s - i\omega)(s + i\omega)} - \frac{s - i\omega}{(s + i\omega)(s - i\omega)}\right] \\
&= \frac{1}{2i}\left[\frac{(s + i\omega) - (s - i\omega)}{s^2 - i^2\omega^2}\right] \\
&= \frac{1}{2i}\left[\frac{2i\omega}{s^2 - i^2\omega^2}\right] \quad ,
\end{aligned}$$

which immediately simplifies to

$$\mathcal{L}[\sin(\omega t)]|_s = \frac{\omega}{s^2 + \omega^2} \qquad \text{for} \quad s > 0 \quad . \tag{24.10}$$

?►*Exercise 24.5 (transform of the cosine function):* *Show that, for any real value* ω *,*

$$\mathcal{L}[\cos(\omega t)]|_s = \frac{s}{s^2 + \omega^2} \qquad \text{for} \quad s > 0 \quad . \tag{24.11}$$

24.3 Tables and a Few More Transforms

In practice, those using the Laplace transform in applications do not constantly recompute basic transforms. Instead, they refer to tables of transforms (or use software) to look up commonly used transforms, just as so many people use tables of integrals (or software) when computing integrals. We, too, can use tables (or software) *after*

1. you have computed enough transforms on your own to understand the basic principles,

and

2. we have computed the transforms appearing in the table so we know our table is correct.

The table we will use is table 24.1, *Laplace Transforms of Common Functions (Version 1)*, on page 484. Checking that table, we see that we have already verified all but two or three of the entries, with those being the transforms of fairly arbitrary powers of t , t^α , and the "shifted step function", $\text{step}(t - \alpha)$. So let's compute them now.

Table 24.1: Laplace Transforms of Common Functions (Version 1)

In the following, α and ω are real-valued constants, and, unless otherwise noted, $s > 0$.

$f(t)$	$F(s) = \mathcal{L}[f(t)]\big\|_s$	Restrictions
1	$\frac{1}{s}$	
t	$\frac{1}{s^2}$	
t^n	$\frac{n!}{s^{n+1}}$	$n = 1, 2, 3, \ldots$
$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$	
t^α	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	$-1 < \alpha$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$\alpha < s$
$e^{i\alpha t}$	$\frac{1}{s-i\alpha}$	
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	
$\text{step}_\alpha(t)$, $\text{step}(t-\alpha)$	$\frac{e^{-\alpha s}}{s}$	$0 \le \alpha$

Arbitrary Powers (and the Gamma Function)

Earlier, we saw that

$$\mathcal{L}\big[t^n\big]\Big|_s = \int_0^\infty t^n e^{-st}\,dt = \frac{n!}{s^{n+1}} \qquad \text{for} \quad s > 0 \tag{24.12}$$

when n is any nonnegative integer. Let us now consider computing

$$\mathcal{L}\big[t^\alpha\big]\Big|_s = \int_0^\infty t^\alpha e^{-st}\,dt \qquad \text{for} \quad s > 0$$

when α is any real number greater than -1. (When $\alpha \le -1$, you can show that t^α 'blows up' too quickly near $t = 0$ for the integral to be finite.)

The method we used to find $\mathcal{L}[t^n]$ becomes awkward when we try to apply it to find $\mathcal{L}[t^\alpha]$ when α is not an integer. Instead, we will 'cleverly' simplify the above integral for $\mathcal{L}[t^\alpha]$ by using the substitution $u = st$. Since t is the variable in the integral, this means

$$t = \frac{u}{s} \qquad \text{and} \qquad dt = \frac{1}{s}\,du \quad .$$

So, assuming $s > 0$ and $\alpha > -1$,

$$\begin{aligned}\mathcal{L}[t^\alpha]\big|_s &= \int_0^\infty t^\alpha e^{-st}\,dt \\ &= \int_0^\infty \left(\frac{u}{s}\right)^\alpha e^{-u}\frac{1}{s}\,du \\ &= \int_0^\infty \frac{u^\alpha}{s^{\alpha+1}}e^{-u}\,du \;=\; \frac{1}{s^{\alpha+1}}\int_0^\infty u^\alpha e^{-u}\,du \quad .\end{aligned}$$

Notice that the last integral depends only on the constant α — we've 'factored out' any dependence on the variable s. Thus, we can treat this integral as a constant (for each value of α) and write

$$\mathcal{L}[t^\alpha]\big|_s = \frac{C_\alpha}{s^{\alpha+1}} \qquad \text{where} \qquad C_\alpha = \int_0^\infty u^\alpha e^{-u}\,du \quad .$$

It just so happens that the above formula for C_α is very similar to the formula for something called the "Gamma function". This is a function that crops up in various applications (such as this) and, for $x > 0$, is given by

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du \quad . \tag{24.13}$$

Comparing this with the formula for C_α, we see that

$$C_\alpha = \int_0^\infty u^\alpha e^{-u}\,du = \int_0^\infty u^{(\alpha+1)-1}e^{-u}\,du = \Gamma(\alpha+1) \quad .$$

So our formula for the Laplace transform of t^α (with $\alpha > -1$) can be written as

$$\mathcal{L}[t^\alpha]\big|_s = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \qquad \text{for} \quad s > 0 \quad . \tag{24.14}$$

This is normally considered the preferred way to express $\mathcal{L}[t^\alpha]$ because the Gamma function is considered to be a "well-known" function. Perhaps you don't yet consider it "well known", but you can find tables for evaluating $\Gamma(x)$, and it is probably one of the functions already defined in your favorite computer math package. That makes graphing $\Gamma(x)$, as done in figure 24.2, relatively easy.

As it is, we can readily determine the value of $\Gamma(x)$ when x is a positive integer by comparing our two formulas for $\mathcal{L}[t^n]$ when n is a nonnegative integer — the one mentioned at the start of our discussion (formula (24.12)), and the more general formula (formula (24.14)) just derived for $\mathcal{L}[t^\alpha]$ with $\alpha = n$:

$$\frac{n!}{s^{n+1}} = \mathcal{L}[t^n]\big|_s = \frac{\Gamma(n+1)}{s^{n+1}} \qquad \text{when} \quad n = 0, 1, 2, 3, 4, \dots \quad .$$

Thus,

$$\Gamma(n+1) = n! \qquad \text{when} \quad n = 0, 1, 2, 3, 4, \dots \quad .$$

Letting $x = n + 1$, this becomes

$$\Gamma(x) = (x-1)! \qquad \text{when} \quad x = 1, 2, 3, 4, \dots \quad . \tag{24.15}$$

In particular:

$$\Gamma(1) = (1-1)! = 0! = 1 \quad ,$$

$$\Gamma(2) = (2-1)! = 1! = 1 \quad ,$$

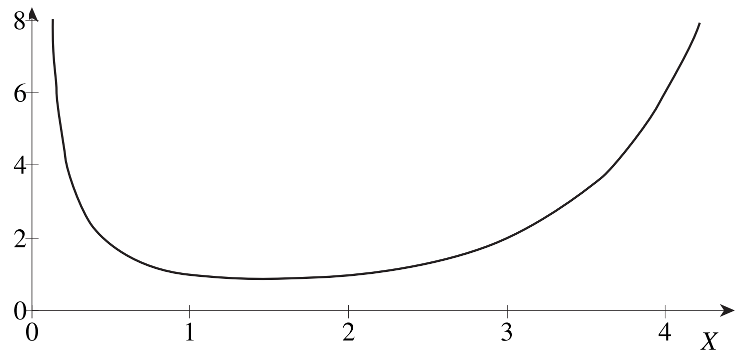

Figure 24.2: The graph of graph of the Gamma function over the interval $(0,4)$. As $x \to 0^+$ or $x \to +\infty$, $\Gamma(x) \to +\infty$ very rapidly.

$$\Gamma(3) = (3-1)! = 2! = 2 \quad ,$$

$$\Gamma(4) = (4-1)! = 3! = 6 \quad ,$$

and

$$\Gamma(12) = (12-1)! = 11! = 39{,}916{,}800 \quad .$$

This shows that the Gamma function can be viewed as a generalization of the factorial. Indeed, you will find texts where the factorial is redefined for all positive numbers (not just integers) by

$$x! = \Gamma(x+1) \quad .$$

We won't do that.

Computing $\Gamma(x)$ when x is not an integer is not so simple. It can be shown that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad . \tag{24.16}$$

Also, using integration by parts (just as you did in exercise 24.2 on page 480), you can show that

$$\Gamma(x+1) = x\Gamma(x) \qquad \text{for} \quad x > 0 \quad , \tag{24.17}$$

which is analogous to the factorial identity $(n+1)! = (n+1)n!$. We will leave the verification of these to the more adventurous (see exercise 24.13 on page 504), and go on to the computation of a few more transforms.

!►*Example 24.7:* *Consider finding the Laplace transforms of*

$$\frac{1}{\sqrt{t}} \quad , \quad \sqrt{t} \quad \text{and} \quad \sqrt[3]{t} \quad .$$

For the first, we use formulas (24.14) with $\alpha = -1/2$, along with equation (24.16):

$$\mathcal{L}\left[\frac{1}{\sqrt{t}}\right]\Big|_s = \mathcal{L}\left[t^{-1/2}\right]\Big|_s = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-1/2+1}} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} \quad .$$

For the second, formula (24.14) with $\alpha = 1/2$ gives

$$\mathcal{L}\left[\sqrt{t}\right]\Big|_s = \mathcal{L}\left[t^{1/2}\right]\Big|_s = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{1/2+1}} = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} \quad .$$

Using formulas (24.17) and (24.16), we see that

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} \quad .$$

Thus

$$\mathcal{L}\left[\sqrt{t}\right]\Big|_s = \mathcal{L}\left[t^{1/2}\right]\Big|_s = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}} \quad .$$

For the transform of $\sqrt[3]{t}$ *, we simply have*

$$\mathcal{L}\left[\sqrt[3]{t}\right]\Big|_s = \mathcal{L}\left[t^{1/3}\right]\Big|_s = \frac{\Gamma\left(\frac{1}{3}+1\right)}{s^{1/3+1}} = \frac{\Gamma\left(\frac{4}{3}\right)}{s^{4/3}} \quad .$$

Unfortunately, there is not a formula analogous to (24.16) for $\Gamma\left(^4/_3\right)$ *or* $\Gamma\left(^1/_3\right)$ *. There is the approximation*

$$\Gamma\left(\frac{4}{3}\right) \approx .8929795121 \quad ,$$

which can be found using either tables or a computer math package, but, since this is just an approximation, we might as well leave our answer as

$$\mathcal{L}\left[\sqrt[3]{t}\right]\Big|_s = \mathcal{L}\left[t^{1/3}\right]\Big|_s = \frac{\Gamma\left(\frac{4}{3}\right)}{s^{4/3}} \quad .$$

The Shifted Unit Step Function

Step functions are the simplest discontinuous functions we can have. The *(basic) unit step function*, which we will denote by $\text{step}(t)$, is defined by

$$\text{step}(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad 0 < t \end{cases} \quad .$$

Its graph has been sketched in figure 24.3a.[1]

For any real value α , the corresponding *shifted unit step function*, which we will denote by step_α , is given by

$$\text{step}_\alpha(t) = \text{step}(t-\alpha) = \left\{\begin{matrix} 0 & \text{if} & t-\alpha < 0 \\ 1 & \text{if} & 0 < t-\alpha \end{matrix}\right\} = \begin{cases} 0 & \text{if} \quad t < \alpha \\ 1 & \text{if} \quad \alpha < t \end{cases} \quad .$$

Its graph, with $\alpha > 0$, has been sketched in figure 24.3b. Do observe that the basic step function and the the step function at zero are the same, $\text{step}(t) = \text{step}_0(t)$.

You may have noted that we've not defined the step functions at their points of discontinuity ($t = 0$ for $\text{step}(t)$, and $t = \alpha$ for $\text{step}_\alpha(t)$). That is because the value of a step function right at

[1] The unit step function is also called the *Heaviside step function*, and, in other texts, is often denoted by u and, occasionally, by h or H .

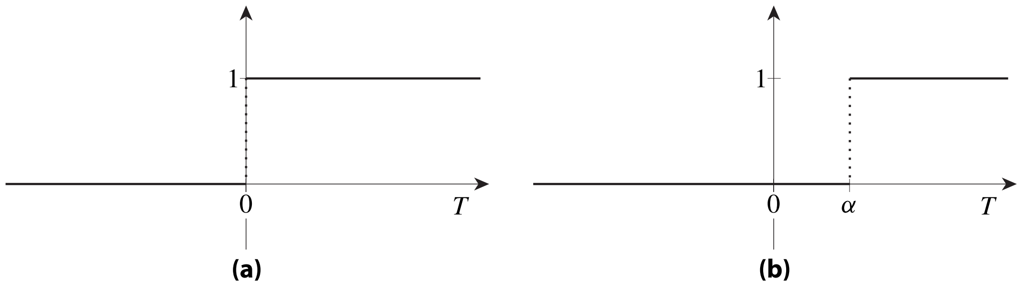

Figure 24.3: The graphs of **(a)** the basic step function $\text{step}(t)$ and **(b)** a shifted step function $\text{step}_\alpha(t)$ with $\alpha > 0$.

its single discontinuity will be completely irrelevant in any of our computations or applications. Observe this fact as we compute the Laplace transform of $\text{step}_\alpha(t)$ when $\alpha \geq 0$:

$$\begin{aligned} \mathcal{L}\big[\text{step}_\alpha(t)\big]\big|_s &= \int_0^\infty \text{step}_\alpha(t) e^{-st}\,dt \\ &= \int_0^\alpha \text{step}_\alpha(t) e^{-st}\,dt + \int_\alpha^\infty \text{step}_\alpha(t) e^{-st}\,dt \\ &= \int_0^\alpha 0 \cdot e^{-st}\,dt + \int_\alpha^\infty 1 \cdot e^{-st}\,dt = \int_\alpha^\infty e^{-st}\,dt \quad . \end{aligned}$$

You can easily show that the above integral is infinite if $s < 0$ or $s = 0$. But if $s > 0$, then the above becomes

$$\begin{aligned} \mathcal{L}\big[\text{step}_\alpha(t)\big]\big|_s = \int_\alpha^\infty e^{-st}\,dt &= \frac{1}{-s} e^{-st}\bigg|_{t=\alpha}^{\infty} \\ &= \lim_{t\to\infty} \frac{1}{-s} e^{-st} - \frac{1}{-s} e^{-s\alpha} = 0 + \frac{1}{s} e^{\alpha s} \quad . \end{aligned}$$

Thus,

$$\mathcal{L}\big[\text{step}_\alpha(t)\big]\big|_s = \frac{1}{s} e^{-\alpha s} \qquad \text{for} \quad s > 0 \quad \text{and} \quad \alpha \geq 0 \quad . \tag{24.18}$$

24.4 The First Translation Identity (And More Transforms)

The linearity of the Laplace transform allows us to construct transforms from linear combinations of known transforms. Other identities allow us to construct new transforms from other formulas involving known transforms. One particularly useful identity is the "first translation identity" (also called the "translation along the S–axis identity" for reasons that will soon be obvious). The derivation of this identity starts with the observation that, in the expression

$$F(s) = \mathcal{L}[f(t)]|_s = \int_0^\infty f(t) e^{-st}\,dt \qquad \text{for} \quad s > s_0 \quad ,$$

the s is simply a place holder. It can be replaced with any symbol, say, X, that does not involve the constant of integration t,

$$F(X) = \mathcal{L}[f(t)]|_X = \int_0^\infty f(t)e^{-Xt}\,dt \qquad \text{for} \quad X > s_0 \quad .$$

In particular, let $X = s - \alpha$ where α is any real constant. Using this for X in the above gives us

$$F(s-\alpha) = \mathcal{L}[f(t)]|_{s-\alpha} = \int_0^\infty f(t)e^{-(s-\alpha)t}\,dt \qquad \text{for} \quad s - \alpha > s_0 \quad .$$

But

$$s - \alpha > s_0 \quad \Longleftrightarrow \quad s > s_0 + \alpha$$

and

$$\begin{aligned}\int_0^\infty f(t)e^{-(s-\alpha)t}\,dt &= \int_0^\infty f(t)e^{-st}e^{\alpha t}\,dt \\ &= \int_0^\infty e^{\alpha t} f(t)e^{-st}\,dt = \mathcal{L}\big[e^{\alpha t} f(t)\big]\big|_s \quad .\end{aligned}$$

So the expression above for $F(s-\alpha)$ can be written as

$$F(s-\alpha) = \mathcal{L}\big[e^{\alpha t} f(t)\big]\big|_s \qquad \text{for} \quad s > s_0 + \alpha \quad .$$

This gives us the following identity:

Theorem 24.2 (First translation identity)

If

$$\mathcal{L}[f(t)]|_s = F(s) \qquad \text{for} \quad s > s_0 \quad ,$$

then, for any real constant α *,*

$$\mathcal{L}\big[e^{\alpha t} f(t)\big]\big|_s = F(s-\alpha) \qquad \text{for} \quad s > s_0 + \alpha \quad . \tag{24.19}$$

This is called a 'translation identity' because the graph of the right side of the identity, equation (24.19), is the graph of $F(s)$ translated to the right by α.[2,3] (The second translation identity, in which f is the function shifted, will be developed later.)

!►*Example 24.8:* *Let us use this translation identity to find the transform of* $t^2 e^{6t}$ *. First we must identify the '* $f(t)$ *' and the '* α *' and then apply the identity:*

$$\mathcal{L}\big[t^2 e^{6t}\big]\big|_s = \mathcal{L}\big[e^{6t} \underbrace{t^2}_{f(t)}\big]\big|_s = \mathcal{L}\big[e^{6t} f(t)\big]\big|_s = F(s-6) \quad . \tag{24.20}$$

Here, $f(t) = t^2$ *and* $\alpha = 6$ *. From either formula (24.7) or table 24.1, we know*

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\big[t^2\big]\big|_s = \frac{2!}{s^{2+1}} = \frac{2}{s^3} \quad .$$

So, for any X *,*

$$F(X) = \frac{2}{X^3} \quad .$$

[2] More precisely, it's shifted to the right by α if $\alpha > 0$, and is shifted to the left by $|a|$ if $\alpha < 0$.

[3] Some authors prefer to use the word "shifting" instead of "translation".

Using this with $X = s-6$ *, the above computation of* $\mathcal{L}\left[t^2 e^{6t}\right]$ *(equation set (24.20)) becomes*

$$\mathcal{L}\left[t^2 e^{6t}\right]\Big|_s = \cdots = F(\underbrace{s-6}_{X}) = F(X) = \frac{2}{X^3} = \frac{2}{(s-6)^3} \quad .$$

Notice that, in the last example, we carefully rewrote the formula for $F(s)$ as a formula of another variable, X , and used that to get the formula for $F(s-3)$,

$$F(s) = \frac{2}{s^3} \quad \Longrightarrow \quad F(X) = \frac{2}{X^3} \quad \Longrightarrow \quad F(\underbrace{s-6}_{X}) = \frac{2}{(s-6)^3} \quad .$$

This helps to prevent dumb mistakes. It replaces the s with a generic placeholder X , which, in turn, is replaced with some formula of s . So long as you remember that the s in the first equation is, itself, simply a placeholder and can be replaced throughout the equation with another formula of s , you can go straight from the formula for $F(s)$ to the formula for $F(s-6)$. Unfortunately, this is often forgotten in the heat of computations, especially by those who are new to these sorts of computations. So I strongly recommend including this intermediate step of replacing $F(s)$ with $F(X)$, and using the formula for $F(X)$ with $X = s-6$ (or $X =$ "whatever formula of s is appropriate").

Let's try another:

!►Example 24.9: *Find* $\mathcal{L}\left[e^{3t}\sin(2t)\right]$ *. Here,*

$$\mathcal{L}\left[e^{3t}\sin(2t)\right]\Big|_s = \mathcal{L}\Big[e^{3t}\underbrace{\sin(2t)}_{f(t)}\Big]\Big|_s = \mathcal{L}\left[e^{3t}f(t)\right]\Big|_s = F(s-3) \quad .$$

In this case, $f(t) = \sin(2t)$ *. Recalling the formula for the transform of such a function (or peeking back at formula (24.10) or table 24.1), we have*

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}[\sin(2t)]|_s = \frac{2}{s^2+2^2} = \frac{2}{s^2+4} \quad .$$

So, for any X *,*

$$F(X) = \frac{2}{X^2+4} \quad .$$

Using this with $X = s-3$ *, the above computation of* $\mathcal{L}\left[e^{3t}\sin(2t)\right]$ *becomes*

$$\mathcal{L}\left[e^{3t}\sin(2t)\right]\Big|_s = \cdots = F(\underbrace{s-3}_{X}) = F(X) = \frac{2}{X^2+4} = \frac{2}{(s-3)^2-4} \quad .$$

In the homework, you'll derive the general formulas for

$$\mathcal{L}\left[t^n e^{\alpha t}\right]\Big|_s \quad , \quad \mathcal{L}\left[e^{\alpha t}\sin(\omega t)\right]\Big|_s \quad \text{and} \quad \mathcal{L}\left[e^{\alpha t}\cos(\omega t)\right]\Big|_s \quad .$$

These formulas are found in most tables of common transforms (but not ours).

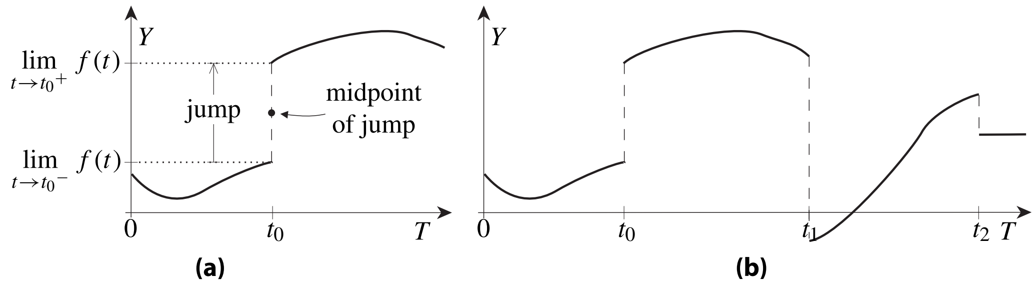

Figure 24.4: The graph of **(a)** a function with a jump dicontinuity at t_0 and **(b)** a function with several jump discontinuities.

24.5 What Is "Laplace Transformable"? (and Some Standard Terminology)

When we say a function f is "Laplace transformable", we simply mean that there is a finite value s_0 such that the integral for $\mathcal{L}[f(t)]|_s$,

$$\int_0^\infty f(t)e^{-st}\,dt \quad ,$$

exists and is finite for every value of s greater than s_0. Not every function is Laplace transformable. For example, t^{-2} and e^{t^2} are not.

Unfortunately, further developing the theory of Laplace transforms assuming nothing more than the "Laplace transformability of our functions" is a bit difficult, and would lead to some rather ungainly wording in our theorems. To simplify our discussions, we will usually insist that our functions are, instead, "piecewise continuous" and "of exponential order". Together, these two conditions will ensure that a function is Laplace transformable, and they will allow us to develop some very general theory that can be applied using the functions that naturally arise in applications. Moreover, these two conditions will be relatively easy to visualize.

So let's find out just what these terms mean.

Jump Discontinuities and Piecewise Continuity

The phrase "piecewise continuity" suggests that we only require continuity on "pieces" of a function. That is a little misleading. For one thing, we want to limit the discontinuities between the pieces to "jump" discontinuities.

Jump Discontinuities

A function f is said to have a *jump discontinuity* at a point t_0 if the left- and right-hand limits

$$\lim_{t\to t_0{}^-} f(t) \qquad \text{and} \qquad \lim_{t\to t_0{}^+} f(t)$$

exist, but are different finite numbers. The *jump* at this discontinuity is the difference of the two limits,

$$\text{jump} \;=\; \lim_{t\to t_0{}^+} f(t) \;-\; \lim_{t\to t_0{}^-} f(t) \quad ,$$

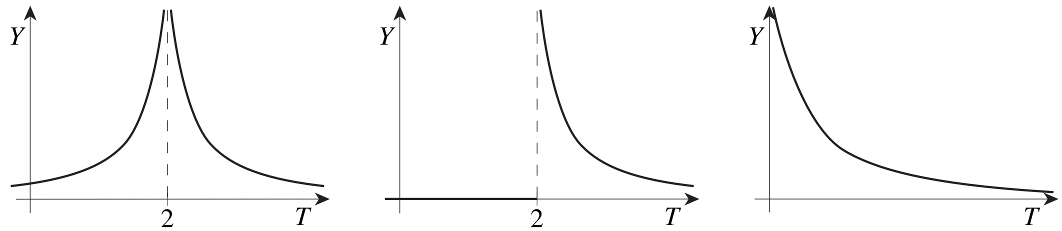

Figure 24.5: Functions having at least one point with an infinite left- or right-hand limit at some point.

and the average of the two limits is the Y-coordinate of the *midpoint* of the jump,

$$y_{\text{midpoint}} = \frac{1}{2}\left[\lim_{t\to t_0{}^+} f(t) + \lim_{t\to t_0{}^-} f(t)\right] .$$

A generic example is sketched in figure 24.4a. And right beside that figure (in figure 24.4b) is the graph of a function with multiple jump discontinuities.

The simplest example of a function with a jump discontinuity is the basic step function, $\text{step}(t)$. Just looking at its graph (figure 24.3a on page 487) you can see that it has a jump discontinuity at $t = 0$ with $\text{jump} = 1$, and $y = 1/2$ as the Y-coordinate of the midpoint.

On the other hand, consider the functions

$$f(t) = \frac{1}{(t-2)^2} \qquad \text{and} \qquad g(t) = \begin{cases} 0 & \text{if} \quad t < 2 \\ \dfrac{1}{(t-2)^2} & \text{if} \quad 2 < t \end{cases} ,$$

sketched in figures 24.5a and 24.5b, respectively. Both have discontinuities as $t = 2$. In each case, however, the limit of the function as $t \to 2$ from the right is infinite. Hence, we do not view these discontinuities as "jump" discontinuities.

Piecewise Continuity

We say that a function f is *piecewise continuous* on an *finite* open interval (a, b) if and only if both of the following hold:

1. f is continuous on the interval except for, at most, a finite number of jump discontinuities in (a, b) .

2. The endpoint limits
$$\lim_{t\to a+} f(t) \qquad \text{and} \qquad \lim_{t\to b-} f(t)$$
exist and are finite.

We extend this concept to functions on infinite open intervals (such as $(0, \infty)$) by defining a function f to be *piecewise continuous* on an *infinite* open interval if and only if f is piecewise continuous on every finite open subinterval. In particular then, a function f being piecewise continuous on $(0, \infty)$ means that

$$\lim_{t\to 0^+} f(t)$$

is a finite value, and that, for every finite, positive value T , $f(t)$ has at most a finite number of discontinuities on the interval $(0, T)$, with each of those being a jump discontinuity.

For some of our discussions, we will only need our function f to be piecewise continuous on $(0, \infty)$. Strictly speaking, this says nothing about the possible value of $f(t)$ when $t = 0$. If, however, we are dealing with initial-value problems, then we may require our function f to be *piecewise continuous on* $[0, \infty)$, which simply means f is piecewise continuous on $(0, \infty)$, defined at $t = 0$, and

$$f(0) = \lim_{t \to 0^+} f(t) \quad .$$

Keep in mind that "a finite number" of jump discontinuities can be zero, in which case f has no discontinuities and is, in fact, continuous on that interval. What is important is that a piecewise continuous function cannot 'blow up' at any (finite) point in or at the ends of the interval. At worst, it has only 'a few' jump discontinuities in each finite subinterval.

The functions sketched in figure 24.4 are piecewise continuous, at least over the intervals in the figures. And any step function is piecewise continuous on $(0, \infty)$. On the other hand, the functions sketched in figures 24.5a and 24.5b, are not piecewise continuous on $(0, \infty)$ because they both "blow up" at $t = 2$. Consider even the function

$$f(t) = \frac{1}{t} \quad ,$$

sketched in figure 24.5c. Even though this function is continuous on the interval $(0, \infty)$, we do not consider it to be piecewise continuous on $(0, \infty)$ because

$$\lim_{t \to 0^+} \frac{1}{t} = \infty \quad .$$

Two simple observations will soon be important to us:

1. If f is piecewise continuous on $(0, \infty)$, and T is any positive finite value, then the integral

$$\int_0^T f(t)\, dt$$

is well defined and evaluates to a finite number. Remember, geometrically, this integral is the "net area" between the graph of f and the T–axis over the interval $(0, T)$. The piecewise continuity of f assures us that f does not "blow up" at any point in $(0, T)$, and that we can divide the graph of f over $(0, T)$ into a finite number of fairly nicely behaved 'pieces' (see figure 24.4b) with each piece enclosing finite area.

2. The product of any two piecewise continuous functions f and g on $(0, \infty)$ will, itself, be piecewise continuous on $(0, \infty)$. You can easily verify this yourself using the fact that

$$\lim_{t \to t_0^{\pm}} f(t)g(t) = \lim_{t \to t_0^{\pm}} f(t) \times \lim_{t \to t_0^{\pm}} g(t) \quad .$$

Combining the above two observations with the obvious fact that, for any real value of s , $g(t) = e^{-st}$ is a piecewise continuous function of t on $(0, \infty)$ gives us:

Lemma 24.3

Let f be a piecewise continuous function on $(0, \infty)$, and let T be any finite positive number. Then the integral

$$\int_0^T f(t)e^{-st}\,dt$$

is a well-defined finite number for each real value s.

Because of our interest in the Laplace transform, we will want to ensure that the above integral converges to a finite number as $T \to \infty$. That is the next issue we will address.

Exponential Order*

Let f be a function on $(0, \infty)$, and let s_0 be some real number. We say that f is *of exponential order* s_0 if and only if there are finite constants M and T such that

$$|f(t)| \leq Me^{s_0 t} \qquad \text{whenever} \quad T \leq t \quad . \tag{24.21}$$

Often, the precise value of s_0 is not particularly important. In these cases we may just say that f is *of exponential order* to indicate that it is of exponential order s_0 for some value s_0.

Saying that f is of exponential order is just saying that the graph of $|f(t)|$ is bounded above by the graph of some constant multiple of some exponential function on some interval of the form $[T, \infty)$. Note that, if this is the case and s is any real number, then

$$\left|f(t)e^{-st}\right| = |f(t)|\,e^{-st} \leq Me^{s_0 t}e^{-st} = Me^{-(s-s_0)t} \qquad \text{whenever} \quad T \leq t \quad .$$

Moreover, if $s > s_0$, then $s - s_0$ is positive, and

$$\left|f(t)e^{-st}\right| \leq Me^{-(s-s_0)t} \to 0 \qquad \text{as} \quad t \to \infty \quad . \tag{24.22}$$

Thus, in the future, we will automatically know that

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

whenever f is of exponential order s_0 and $s > s_0$.

Transforms of Piecewise Continuous Functions of Exponential Order

Now, suppose f is a piecewise continuous function of exponential order s_0 on the interval $(0, \infty)$. As already observed, the piecewise continuity of f assures us that

$$\int_0^T f(t)e^{-st}\,dt$$

is a well-defined finite number for each $T > 0$. And if $s > s_0$, then inequality (24.22), above, tells us that $f(t)e^{-st}$ is shrinking to 0 as $t \to \infty$ at least as fast as a constant multiple of some decreasing exponential. It is easy to verify that this is fast enough to ensure that

$$\lim_{T\to\infty} \int_0^T f(t)e^{-st}\,dt$$

* More precisely: **Exponential Order as $t \to \infty$**. One can have "exponential order as $t \to -\infty$" and even "as $t \to 3$"). However, we are not interested in those cases, and it is silly to keep repeating "as $t \to \infty$".

converges to some finite value. And that gives us the following theorem on conditions ensuring the existence of Laplace transforms.

Theorem 24.4
If f is both piecewise continuous on $(0, \infty)$ and of exponential order s_0, then

$$F(s) = \mathcal{L}[f(t)]|_s = \int_0^\infty f(t)e^{-st}\,dt$$

is a well-defined function for $s > s_0$.

In the next several chapters, we will often assume that our functions of t are both piecewise continuous on $(0, \infty)$ and of exponential order. Mind you, not all Laplace transformable functions satisfy these conditions. For example, we've already seen that t^α with $-1 < \alpha$ is Laplace transformable. But

$$\lim_{t \to 0^+} t^\alpha = \infty \qquad \text{if} \quad \alpha < 0 \quad .$$

So those functions given by t^α with $-1 < \alpha < 0$ (such as $1/\sqrt{t}$) are not not piecewise continuous on $(0, \infty)$, even though they are certainly Laplace transformable. Still, all the other functions on the left side of table 24.1 on page 484 are piecewise continuous on $(0, \infty)$ and are of exponential order. More importantly, the functions that naturally arise in applications in which the Laplace transform may be useful are usually piecewise continuous on $(0, \infty)$ and of exponential order.

By the way, since you've probably just glanced at table 24.1 on page 484, go back at look at the functions on the right side of the table. Observe that

1. these functions have no discontinuities in the intervals on which they are defined,

and

2. they all shrink to 0 as $s \to \infty$.

It turns out that you can extend the work used to obtain the above theorem to show that the above observations hold much more generally. More precisely, the above theorem can be extended to:

Theorem 24.5
If f is both piecewise continuous on $(0, \infty)$ and of exponential order s_0, then

$$F(s) = \mathcal{L}[f(t)]|_s = \int_0^\infty f(t)e^{-st}\,dt$$

is a continuous function on (s_0, ∞) and

$$\lim_{s \to \infty} F(s) = 0 \quad .$$

We will verify this theorem at the end of the next section.

24.6 Further Notes on Piecewise Continuous and Exponentially Bounded Functions

Issues Regarding Piecewise Continuous Functions on $(0, \infty)$

In the next several chapters, we will be concerned mainly with functions that are piecewise continuous on $(0, \infty)$. There are a few small technical issues regarding these functions that could become significant later if we don't deal with them now. These issues concern the values of such functions at jumps.

On the Value of a Function at a Jump

Take a look at figure 24.4b on page 491. Call the function sketched there f , and consider evaluating, say,

$$\int_0^{t_2} f(t)e^{-st}\,dt \quad .$$

The obvious approach is to break up the integral into three pieces,

$$\int_0^{t_2} f(t)e^{-st}\,dt \;=\; \int_0^{t_0} f(t)e^{-st}\,dt \;+\; \int_{t_0}^{t_1} f(t)e^{-st}\,dt \;+\; \int_{t_1}^{t_2} f(t)e^{-st}\,dt \quad ,$$

and use values/formulas for f over the intervals $(0, t_0)$, (t_0, t_1) and (t_1, t_2) to compute the individual integrals in the above sum. What you would not worry about would be the actual values of f at the points of discontinuity, t_0 , t_1 and t_2 . In particular, it would not matter if

$$f(t_0) \;=\; \lim_{t\to t_0{}^-} f(t) \qquad \text{or} \qquad f(t_0) \;=\; \lim_{t\to t_0{}^+} f(t)$$

or

$$f(t_0) \;=\; \text{the } Y\text{-coordinate of the midpoint of the jump} \quad .$$

This extends an observation made when we computed the Laplace transform of the shifted step function. There, we found that the precise value of $\text{step}_\alpha(t)$ at $t = \alpha$ was irrelevant to the computation of $\mathcal{L}\big[\text{step}_\alpha(t)\big]$. And the pseudo-computations in the previous paragraph point out that, in general, the value of any piecewise continuous function at a point of discontinuity will be irrelevant to the integral computations we will be doing with these functions.

Parallel to these observations are the observations of how we use functions with jump discontinuities in applications. Typically, a function with a jump discontinuity at $t = t_0$ is modeling something that changes so quickly around $t = t_0$ that we might as well pretend the change is instantaneous. Consider, for example, the output of a one-lumen incandescent light bulb switched on at $t = 2$: Until is is switched on, the bulb's light output is 0 lumen. For a brief period around $t = 2$ the filament is warming up and the light output increases from 0 to 1 lumen, and remains at 1 lumen thereafter. In practice, however, the warm-up time is so brief that we don't notice it, and are content to describe the light output by

$$\text{light output at time } t \;=\; \begin{Bmatrix} 0 \text{ lumen} & \text{if} & t < 2 \\ 1 \text{ lumen} & \text{if} & 2 < t \end{Bmatrix} \;=\; \text{step}_2(t) \text{ lumen}$$

without giving any real thought as to the value of the light output the very instant we are turning on the bulb.[4]

What all this is getting to is that, for our work involving piecewise continuous functions on $(0,\infty)$,

the value of a function f at any point of discontinuity t_0 in $(0,\infty)$ is irrelevant.

What is important is not $f(t_0)$ but the one-sided limits

$$\lim_{t\to t_0^-} f(t) \qquad \text{and} \qquad \lim_{t\to t_0^+} f(t) \quad .$$

Because of this, we will not normally specify the value of a function at a discontinuity, at least not while developing Laplace transforms. If this disturbs you, go ahead and assume that, unless otherwise indicated, the value of a function at each jump discontinuity is given by the Y-coordinate of the jump's midpoint. It's as good as any other value.

Equality of Piecewise Continuous Functions

Because of the irrelevance of the value of a function at a discontinuity, we need to slightly modify what it means to say " $f = g$ on some interval". Henceforth, let us say that

$$f \;=\; g \text{ on some interval (as piecewise continuous functions)}$$

means

$$f(t) \;=\; g(t)$$

for every t in the interval at which f and g are continuous. We will not insist that f and g be equal at the relatively few points of discontinuity in the functions. But do note that we will still have

$$\lim_{t\to t_0^\pm} f(t) \;=\; \lim_{t\to t_0^\pm} g(t)$$

for every t_0 in the interval. Consequently, the graphs of f and g will have the same 'jumps' in the interval.

By the way, the phrase "as piecewise continuous functions" in the above definition is recommended, but is often forgotten.

!►*Example 24.10:* *The functions*

$$\text{step}_2(t) \quad , \quad f(t) \;=\; \begin{cases} 0 & \textit{if} \quad t \le 2 \\ 1 & \textit{if} \quad 2 < t \end{cases} \quad \textit{and} \quad g(t) \;=\; \begin{cases} 0 & \textit{if} \quad t < 2 \\ 1 & \textit{if} \quad 2 \le t \end{cases}$$

all satisfy

$$\text{step}_2(t) \;=\; f(t) \;=\; g(t)$$

for all values of t in $(0,\infty)$ except $t = 2$, at which each has a jump. So, as piecewise continuous functions,

$$\text{step}_2 \;=\; f \;=\; g \qquad \textit{on} \quad (0,\infty) \quad .$$

[4] On the other hand, "What is the light output of a one-lumen light bulb the very instant the light is turned on?" is a nice question to meditate upon if you are studying Zen.

Conversely, if we know $h = \text{step}_2$ *on* $(0, \infty)$ *(as piecewise continuous functions), then we know*

$$h(t) = \begin{cases} 0 & \text{if} \quad 0 < t < 2 \\ 1 & \text{if} \quad 2 < t \end{cases} \quad .$$

We do not know (nor do we care about) the value of $h(t)$ *when* $t = 2$ *(or when* $t < 0$ *).*

Testing for Exponential Order

Before deriving this test for exponential order, it should be noted that the "order" is not unique. After all, if

$$|f(t)| \leq Me^{s_0 t} \qquad \text{whenever} \quad T \leq t \quad ,$$

and $s_0 \leq s_1$, then

$$|f(t)| \leq Me^{s_0 t} \leq Me^{s_1 t} \qquad \text{whenever} \quad T \leq t \quad ,$$

proving the following little lemma:

Lemma 24.6
If f *is of exponential order* s_0 *, then* f *is of exponential order* s_1 *for every* $s_1 \geq s_0$ *.*

Now here is the test:

Lemma 24.7 (test for exponential order)
Let f *be a function on* $(0, \infty)$ *.*

1. *If there is a real value* s_0 *such that*

$$\lim_{t \to \infty} f(t)e^{-s_0 t} = 0 \quad ,$$

then f *is of exponential order* s_0 *.*

2. *If*

$$\lim_{t \to \infty} f(t)e^{-st}$$

does not converge to 0 *for any real value* s *, then* f *is not of exponential order.*

PROOF: First, assume

$$\lim_{t \to \infty} f(t)e^{-s_0 t} = 0$$

for some real value s_0 , and let M be any finite positive number you wish (it could be 1 , $1/2$, 827 , whatever). By the definition of "limits", the above assures us that, if t is large enough, then $f(t)e^{-s_0 t}$ is within M of 0 . Letting T be any single "large enough" value of t , we then must have

$$t \geq T \quad \Longrightarrow \quad \left| f(t)e^{-s_0 t} - 0 \right| \leq M \quad .$$

By elementary algebra, we can rewrite this as

$$|f(t)| \leq Me^{s_0 t} \qquad \text{whenever} \quad T \leq t \quad ,$$

which is exactly what we mean when we say " f is of exponential order s_0 ". That confirms the first part of the lemma.

To verify the second part of the lemma, assume

$$\lim_{t\to\infty} f(t)e^{-st}$$

does not converge to 0 for any real value s. If f were of exponential order, then it is of exponential order s_0 for some finite real number s_0, and, as noted in the discussion of expression (24.22) on page 494, we would then have that

$$\lim_{t\to\infty} f(t)e^{-st} = 0 \qquad \text{for} \quad s > s_0 \quad .$$

But we've assumed this is not possible; thus, it is not possible for f to be of exponential order. ∎

Proving Theorem 24.5
The Theorem and a Bad Proof

The basic premise of theorem 24.5 is that we have a piecewise continuous function f on $(0, \infty)$ which is also of exponential order s_0. From the previous theorem, we know

$$F(s) = \mathcal{L}[f(t)]|_s = \int_0^\infty f(t)e^{-st}\,dt$$

is a well-defined function on (s_0, ∞). Theorem 24.5 further claims that

1. $F(s) = \mathcal{L}[f(t)]|_s$ is continuous on (s_0, ∞). That is,
$$\lim_{s\to s_1} F(s) = F(s_1) \qquad \text{for each} \quad s_1 > s_0 \quad .$$

and

2. $\displaystyle\lim_{s\to\infty} F(s) = 0 \quad .$

In a naive attempt to verify these claims, you might try

$$\begin{aligned}\lim_{s\to s_1} F(s) &= \lim_{s\to s_1}\int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^\infty \lim_{s\to s_1} f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-s_1 t}\,dt = F(s_1) \quad ,\end{aligned}$$

and

$$\begin{aligned}\lim_{s\to\infty} F(s) &= \lim_{s\to\infty}\int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^\infty \lim_{s\to\infty} f(t)e^{-st}\,dt = \int_0^\infty f(t)\cdot 0\,dt = 0 \quad .\end{aligned}$$

Unfortunately, these computations assume

$$\lim_{s\to\alpha}\int_0^\infty g(t,s)\,dt = \int_0^\infty \lim_{s\to\alpha} g(t,s)\,dt$$

which is NOT always true. Admittedly, it often is true. But there are exceptions. And because there are exceptions, we cannot rely on this sort of "switching of limits with integrals" to prove our claims.

Preliminaries

There are two small observations that will prove helpful here and elsewhere.

The first concerns any function f which is piecewise continuous on $(0, \infty)$ and satisfies

$$|f(t)| \leq M_T e^{s_0 t} \qquad \text{whenever} \quad T \leq t \quad ,$$

for two positive values M_T and T. For convenience, let

$$g(t) = f(t)e^{-s_0 t} \qquad \text{for} \quad t > 0 \quad .$$

This is another piecewise continuous function on $(0, \infty)$, but it satisfies

$$|g(t)| = \left|f(t)e^{-s_0 t}\right| = |f(t)|\, e^{-s_0 t} \leq M e^{s_0 t} e^{-s_0 t} = M \qquad \text{for} \quad T < t \quad ,$$

On the other hand, the piecewise continuity of g on $(0, \infty)$ means that g does not "blow up" anywhere in or at the endpoints of $(0, T)$. So it is easy to see (and to prove) that there is a constant B such that

$$|g(t)| \leq B \qquad \text{for} \quad 0 < t < T \quad .$$

Letting M_0 be the larger of B and M_T, we now have that

$$|g(t)| \leq M_0 \qquad \text{if} \quad 0 < t < T \quad \text{or} \quad T \leq t \quad .$$

So,

$$|f(t)|\, e^{-s_0 t} = \left|f(t)e^{-s_0 t}\right| = |g(t)| \leq M_0 \qquad \text{for} \quad 0 < t \quad .$$

Multiply through by the exponential, and you've got:

Lemma 24.8
Assume f is a piecewise continuous function on $(0, \infty)$ which is also of exponential order s_0. Then there is a constant M_0 such that

$$|f(t)| \leq M_0 e^{s_0 t} \qquad \text{for} \quad 0 < t \quad .$$

The above lemma will let us use the exponential bound $M_0 e^{s_0 t}$ over all of $(0, \infty)$, and not just (M_T, ∞). The next lemma is one you should either already be acquainted with, or can easily confirm on your own.

Lemma 24.9
If g is an integrable function on the interval (a, b), then

$$\left|\int_a^b g(t)\, dt\right| \leq \int_a^b |g(t)|\, dt \quad .$$

The Proof of Theorem 24.5

Now we will prove the two claims of theorem 24.5. Keep in mind that f is a piecewise continuous function on $(0, \infty)$ of exponential order s_0, and that

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad \text{for} \quad s > s_0 \quad .$$

We will make repeated use of the fact, stated in lemma 24.8 just above, that there is a constant M_0 such that

$$|f(t)| \leq M_0 e^{s_0 t} \qquad \text{for} \quad 0 < t \quad . \tag{24.23}$$

Since the second claim is a little easier to verify, we will start with that.

Proof of the Second Claim

The second claim is that

$$\lim_{s\to\infty} F(s) = 0 \quad ,$$

which, of course, can be proven by showing

$$\lim_{s\to\infty} |F(s)| \leq 0 \quad .$$

Now let $s > s_0$. Using inequality (24.23) with the integral inequality form lemma 24.9, we have

$$\begin{aligned} |F(s)| = \left|\int_0^\infty f(t)e^{-st}\,dt\right| &\leq \int_0^\infty \left|f(t)e^{-st}\right|\,dt \\ &= \int_0^\infty |f(t)|\,e^{-st}\,dt \\ &\leq \int_0^\infty M_0 e^{s_0 t}e^{-st}\,dt = M_0 \mathcal{L}\big[e^{s_0 t}\big]\big|_s = \frac{M_0}{s - s_0} \quad . \end{aligned}$$

Thus,

$$\lim_{s\to\infty} |F(s)| \leq \lim_{s\to\infty} \frac{M_0}{s - s_0} = 0 \quad ,$$

confirming the claim.

Proof of the First Claim

The first claim is that F is continuous on (s_0, ∞). To prove this, we need to show that, for each $s_1 > s_0$,

$$\lim_{s\to s_1} F(s) = F(s_1) \quad .$$

Note that this limit can be verified by showing

$$\lim_{s\to s_1} |F(s) - F(s_1)| \leq 0 \quad .$$

Now let s and s_1 be two different points in (s_0, ∞). Again using inequality (24.23) with the integral inequality from lemma 24.9,

$$
\begin{aligned}
|F(s) - F(s_1)| &= \left| \int_0^\infty f(t)e^{-st}\,dt - \int_0^\infty f(t)e^{-s_1 t}\,dt \right| \\
&= \left| \int_0^\infty f(t)\left[e^{-st} - e^{-s_1 t}\right] dt \right| \\
&\leq \int_0^\infty |f(t)| \left|e^{-st} - e^{-s_1 t}\right| dt \;\leq\; \int_0^\infty M_0 e^{s_0 t} \left|e^{-st} - e^{-s_1 t}\right| dt
\end{aligned}
$$

Observe that

$$
\left|e^{-st} - e^{-s_1 t}\right| = \left\{ \begin{array}{ll} +\left[e^{-st} - e^{-s_1 t}\right] & \text{if} \quad s < s_1 \\ -\left[e^{-st} - e^{-s_1 t}\right] & \text{if} \quad s_1 < s \end{array} \right\} = \pm\left[e^{-st} - e^{-s_1 t}\right]
$$

with the sign chosen appropriately. Using this with the preceding sequence of inequalities, we get

$$
\begin{aligned}
|F(s) - F(s_1)| &\leq \int_0^\infty M_0 e^{s_0 t} \left|e^{-st} - e^{-s_1 t}\right| dt \\
&\leq \pm M_0 \int_0^\infty M_0 e^{s_0 t} \left[e^{-st} - e^{-s_1 t}\right] dt \\
&\leq \pm M_0 \left[\int_0^\infty e^{s_0 t} e^{-st}\,dt - \int_0^\infty e^{s_0 t} e^{-s_1 t}\,dt \right] \\
&\leq \pm M_0 \left[\mathcal{L}\left[e^{s_0 t}\right]\Big|_s - \mathcal{L}\left[e^{s_0 t}\right]\Big|_{s_1} \right] \\
&\leq \pm M_0 \left[\frac{1}{s_0 - s} - \frac{1}{s_0 - s_1} \right] \quad .
\end{aligned}
$$

Thus,

$$
\lim_{s \to s_1} |F(s) - F(s_1)| \leq \lim_{s \to s_1} \pm M_0 \left[\frac{1}{s_0 - s} - \frac{1}{s_0 - s_1} \right] = \pm M_0 \left[\frac{1}{s_0 - s_1} - \frac{1}{s_0 - s_1} \right] = 0 \quad ,
$$

which is all we needed to show to confirm the first claim.

Additional Exercises

24.6. *Sketch the graph of each of the following choices of $f(t)$, and then find that function's Laplace transform by direct application of the definition, formula (24.1) on page 475 (i.e., compute the integral). Also, if there is a restriction on the values of s for which the formula of the transform is valid, state that restriction.*

a. $f(t) = 4$

b. $f(t) = 3e^{2t}$

c. $f(t) = \begin{cases} 2 & \text{if} \quad t \leq 3 \\ 0 & \text{if} \quad 3 < t \end{cases}$

d. $f(t) = \begin{cases} 0 & \text{if} \quad t \leq 3 \\ 2 & \text{if} \quad 3 < t \end{cases}$

e. $f(t) = \begin{cases} e^{2t} & \text{if} \quad t \le 4 \\ 0 & \text{if} \quad 4 < t \end{cases}$ **f.** $f(t) = \begin{cases} e^{2t} & \text{if} \quad 1 < t \le 4 \\ 0 & \text{otherwise} \end{cases}$

g. $f(t) = \begin{cases} t & \text{if} \quad 0 < t \le 1 \\ 0 & \text{otherwise} \end{cases}$ **h.** $f(t) = \begin{cases} 0 & \text{if} \quad 0 < t \le 1 \\ t & \text{otherwise} \end{cases}$

24.7. *Find the Laplace transform of each, using either formula (24.7) on page 480, formula (24.9) on page 481 or formula (24.4) on page 481, as appropriate:*

a. t^4 **b.** t^9 **c.** e^{7t} **d.** e^{-7t} **e.** e^{i7t}

24.8. *Find the Laplace transform of each of the following, using table 24.1 on page 484 (Transforms of Common Functions) and the linearity of the Laplace transform:*

a. $\sin(3t)$ **b.** $\cos(3t)$ **c.** 7

d. $\cosh(3t)$ **e.** $\sinh(4t)$ **f.** $3t^2 - 8t + 47$

g. $6e^{2t} + 8e^{-3t}$ **h.** $3\cos(2t) + 4\sin(6t)$ **i.** $3\cos(2t) - 4\sin(2t)$

24.9. *Compute the following Laplace transforms:*

a. $t^{3/2}$ **b.** $t^{5/2}$ **c.** $t^{-1/3}$ **d.** $\sqrt[4]{t}$ **e.** $\text{step}_2(t)$

24.10. *For the following, let*

$$f(t) = \begin{cases} 1 & \text{if} \quad t < 2 \\ 0 & \text{if} \quad 2 \le t \end{cases} .$$

a. *Verify that* $f(t) = 1 - \text{step}_2(t)$ *, and using this and linearity,*

b. *compute* $\mathcal{L}[f(t)]|_s$.

24.11. *Find the Laplace transform of each of the following, using table 24.1 on page 484 (Transforms of Common Functions) and the first translation identity:*

a. te^{4t} **b.** $t^4 e^t$ **c.** $e^{2t}\sin(3t)$

d. $e^{2t}\cos(3t)$ **e.** $e^{3t}\sqrt{t}$ **f.** $e^{3t}\,\text{step}_2(t)$

24.12. *Verify each of the following using table 24.1 on page 484 (Transforms of Common Functions) and the first translation identity (assume* α *and* ω *are real-valued constants and* n *is a positive integer):*

a. $\mathcal{L}\left[t^n e^{\alpha t}\right]\Big|_s = \dfrac{n!}{(s-\alpha)^{n+1}} \qquad \text{for} \quad s > \alpha$

b. $\mathcal{L}\left[e^{\alpha t}\sin(\omega t)\right]\Big|_s = \dfrac{\omega}{(s-\alpha)^2 + \omega^2} \qquad \text{for} \quad s > \alpha$

c. $\mathcal{L}\left[e^{\alpha t}\cos(\omega t)\right]\Big|_s = \dfrac{s-\alpha}{(s-\alpha)^2 + \omega^2} \qquad \text{for} \quad s > \alpha$

d. $\mathcal{L}\left[e^{\alpha t}\,\text{step}_\omega(t)\right]\Big|_s = \dfrac{1}{s-\alpha}e^{-\omega(s-\alpha)} \qquad \text{for} \quad s > \alpha \quad \text{and} \quad \omega \ge 0$

24.13. *The following problems all concern the Gamma function,*

$$\Gamma(\sigma) = \int_0^\infty e^{-u} u^{\sigma-1}\, du \quad .$$

a. *Using integration by parts, show that* $\Gamma(\sigma+1) = \sigma\Gamma(\sigma)$ *whenever* $\sigma > 0$.

b i. *By using an appropriate change of variables and symmetry, verify that*

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-\tau^2}\, d\tau \quad .$$

ii. *Starting with the observation that, by the above,*

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \left(\int_{-\infty}^\infty e^{-x^2}\, dx\right)\left(\int_{-\infty}^\infty e^{-y^2}\, dy\right) = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2}\, dx\, dy \quad ,$$

show that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad .$$

(Hint: Use polar coordinates to integrate the double integral.)

24.14. *Several functions are given below. Sketch the graph of each over an appropriate interval, and decided whether each is or is not piecewise continuous on* $(0, \infty)$.

a. $f(t) = 2\,\text{step}_3(t)$

b. $g(t) = \text{step}_2(t) - \text{step}_3(t)$

c. $\sin(t)$

d. $\dfrac{\sin(t}{t}$

e. $\tan(t)$

f. $\sqrt{t}$

g. $\dfrac{1}{\sqrt{t}}$

h. $t^2 - 1$

i. $\dfrac{1}{t^2-1}$

j. $\dfrac{1}{t^2+1}$

k. *The "ever increasing stair" function,*

$$\text{stair}(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad 0 < t < 1 \\ 2 & \text{if} \quad 2 < t < 3 \\ 3 & \text{if} \quad 3 < t < 4 \\ 4 & \text{if} \quad 4 < t < 5 \\ \vdots & \qquad \vdots \end{cases}$$

24.15. *Assume* f *and* g *are two piecewise continuous functions on an interval* (a, b) *containing the point* t_0 . *Assume further that* f *has a jump discontinuity at* t_0 *while* g *is continuous at* t_0 . *Verify that the jump in the product* fg *at* t_0 *is given by*

$$\text{"the jump in } f \text{ at } t_0\text{"} \times g(t_0) \quad .$$

24.16. *Using the test for exponential order (lemma 24.7 on page 498), determine which of the following are of exponential order, and, for each which is of exponential order, determine the possible values for the order.*

a. e^{3t} **b.** t^2 **c.** te^{3t}

d. e^{t^2} **e.** $\sin(t)$

24.17. *For the following, let* α *and* σ *be any two positive numbers.*

a. *Using basic calculus, show that* $t^{\alpha}e^{-\sigma t}$ *has a maximum value* $M_{\alpha,\sigma}$ *on the interval* $[0,\infty)$ *. Also, find both where this maximum occurs and the value of* $M_{\alpha,\sigma}$ *.*

b. *Explain why this confirms that*

i. $t^{\alpha} \leq M_{\alpha,\sigma}e^{\sigma t}$ *whenever* $t > 0$*, and that*

ii. t^{α} *is of exponential order* σ *for any* $\sigma > 0$ *.*

24.18. *Assume* f *is a piecewise continuous function on* $(0,\infty)$ *of exponential order* s_0 *, and let* α *and* σ *be any two positive numbers. Using the results of the last exercise, show that* $t^{\alpha}f(t)$ *is piecewise continuous on* $(0,\infty)$ *and of exponential order* $s_0 + \sigma$ *.*