22

Springs: Part II (Forced Vibrations)

Let us look, again, at those mass/spring systems discussed in chapter 17. Remember, in such a system we have a spring with one end attached to an immobile wall and the other end attached to some object that can move back and forth under the influences of the spring and whatever friction may be in the system. Now that we have methods for dealing with nonhomogeneous differential equations (in particular, the method of educated guess), we can expand our investigations to mass/spring systems that are under the influence of outside forces such as gravity or of someone pushing and pulling the object. Of course, the limitations of the method of guess will limit the forces we can consider. Still, these forces happen to be particularly relevant to mass/spring systems, and our analysis will lead to some very interesting results — results that can be extremely useful not just when considering springs, but also when considering other systems in which things vibrate or oscillate.

22.1 The Mass/Spring System

In chapter 17, we derived

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = F \quad .$$

to model the mass/spring system. In this differential equation:

1. $y = y(t)$ is the position (in meters) at time t (in seconds) of the object attached to the spring. As before, the Y–axis is positioned so that

   (a) $y = 0$ is the location of the object when the spring is at its natural length. (This is the "equilibrium point" of the object, at least when $F = 0$.)

   (b) $y > 0$ when the spring is stretched.

   (c) $y < 0$ when the spring is compressed.

   In chapter 17 we visualized the spring as laying horizontally as in figure 22.1a, but that was just to keep us from thinking about the effect of gravity on this mass/spring system. Now, we can allow the spring (and Y–axis) to be either horizontal or vertical or even at some other angle. All that is important is that the motion of the object only be along the Y–axis. (Do note, however, that if the spring is hanging vertically, as in figure 22.1c, then the Y–axis is actually pointing *downward*.)

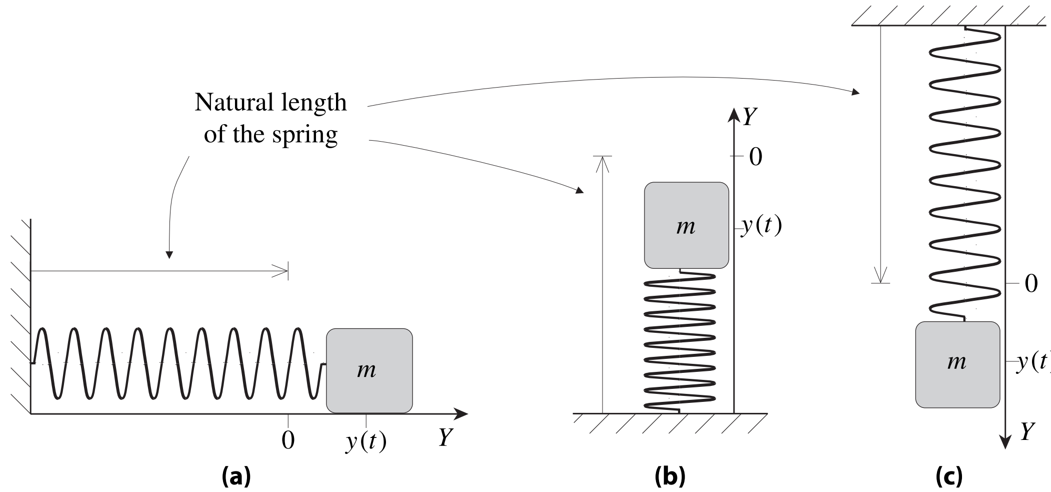

Figure 22.1: Three equivalent mass/spring systems with slightly different orientations.

2. m is the mass (in kilograms) of the object attached to the spring.
3. κ is the spring constant, a positive quantity describing the "stiffness" of the spring (with "stiffer" springs having larger values for κ).
4. γ is the damping constant, a nonnegative quantity describing how much friction is in the system resisting the motion (with $\gamma = 0$ corresponding to an ideal system with no friction whatsoever).
5. F is the sum of all forces acting on the spring other than those due to the spring responding to being compressed and stretched, and the frictional forces in the system resisting motion.

Since we are expanding on the results from chapter 17, let us recall some of the major results derived there regarding the general solution y_h to the corresponding homogeneous equation

$$m\frac{d^2 y_h}{dt^2} + \gamma\frac{dy_h}{dt} + \kappa y = 0 \quad . \tag{22.1}$$

If there is no friction in the system then we say the system is undamped, and the solution to equation (22.1) is

$$y_h(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

or, equivalently,

$$y_h(t) = A\cos(\omega_0 t - \phi)$$

where

$$\omega_0 = \sqrt{\frac{\kappa}{m}}$$

is the natural angular frequency of the system, and the other constants are related by

$$A = \sqrt{(c_1)^2 + (c_2)^2} \quad , \quad \cos(\phi) = \frac{c_1}{A} \quad \text{and} \quad \sin(\phi) = \frac{c_2}{A} \quad .$$

When convenient, we can rewrite the above formulas for y_h in terms of the system's natural frequency ν_0 by simply replacing each ω_0 with $2\pi\nu_0$.

If there is friction resisting the object's motion (i.e., $0 < \gamma$), then we say the system is damped, and we can further classify the system as being underdamped, critically damped and overdamped, depending on the precise relation between γ , κ and m . Slightly different solutions to equation (22.1) arise, depending on whether we are dealing with an under-, critically or overdamped system. In any of these cases, though, each term of the solution $y_h(t)$ has an exponentially decreasing factor. This factor ensures that

$$y_h(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad .$$

That is what will be particularly relevant in this chapter.

(At this point, you may want to go back and quickly review chapter 17 yourself, verifying the above and filling in some of the details glossed over. In particular, you may want to glance back over the brief note on 'units' starting on page 360.)

22.2 Constant Force

Let us first consider the case where the external force is constant. For example, the spring might be hanging vertically and the external force is the force of gravity on the object. Letting F_0 be that constant, the differential equation for $y = y(t)$ is

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = F_0 \quad .$$

From our development of the method of guess, we know the general solution is

$$y(t) = y_h(t) + y_p(t)$$

where y_h is as described in the previous section, and the particular solution, y_p , is some constant,

$$y_p(t) = y_0 \qquad \text{for all} \quad t \quad .$$

Plugging this constant solution into the differential equation, we get

$$m \cdot 0 + \gamma \cdot 0 + \kappa y_0 = F_0 \quad .$$

Hence,

$$y_0 = \frac{F_0}{\kappa} \quad . \tag{22.2}$$

If the system is undamped, then

$$y(t) = y_h(t) + y_0 = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + y_0 \quad ,$$

which tells us that the object is oscillating about $y = y_0$. On the other hand, if the system is damped, then

$$\lim_{t\rightarrow\infty} y(t) = \lim_{t\rightarrow\infty} [y_h(t) + y_0] = 0 + y_0 \quad .$$

In this case, $y = y_0$ is where the object finally ends up. Either way, the effect of this constant force is to change the object's equilibrium point from $y = 0$ to $y = y_0$. Accordingly, if L is the natural length of the spring, then we call $L + y_0$ the *equilibrium length* of the spring in this mass/spring system under the constant force F_0 .

It's worth noting that, in practice, y_0 is a quantity that can often be measured. If we also know the force, then relation (22.2) can be used to determine the spring constant κ .

!►***Example 22.1:*** *Suppose we have a spring whose natural length is 1 meter. We attach a 2 kilogram mass to its end and hang it vertically (as in figure 22.1c), letting the force of gravity (near the Earth's surface) act on the mass. After the mass stops bobbing up and down, we measure the spring and find that its length is now 1.4 meters, 0.4 meters longer than its natural length. This gives us y_0 (as defined above), and since we are near the Earth's surface,*

$$F_0 = \text{force of gravity on the mass} = mg = 2 \times 9.8 \left(\frac{kg{\cdot}meter}{sec^2}\right) \quad .$$

Solving equation (22.2) for the spring constant and plugging in the above values, we get

$$\kappa = \frac{F_0}{y_0} = \frac{2 \times 9.8}{.4} = 49 \quad \left(\frac{kg}{sec^2}\right) \quad .$$

22.3 Resonance and Sinusoidal Forces

The mass/spring systems being considered here are but a small subset of all the things that naturally vibrate or oscillate at or around fixed frequencies — consider the swinging of a pendulum after being pushed, the vibrations of a guitar string or a steel beam after being plucked or struck — even an ordinary drinking glass may vibrate when lightly struck. And if these vibrating/oscillating systems are somehow forced to move using a force that, itself, varies periodically, then we may see *resonance*. This is the tendency of the system's vibrations or oscillations to become very large when the frequency of the force is at certain frequencies. Sometimes, these oscillations can be so large that the system breaks. Because of resonance, bridges have collapsed, singers have shattered glass, and small but vital parts of motors have broken off at inconvenient moments. (On the other hand, if you are in a swing, you use resonance in pumping the swing to swing as high as possible, and if you are a musician, your instrument may well use resonance to amplify the mellow tones you want amplified. So resonance is not always destructive.)

We can investigate to phenomenon of resonance in our mass/spring system by looking at the solutions to

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = F(t)$$

when $F(t)$ is a sinusoidal, that is,

$$F = F(t) = a\cos(\eta t) + b\sin(\eta t)$$

where a , b and η are constants with $\eta > 0$. Naturally, we call η the *forcing angular frequency*, and the corresponding frequency, $\mu = {}^{\eta}/_{2\pi}$, the *forcing frequency*. To simplify our imagery, let us use an appropriate trigonometric identity (see page 362), and rewrite this function as a shifted cosine function,

$$F(t) = F_0\cos(\eta t - \phi)$$

where

$$F_0 = \sqrt{a^2 + b^2} \quad , \quad \cos(\phi) = \frac{a}{F_0} \quad \text{and} \quad \sin(\phi) = \frac{b}{F_0} \quad .$$

(Such a force can be generated by an unbalanced flywheel on the object spinning with angular velocity η about an axis perpendicular to the Y–axis. Another way to generate such a force[1] is described in exercise 22.4 on page 453.)

The value of ϕ is relatively unimportant to our investigations, so let's set $\phi = 0$ and just consider the system modeled by

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = F_0\cos(\eta t) \quad . \tag{22.3}$$

You can easily verify, at your leisure, that completely analogous results are obtained using $\phi \neq 0$. The only change is that each particular solution y_p will have a corresponding nonzero shift.

In all that follows, keep in mind that F_0 and η are positive constants. You might even want to observe that letting $\eta \to 0$ leads to the constant force case just considered in the previous section.

It is convenient to consider the undamped and damped systems separately. We'll start with an ideal mass/spring system in which there is no friction to dampen the motion.

Sinusoidal Force in Undamped Systems

If the system is undamped, equation (22.3) reduces to

$$m\frac{d^2y}{dt^2} + \kappa y = F_0\cos(\eta t) \quad ,$$

and the general solution to the corresponding homogeneous equation is

$$y_h(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) \qquad \text{with} \quad \omega_0 = \sqrt{\frac{\kappa}{m}} \quad .$$

To save a little effort later, let's observe that the equation for the natural angular frequency ω_0 can be rewritten as $\kappa = m(\omega_0)^2$. This and a little algebra allows us to rewrite above differential equation as

$$\frac{d^2y}{dt^2} + (\omega_0)^2 y = \frac{F_0}{m}\cos(\eta t) \quad . \tag{22.4}$$

The general solution to this is

$$y(t) = y_h(t) + y_p(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + y_p(t)$$

where y_p is of the form

$$y_p(t) = \begin{cases} A\cos(\eta t) + B\sin(\eta t) & \text{if} \quad \eta \neq \omega_0 \\ At\cos(\omega_0 t) + Bt\sin(\omega_0 t) & \text{if} \quad \eta = \omega_0 \end{cases} \quad .$$

We now have two cases to consider: the case where $\eta = \omega_0$, and the case where $\eta \neq \omega_0$. Let's start with the most interesting of these two cases.

[1] using a chicken

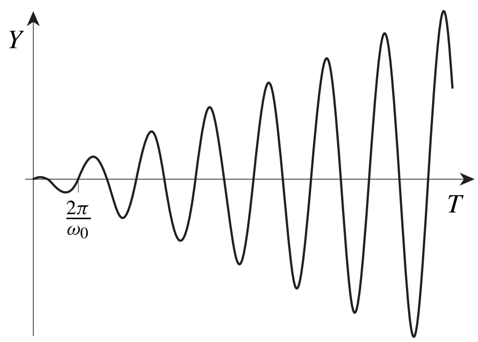

Figure 22.2: Graph of a particular solution exhibiting the "runaway" resonance in an undamped mass/spring system having natural angular frequency ω_0 .

The Case Where $\eta = \omega_0$

If the forcing angular frequency η is the same as the natural angular frequency ω_0 of our mass/spring system, then

$$y_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t) \quad .$$

Right off, you can see that this is describing oscillations of larger and larger amplitude as time goes on. To get a more precise picture of the motion, plug the above formula for $y = y_p$ into differential equation (22.4). You can easily verify that the result is

$$\left[2B\omega_0 - At(\omega_0)^2\right]\cos(\omega_0 t) + \left[-2A\omega_0 - (Bt\omega_0)^2\right]\sin(\omega_0 t)$$
$$+ (\omega_0)^2\left[At\cos(\omega_0 t) + Bt\sin(\omega_0 t)\right] = \frac{F_0}{m}\cos(\omega_0 t) \quad ,$$

which simplifies to

$$2B\omega_0\cos(\omega_0 t) - 2A\omega_0\sin(\omega_0 t) = \frac{F_0}{m}\cos(\omega_0 t) \quad .$$

Comparing the cosine terms and the sine terms on either side of this equation then gives us the pair

$$2B\omega_0 = \frac{F_0}{m} \quad \text{and} \quad -2A\omega_0 = 0 \quad .$$

Thus,

$$B = \frac{F_0}{2m\omega_0} \quad \text{and} \quad A = 0 \quad ,$$

the particular solution is

$$y_p(t) = \frac{F_0}{2m\omega_0}t\sin(\omega_0 t) \quad , \tag{22.5}$$

and the general solution is

$$y(t) = y_h(t) + y_p(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0}t\sin(\omega_0 t) \quad .$$

The graph of y_p is sketched in figure 22.2. Clearly we have true, "run-away" resonance here. As time increases, the size of the oscillations are becoming steadily larger, dwarfing those in the y_h term. With each oscillation, the object moves further and further from its equilibrium point, stretching and compressing the spring more and more (try visualizing that motion!). Wait long enough, and, according to our model, the magnitude of the oscillations will exceed any size desired ... unless the spring breaks.

The Case Where $\eta \neq \omega_0$

Plugging

$$y_p(t) \;=\; A\cos(\eta t) \;+\; B\sin(\eta t)$$

into equation (22.4) yields

$$-\eta^2\,[A\cos(\eta t) + B\sin(\eta t)] \;+\; (\omega_0)^2\,[A\cos(\eta t) + B\sin(\eta t)] \;=\; \frac{F_0}{m}\cos(\eta t) \quad ,$$

which simplifies to

$$\left[(\omega_0)^2 - \eta^2\right] A\cos(\eta t) \;+\; \left[(\omega_0)^2 - \eta^2\right] B\sin(\eta t) \;=\; \frac{F_0}{m}\cos(\eta t) \quad .$$

Comparing the cosine terms and the sine terms on either side of this equation then gives us the pair

$$\left[(\omega_0)^2 - \eta^2\right] A \;=\; \frac{F_0}{m} \qquad \text{and} \qquad \left[(\omega_0)^2 - \eta^2\right] B \;=\; 0 \quad .$$

Thus,

$$A \;=\; \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]} \qquad \text{and} \qquad B \;=\; 0 \quad ,$$

the particular solution is

$$y_p(t) \;=\; \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\cos(\eta t) \quad , \tag{22.6}$$

and the general solution is

$$\begin{aligned} y(t) \;&=\; y_h(t) \;+\; y_p(t) \\ &=\; c_1\cos(\omega_0 t) \;+\; c_2\sin(\omega_0 t) \;+\; \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\cos(\eta t) \quad . \end{aligned} \tag{22.7}$$

Here, the oscillations in the y_p term are not increasing with time. However, if the forcing angular frequency η is close to the natural angular frequency ω_0 of the system (and $F_0 \neq 0$), then

$$(\omega_0)^2 - \eta^2 \approx 0$$

and, so, the amplitude of the oscillations in y_p,

$$\left|\frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\right| \quad .$$

will be very large. If we can adjust the forcing angular frequency η (but keeping F_0 constant), then we can make the amplitude of the oscillations in y_p as large as we could wish. So, again, our solutions are exhibiting “resonance” (perhaps we should call this “near resonance”).

Some Comments About What We've Just Derived

1. *Relevance of the y_h term*: Because the oscillations in the y_p term are not increasing with time, every term in formula (22.7) can play a relatively significant role in the long-term motion of the object in an undamped mass/spring system. In addition, the oscillations in the y_h term can "interfere" with the y_p term to prevent $y(t)$ from reaching its maximum value within the first oscillation from when the object is initially still. In fact, the interaction of the y_h terms with the y_p term can lead to some very interesting motion. However, exploring how y_h and y_p can interact goes a little outside of our current discussions of "resonance". Accordingly, we will delay a more complete discussion of this interaction to section 22.4, after finishing our discussion of resonance.

2. *The limit as near resonance approaches true resonance*: The *resonant frequency* of a system is the forcing frequency at which resonance is most pronounced for that system. The above analysis tells us that the resonant frequency for an undamped mass/spring system is the same as the system's natural frequency. At least, it tells us that when the forcing function is given by a cosine function. It turns out that, using more advanced tools, we can show that we get those ever-increasing oscillations whenever the force is given by a periodic function having the same frequency as the natural frequency of that undamped mass/spring system.

   Something you might expect is that, as η gets closer and closer to the natural angular frequency ω_0, the corresponding solution y of equation 22.4 satisfying some given initial values will approach that obtained when $\eta = \omega_0$. This is, indeed, the case, and its verification will be left as an exercise (exercise 22.5 on page 454).

3. *Limitations in our model*: Keep in mind that our model for the mass/spring system was based on certain assumptions regarding the behavior of springs. In particular, the κ term in our differential equation came from Hooke's law,

   $$F_{\text{spring}}(y) = -\kappa y \quad ,$$

   relating the spring's force to the object's position. As we noted after deriving Hooke's law (page 359), this is a good model for the spring force, *provided the spring is not stretched or compressed too much.* So if our formulas for $y(t)$ have $|y(t)|$ becoming too large for Hooke's law to remain valid, then these formulas are probably are not that accurate after $|y(t)|$ becomes that large. Precisely what happens after the oscillations become so large that our model is no longer valid will depend on the spring and the force.

Sinusoidal Force in Damped Systems

If the system is damped, then we need to consider equation (22.3),

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = F_0\cos(\eta t) \quad , \tag{22.3$'$}$$

assuming $0 < \gamma$. As noted a few pages ago, the terms in y_h, the solution to the corresponding homogeneous differential equation, all contain decaying exponential factors. Hence,

$$y_h(t) \to 0 \quad \text{as} \quad t \to \infty \quad ,$$

and we can assume a particular solution of the form

$$y_p(t) \;=\; A\cos(\eta t) \;+\; B\sin(\eta t) \quad .$$

We can then write the general solution to our nonhomogeneous differential equation as

$$y(t) \;=\; y_h(t) \;+\; y_p(t) \;=\; y_h(t) \;+\; A\cos(\eta t) \;+\; B\sin(\eta t) \quad ,$$

and observe that, as $t \to \infty$,

$$y(t) \;=\; y_h(t) \;+\; y_p(t) \;\to\; 0 \;+\; y_p(t) \;=\; A\cos(\eta t) \;+\; B\sin(\eta t) \quad .$$

This tells us that any long-term behavior of y depends only on y_p, and may explain why, in these cases, we refer to $y_h(t)$ as the *transient* part of the solution and $y_p(t)$ as the *steady-state* part of the solution.

The analysis of the particular solution,

$$y_p(t) \;=\; A\cos(\eta t) \;+\; B\sin(\eta t) \quad ,$$

is relatively straightforward, but a little tedious. We'll leave the computational details to the interested reader (exercise 22.6 on page 455), and quickly summarize the high points.

Plugging in the above formula for y_p into our differential equation and solving for A and B yields

$$y_p(t) \;=\; A\cos(\eta t) \;+\; B\sin(\eta t) \tag{22.8a}$$

with

$$A \;=\; \frac{\eta\gamma F_0}{\left[\kappa - m\eta^2\right]^2 + \eta^2\gamma^2} \qquad \text{and} \qquad B \;=\; -\frac{\left[\kappa - m\eta^2\right] F_0}{\left[\kappa - m\eta^2\right]^2 + \eta^2\gamma^2} \quad . \tag{22.8b}$$

Using a little trigonometry, we can rewrite this as

$$y_p(t) \;=\; C\cos(\eta t - \phi) \tag{22.9a}$$

where the amplitude of these forced vibrations is

$$C \;=\; \frac{F_0}{\sqrt{\left[\kappa - m\eta^2\right]^2 + \eta^2\gamma^2}} \tag{22.9b}$$

and ϕ satisfies

$$\cos(\phi) \;=\; \frac{A}{C} \qquad \text{and} \qquad \sin(\phi) \;=\; \frac{B}{C} \quad . \tag{22.9c}$$

Recalling that the natural angular frequency ω_0 of the corresponding undamped system is related to the spring constant and object's mass by

$$(\omega_0)^2 \;=\; \frac{\kappa}{m} \quad ,$$

we see that the above formula for the amplitude of the forced oscillations can be rewritten as

$$C \;=\; \frac{F_0}{\sqrt{m^2\left[(\omega_0)^2 - \eta^2\right]^2 + \eta^2\gamma^2}} \quad .$$

This value does not blow up with time, nor does it become infinite for any forcing angular frequency η. So we do not have the "run-away" resonance exhibited by an undamped mass/spring system. Still this amplitude does vary with the forcing frequency. With a little work, you can show that, for a given damped mass/spring system, the amplitude of the forced vibrations has a maximum value provided the friction is not too great. To be specific, if

$$\gamma < m\omega_0\sqrt{2} \quad ,$$

then the maximum amplitude occurs when the forcing angular frequency is

$$\eta_0 = \sqrt{(\omega_0)^2 - \frac{1}{2}\left(\frac{\gamma}{m}\right)^2} \quad .$$

This is the resonant angular frequency for the corresponding damped mass/spring system. Observe that it is less than the natural angular frequency of the corresponding undamped system, and that it decreases further as the friction in the system increases. The corresponding maximum amplitude, obtained by letting $\eta = \eta_0$ in the last formula for C, is

$$C_{\text{max}} = \frac{2mF_0}{\gamma\sqrt{(2m\omega_0)^2 - \gamma^2}} \quad .$$

On the other hand, if

$$m\omega_0\sqrt{2} \leq \gamma \quad ,$$

then the amplitude decreases as the forcing angular frequency increases from $\eta = 0$; hence, no there is no resonant frequency.

22.4 More on Undamped Motion Under Nonresonant Sinusoidal Forces

When two or more sinusoidal functions of different frequencies are added together, they can alternatively amplify and interfere with each other to produce a graph that looks somewhat like a single sinusoidal function whose amplitude varies in some regular fashion. This is illustrated in figure 22.3 in which graphs of

$$\cos(\eta t) - \cos(\omega_0 t)$$

have been sketched using one value for ω_0 and two values for η. The first figure (figure 22.3a) illustrates what is commonly called the *beat phenomenon*, in which we appear to have a fairly high frequency sinusoidal whose amplitude seems to be given by another, more slowly varying sinusoidal. This slowly varying sinusoidal gives us the individual "beats" in which the high frequency function intensifies and fades (figure 22.3a shows three beats).

This beat phenomenon is typical of the sum (or difference) of two sinusoidal functions of almost the same frequency, and can be analyzed somewhat using trigonometric identities. For the functions graphed in figure 22.3a we can use basic trigonometric identities to show that

$$\cos(\eta t) - \cos(\omega_0 t) = -2\sin\left(\frac{\eta + \omega_0}{2}t\right)\sin\left(\frac{\eta - \omega_0}{2}t\right) \quad .$$

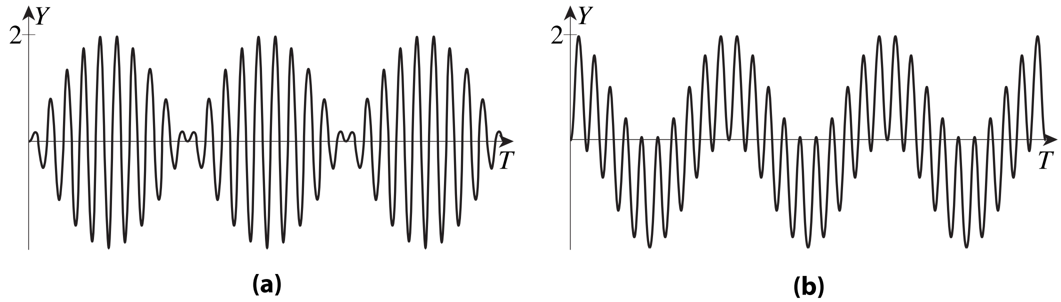

Figure 22.3: Graph of $\cos(\eta t) - \cos(\omega_0 t)$ with $\omega_0 = 2$ and with **(a)** $\eta = 0.9\,\omega_0$ and **(b)** $\eta = 0.1\,\omega_0$. (Drawn using the same horizontal scales for both graphs).

Thus, we have

$$\cos(\eta t) \;-\; \cos(\omega_0 t) \;=\; A(t)\sin\bigl(\omega_{\text{high}} t\bigr) \qquad \text{with} \quad \omega_{\text{high}} \;=\; \frac{\eta + \omega_0}{2}$$

where

$$A(t) \;=\; \pm 2\sin(\omega_{\text{low}} t) \qquad \text{with} \quad \omega_{\text{low}} \;=\; \left|\frac{\omega_0 - \eta}{2}\right| \quad .$$

The angular frequency of the high-frequency wiggles in figure 22.3a are approximately ω_{high} , while ω_{low} corresponds to the angular frequency of pairs of beats. (Visualizing $A(t)$ as a slowly varying amplitude only makes sense if $A(t)$ varies much more slowly than $\sin\bigl(\omega_{\text{high}} t\bigr)$. And, if you think about it, you will realize that, if $\eta \approx \omega_0$, then

$$\omega_{\text{high}} \;=\; \frac{\eta + \omega_0}{2} \approx \omega_0 \qquad \text{and} \qquad \omega_{\text{low}} \;=\; \left|\frac{\omega_0 - \eta}{2}\right| \approx 0 \quad .$$

So this analysis is justified if the forcing frequency is close, but not equal, to the resonant frequency.)

The general phenomenon just described (with or without “beats”) occurs whenever we have a linear combination of sinusoidal functions. In particular, it becomes relevant whenever describing the behavior of an undamped mass/spring system with a sinusoidal forcing function not at resonant frequency. Let’s do one general example:

!►*Example 22.2:* *Consider an undamped mass/spring system having resonant angular frequency ω_0 under the influence of a force given by*

$$F(t) \;=\; F_0\cos(\eta t)$$

where $\eta \neq \omega_0$. Assume, further that the object in the system (with mass m) is initially at rest. In other words, we want to find the solution to the initial-value problem

$$\frac{d^2y}{dt^2} \;+\; (\omega_0)^2 y \;=\; \frac{F_0}{m}\cos(\eta t) \qquad \textit{with} \quad y(0) = 0 \quad \textit{and} \quad y'(0) = 0 \quad .$$

From our work a few pages ago (see equation (22.7)), we know

$$y(t) \;=\; c_1\cos(\omega_0 t) \;+\; c_2\sin(\omega_0 t) \;+\; \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\cos(\eta t) \quad .$$

To satisfy the initial conditions, we must then have

$$0 = y(0) = c_1 \cos(0) + c_2 \sin(0) + \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]} \cos(0)$$

and

$$0 = y'(0) = -c_1\omega_0 \sin(0) + c_2\omega_0 \cos(0) - \frac{F_0\eta}{m\left[(\omega_0)^2 - \eta^2\right]} \sin(0) \quad ,$$

which simplifies to the pair

$$0 = c_1 + \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]} \qquad \text{and} \qquad 0 = c_2\omega_0 \quad .$$

So

$$c_1 = -\frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]} \quad , \quad c_2 = 0 \quad ,$$

and

$$\begin{aligned} y(t) &= -\frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]} \cos(\omega_0 t) + \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]} \cos(\eta t) \\ &= \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\left[\cos(\eta t) - \cos(\omega_0 t)\right] \quad . \end{aligned}$$

If $\eta = 0.9\,\omega_0$ *, the last formula for* y *reduces to*

$$y(t) = \frac{100}{19} \cdot \frac{F_0}{m(\omega_0)^2}\left[\cos(\eta t) - \cos(\omega_0 t)\right] \quad ,$$

and the graph of the object's position at time t *is the same as the graph in figure 22.3a with the amplitude multiplied by*

$$\frac{100}{19} \cdot \frac{F_0}{m(\omega_0)^2} \quad .$$

If $\eta = 0.1\,\omega_0$ *, then*

$$y(t) = \frac{100}{99} \cdot \frac{F_0}{m(\omega_0)^2}\left[\cos(\eta t) - \cos(\omega_0 t)\right] \quad ,$$

and the graph of the object's position at time t *is the same as the graph in figure 22.3a with the amplitude multiplied by*

$$\frac{100}{99} \cdot \frac{F_0}{m(\omega_0)^2}$$

(which, it should be noted, is approximately $^1/_5$ *the amplitude when* $\eta = 0.9\omega_0$ *).*

?► ***Exercise 22.1:*** *Consider the mass/spring system just discussed in the last example. Using the graphs in figure 22.3, try to visualize the motion of the object in this system*

a: *when the forcing frequency is* 0.9 *the natural frequency.*

b: *when the forcing frequency is* 0.1 *the natural frequency.*

Additional Exercises

22.2. *A spring, whose natural length is* 0.1 *meter, is stretched to an equilibrium length of* 0.12 *meter when suspended vertically (near the Earth's surface) with a* 0.01 *kilogram mass at the end.*

a. *Find the spring constant* κ *for this spring.*

b. *Find the natural angular frequency* ω_0 *and the natural frequency* ν_0 *for this mass/spring system, assuming the system is undamped.*

22.3. *All of the following concern a single spring of natural length* 1 *meter mounted vertically with one end attached to the floor (as in figure 22.1b on page 442).*

a. *Suppose we place a* 25 *kilogram box of frozen ducks on top of the spring, and, after moving the box down to its equilibrium point, we find that the length of the spring is now* 0.9 *meter.*

i. *What is the spring constant for this spring?*

ii. *What is the natural angular frequency of the mass/spring system assuming the system is undamped?*

iii. *Approximately how many times per second will this box bob up and down assuming the system is undamped, and the box is moved from its equilibrium point and released? (i.e., what is the natural frequency?)*

b. *Suppose we replace the box of frozen ducks with a single* 2 *kilogram chicken.*

i. *Now what is the equilibrium length of the spring?*

ii. *What is the natural angular frequency of the undamped chicken/spring system?*

iii. *Assuming the system is undamped, not initially at equilibrium, and the chicken is not flapping its wings, how many times per second does this bird bob up and down?*

c. *Next, the chicken is replaced with a box of imported fruit. After the box stops bobbing up and down, we find that the length of the spring is* 0.85 *meter. What is the mass of this box of fruit?*

d. *Finally, everything is taken off the spring, and a bunch of red, helium filled balloons are tied onto the end of the spring, stretching it to an new equilibrium length of* 1.02 *meters. What is the buoyant force of this bunch of balloons?*

22.4. *A live* 2 *kilogram chicken is securely attached to the top of the the floor-mounted spring of natural length* 1 *meter (similar to that described in exercise 22.3, above). Nothing else is on the spring. Knowing that the spring will break if it is stretched or compressed by half its natural length, and hoping to use the resonance of the system to stretch or compress the spring to its breaking point, the chicken starts flapping its wings. The force generated by the chicken's flapping wings* t *seconds after it starts to flap is*

$$F(t) = F_0 \cos(2\pi \mu t)$$

where μ is the frequency of the wing flapping (flaps/second) and

$$F_0 = 3 \quad \left(\frac{\text{kg}\cdot\text{meter}}{\text{sec}^2}\right) \quad .$$

For the following exercises, also assume the following:

1. *This chicken/spring system is undamped and has natural frequency $\nu_0 = 6$ (hertz).*

2. *The model given by differential equation (22.4) on page 445 is valid for this chicken/spring system right up to the point where the spring breaks.*

3. *The chicken's position at time t, $y(t)$ is just given by the particular solution y_p found by the method of educated guess (formula (22.5) on page 446 or formula (22.6) on page 447, depending on μ).*

a. *Suppose the chicken flaps at the natural frequency of the system.*

i. *What is the formula for the chicken's position at time t?*

ii. *When does does the amplitude of the oscillations become large enough to break the spring?*

b. *Suppose that the chicken manages to consistently flap its wing 3 times per second.*

i. *What is the formula for the chicken's position at time t?*

ii. *Does the chicken break the spring? If so, when.*

c. *What is the range of values for μ, the flap frequency, that the chicken can flap at, eventually breaking the spring? (That is, find the minimum and maximum values of μ so that the corresponding near resonance will stretch or compress the spring enough to break it.)*

22.5. *For each $\eta > 0$, let y_η be the solution to*

$$\frac{d^2 y_\eta}{dt^2} + (\omega_0)^2 y_\eta = \frac{F_0}{m}\cos(\eta t) \qquad \text{with} \quad y_\eta(0) = 0 \quad \text{and} \quad {y_\eta}'(0) = 0 \quad .$$

Note that this describes an undamped mass/spring system in which the mass is initially at rest.

a. *Find $y_\eta(t)$ assuming $\eta \neq \omega_0$.*

b. *Find $y_\eta(t)$ assuming $\eta = \omega_0$.*

c. *Verify that*

$$\lim_{\eta \to \omega_0} y_\eta(t) = y_{\omega_0}(t) \quad .$$

d. *Using a computer math package (Maple, Mathematica, etc.) sketch the graph of y_η from $t = 0$ to $t = 100$ when $\omega_0 = 5$, $F_0/m = 1$, and*

i. $\eta = 0.1\,\omega_0$ **ii.** $\eta = 0.5\,\omega_0$ **iii.** $\eta = 0.75\,\omega_0$

iv. $\eta = 0.9\,\omega_0$ **v.** $\eta = 0.99\,\omega_0$ **vi.** $\eta = 0.99\,\omega_0$

vii. $\eta = \omega_0$ **viii.** $\eta = 1.1\,\omega_0$ **ix.** $\eta = 2\,\omega_0$

In particular, observe what happens when $\eta \approx \omega_0$ *, and how these graphs illustrate the result given in part c of this exercise.*

22.6. *Consider an damped mass/spring system given by*

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = F_0 \cos(\eta t) \quad .$$

(This is the same as equation (22.3′).)

a. *Using the method of guess, derive the particular solution given by equation set (22.8) on page 449.*

b. *Then show that the solution in the previous part can be rewritten as described by equation set (22.9) on page 449.*

c. *Finally, show that the resonant frequency of the system is*

$$\eta_0 = \sqrt{(\omega_0)^2 - \frac{1}{2}\left(\frac{\gamma}{m}\right)^2}$$

provided $\gamma < m\omega_0\sqrt{2}$ *. What happens if, instead,* $\gamma \geq m\omega_0\sqrt{2}$ *?*

22.7 a. *Show that neither a critically damped nor an overdamped mass/spring system can have a resonant frequency.*

b. *Does every underdamped mass/spring system have a resonant frequency?*

22.8. *Assume we have an underdamped mass/spring system with resonant angular frequency* η_0 *. Show that*

$$\eta_0 = \sqrt{\omega^2 - \left(\frac{\gamma}{2m}\right)^2}$$

where m *is the mass of the object in the system,* γ *is the damping constant, and* ω *is the angular quasi-frequency of the damped system. Also, verify that*

$$\eta_0 < \omega < \omega_0$$

by showing that

$$(\eta_0)^2 + \left(\frac{\gamma}{2m}\right)^2 = \omega^2 = (\omega_0)^2 - \left(\frac{\gamma}{2m}\right)^2$$

where ω_0 *is the natural angular frequency of the corresponding undamped mass/spring system. (Hint: See exercise 17.8 on page 372.)*