17

Springs: Part I

Second-order differential equations arise in a number of applications. We saw one involving a falling object at the beginning of this text (the falling frozen duck example in section 1.2). In fact, since acceleration is given by the second derivative of position, any application requiring Newton's equation $F = ma$ has the potential to be modeled by a second-order differential equation.

In this chapter we will consider a class of applications involving masses bouncing up and down at the ends of springs. This is a particularly good class of examples for us to examine. For one thing, the basic model is relatively easy to derive, and is given by a second-order differential equation with constant coefficients. So we will be able to apply what we learned in the last chapter to derive reasonably accurate descriptions of the motion under a variety of situations. Moreover, most of us already have an intuitive idea of how these "mass/spring systems" behave. Hopefully, what we derive will correspond to what we expect, and may even refine our intuitive understanding.

Another good point about the work we are about to begin is that many of the notions and results we will develop here can carry over to the analysis of other applications involving things that vibrate or oscillate in some manner. For example, the analysis of current in basic electric circuits is completely analogous to the analysis we'll carry out for masses on springs.

17.1 Modeling the Action

The Mass/Spring System

Imagine a horizontal spring with one end attached to an immobile wall and the other end attached to some object of interest (say, a box of frozen ducks) which can slide along the floor, as in figure 17.1. For brevity, this entire assemblage of spring, object, wall, etc. will be called a *mass/spring system*. Let us assume that:

1. The object can only move back and forth in the one horizontal direction.

2. Newtonian physics apply.

3. The total force acting on the object is the sum of:

   (a) The force from the spring responding to the spring being compressed and stretched.

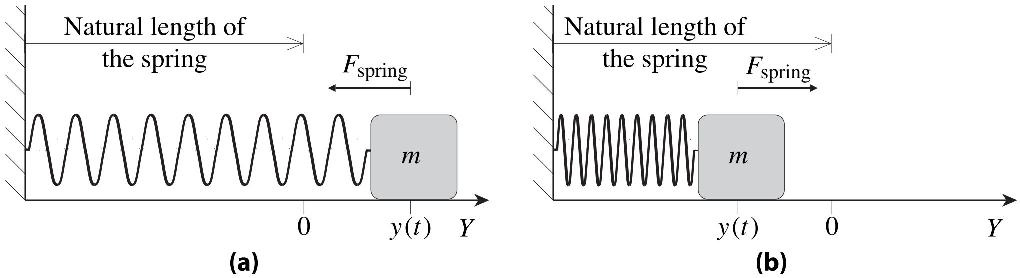

Figure 17.1: The mass/spring system with the direction of the spring force F_{spring} on the mass **(a)** when the spring is extended ($y(t) > 0$), and **(b)** when the spring is compressed ($y(t) < 0$).

(b) The forces resisting motion because of air resistance and friction between the box and the floor.

(c) Any other forces acting on the object. (This term will usually be zero in this chapter. We include it here for use in later chapters, so we don't have to re-derive the equation for the spring to include other forces.)

All forces are assumed to be directed parallel to the direction of the object's motion.

4. The spring is an "ideal spring" with no mass. It has some natural length at which it is neither compressed nor stretched, and it can be both stretched and compressed. (So the coils are not so tightly wound that they are pressed against each other, making compression impossible.)

Our goal is to describe how the position of the object varies with time, and to see how this object's motion depends on the different parameters of our mass/spring system (the object's mass, the strength of the spring, the slipperiness of the floor, etc.).

To set up the general formulas and equations, we'll first make the following traditional symbolic assignments:

$$m = \text{the mass (in kilograms) of the object} \quad ,$$

$$t = \text{the time (in seconds) since the mass/spring system was set into motion} \quad ,$$

and

$$y = \text{the position (in meters) of the object when the spring is at its natural length} \quad .$$

This means our Y–axis is horizontal (nontraditional, maybe, but convenient for this application), and positioned so that $y = 0$ is the "equilibrium position" of the object. Let us also direct the Y–axis so that the spring is stretched when $y > 0$, and compressed when $y < 0$ (again, see figure 17.1).

Modeling the Forces

The motion of the object is governed by Newton's law $F = ma$ with F being the force acting on the box and

$$a = a(t) = \text{acceleration of the box at time } t = \frac{d^2y}{dt^2} \quad .$$

By our assumptions,

$$F = F_{\text{resist}} + F_{\text{spring}} + F_{\text{other}}$$

where

$$F_{\text{resist}} = \text{force due to the air resistance and friction} \quad ,$$

$$F_{\text{spring}} = \text{force from the spring due to it being compressed or stretched} \quad ,$$

and

$$F_{\text{other}} = \text{any other forces acting on the object} \quad .$$

Thus,

$$F_{\text{resist}} + F_{\text{spring}} + F_{\text{other}} = F = ma = m\frac{d^2y}{dt^2} \quad ,$$

which, for convenient reference later, we will rewrite as

$$m\frac{d^2y}{dt^2} - F_{\text{resist}} - F_{\text{spring}} = F_{\text{other}} \quad . \tag{17.1}$$

The resistive force, F_{resist}, is basically the same as the force due to air resistance discussed in the *Better Falling Object Model* in chapter 1 (see page 12) — we are just including friction from the floor along with the friction with the air (or whatever medium surrounds our mass/spring system). So let us model the total resistive force here the same way we modeled the force of air resistance in chapter 1:

$$F_{\text{resist}} = -\gamma \times \text{velocity of the box} = -\gamma\frac{dy}{dt} \tag{17.2}$$

where γ is some nonnegative constant. Because of the role it will play in determining how much the resistive forces "dampens" the motion, we call γ the *damping constant*. It will be large if the air resistance is substantial (possibly because the mass/spring system is submerged in water instead of air) or if the object does not slide easily on the floor. It will be small if there is little air resistance and the floor is very slippery. And it will be zero if there is no air resistance and no friction with the floor (a very idealized situation).

Now consider what we know about the spring force, F_{spring}. At any given time t, this force depends only on how much the spring is stretched or compressed at that time, and that, in turn, is completely described by $y(t)$. Hence, we can describe the spring force as a function of y, $F_{\text{spring}} = F_{\text{spring}}(y)$. Moreover:

1. If $y = 0$, then the spring is at its natural length, neither stretched nor compressed, and exerts no force on the box. So $F_{\text{spring}}(0) = 0$.
2. If $y > 0$, then the spring is stretched and exerts a force on the box pulling it backwards. So $F_{\text{spring}}(y) < 0$ whenever $y > 0$.
3. Conversely, if $y < 0$, then the spring is compressed and exerts a force on the box pushing it forwards. So $F_{\text{spring}}(y) > 0$ whenever $y < 0$.

Knowing nothing more about the spring force, we might as well model it using the simplest mathematical formula satisfying the above:

$$F_{\text{spring}}(y) = -\kappa y \tag{17.3}$$

where κ is some positive constant.

Formula (17.3) is the famous *Hooke's law* for springs. Experiment has shown it to be a good model for the spring force, provided the spring is not stretched or compressed too much. The constant κ in this formula is called the *spring constant*. It describes the "stiffness" of the spring (i.e., how strongly it resists being stretched), and can be determined by compressing or stretching the spring by some amount y_0, and then measuring the corresponding force F_0 at the end of the spring. Hooke's law then says that

$$\kappa = -\frac{F_0}{y_0} \quad .$$

And because κ is a positive constant, we can simplify things a little bit more to

$$\kappa = \frac{|F_0|}{|y_0|} \quad . \tag{17.4}$$

!▶ ***Example 17.1:*** *Assume that, when our spring is pulled out 2 meters beyond its natural length, we measure that the spring is pulling back with a force F_0 of magnitude*

$$|F_0| = 18 \quad \left(\frac{\mathit{kg{\cdot}meter}}{\mathit{sec}^2}\right) \quad .$$

Then,

$$\kappa = \frac{|F_0|}{|y_0|} = \frac{18}{2} \quad \left(\frac{\mathit{kg{\cdot}meter/sec}^2}{\mathit{meter}}\right) \quad .$$

That is,

$$\kappa = 9 \quad \left(\frac{\mathit{kg}}{\mathit{sec}^2}\right) \quad .$$

(This is a pretty weak spring.)

A Note on Units

We defined m, t, and y to be numerical values describing mass, position, and time in terms of kilograms, seconds, and meters. Consequently, everything derived from these quantities — velocity, acceleration, the resistance coefficient γ, and the spring constant κ — are numerical values describing physical parameters in terms of corresponding units. Of course, velocity and acceleration are in terms of meters/second and meters/second^2, respectively. And, because "$F = ma$", any value for force should be interpreted as being in terms of kilogram·meter/second^2 (also called "newtons"). As indicated in the example above, the corresponding units associated with the spring constant are kilogram/second^2, and as you can readily verify, the the resistance coefficient γ is in terms of kilograms/second.

In the above example, the units involved in every calculation were explicitly given in parenthesis. In the future, we will not explicitly state the units in every calculation, and trust that you, the reader, can determine the units appropriate for the final results from the information given.

Indeed, for our purposes, the actual choice of the units is not important. The formulas we have developed and illustrated (along with those we will later develop and illustrate) remain just as valid if m, t, and y are in terms of, say, grams, weeks, and miles, respectively, provided it is understood that the values of the corresponding velocities, accelerations, etc. are in terms of miles/week, miles/week^2, etc.

!► *Example 17.2:* *Pretend that, when our spring is pulled out* 2 *miles beyond its natural length, we measure that the spring is pulling back with a force* F_0 *of magnitude*

$$|F_0| = 18 \quad \left(\frac{\text{gram}\cdot\text{mile}}{\text{week}^2}\right) \quad .$$

Then,

$$\kappa = \frac{|F_0|}{|y_0|} = \frac{18}{2} \quad \left(\frac{\text{gram}\cdot\text{mile}/\text{week}^2}{\text{meter}}\right) \quad .$$

That is,

$$\kappa = 9 \quad \left(\frac{\text{gram}}{\text{week}^2}\right) \quad .$$

17.2 The Mass/Spring Equation and Its Solutions

The Differential Equation

Replacing F_{resist} and F_{spring} in equation (17.1) with the formulas for these forces from equations (17.2) and (17.3), we get

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = F_{\text{other}} \quad . \tag{17.5}$$

This is the differential equation for $y(t)$, the position y of the object in the system at time t.

For the rest of this chapter, let us assume the object is moving "freely" under the influence of no forces except those from friction and from the spring's compression and expansion.[1] Thus, for the rest of this chapter, we will restrict our interest to the above differential equation with $F_{\text{other}} = 0$,

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = 0 \quad . \tag{17.6}$$

This is a second-order, homogeneous, linear differential equation with constant coefficients; so we can solve it by the methods discussed in the previous chapter. The precise functions in these solutions (sine/cosines, exponentials, etc.) will depend on the coefficients. We will go through all the possible cases soon.

Keep in mind that the mass, m, and the spring constant, κ, are positive constants for a real spring. On the other hand, the damping constant, γ, can be positive or zero. This is significant. Because $\gamma = 0$ when there is no resistive force to dampen the motion, we say the mass/spring system is *undamped* when $\gamma = 0$. We will see that the motion of the mass in this case is relatively simple.

If, however, there is a nonzero resistive force to dampen the motion, then $\gamma > 0$. Accordingly, in this case, we say mass/spring system is *damped*. We will see that there are three subcases to consider, according to whether $\gamma^2 - 4\kappa m$ is negative, zero or positive.

Let's now carefully examine, case by case, the solutions that can arise.

[1] We will introduce other forces in later chapters.

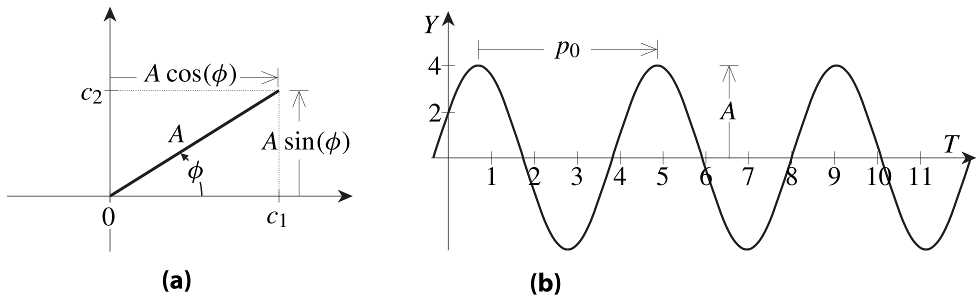

Figure 17.2: **(a)** Expressing c_1 and c_2 as $A\cos(\phi)$ and $A\sin(\phi)$. **(b)** The graph of $y(t)$ for the undamped mass/spring system of example 17.3.

Undamped Systems

If $\gamma = 0$, differential equation (17.6) reduces to

$$m\frac{d^2y}{dt^2} + \kappa y = 0 \quad . \tag{17.7}$$

The corresponding characteristic equation,

$$mr^2 + \kappa = 0 \quad ,$$

has roots

$$r_{\pm} = \pm\frac{\sqrt{-\kappa m}}{m} = \pm i\omega_0 \qquad \text{where} \quad \omega_0 = \sqrt{\frac{\kappa}{m}} \quad .$$

From our discussions in the previous chapter, we know the general solution to our differential equation is given by

$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

where c_1 and c_2 are arbitrary constants. However, for graphing purposes (and a few other purposes) it is convenient to write our general solution in yet another form. To derive this form, plot (c_1, c_2) as a point on a Cartesian coordinate system, and let A and ϕ be the corresponding polar coordinates of this point (see figure 17.2a). That is, let

$$A = \sqrt{(c_1)^2 + (c_2)^2}$$

and let ϕ be the angle in the range $[0, 2\pi)$ with

$$c_1 = A\cos(\phi) \qquad \text{and} \qquad c_2 = A\sin(\phi) \quad .$$

Using this and the well-known trigonometric identity

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi) \quad ,$$

we get

$$\begin{aligned} c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) &= [A\cos(\phi)]\cos(\omega_0 t) + [A\sin(\phi)]\sin(\omega_0 t) \\ &= A\big[\cos(\omega_0 t)\cos(\phi) + \sin(\omega_0 t)\sin(\phi)\big] \\ &= A\cos(\omega_0 t - \phi) \quad . \end{aligned}$$

Thus, our general solution is given by either

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \tag{17.8a}$$

or, equivalently,

$$y(t) = A \cos(\omega_0 t - \phi) \tag{17.8b}$$

where

$$\omega_0 = \sqrt{\frac{\kappa}{m}}$$

and the other constants are related by

$$A = \sqrt{(c_1)^2 + (c_2)^2} \quad , \quad \cos(\phi) = \frac{c_1}{A} \quad \text{and} \quad \sin(\phi) = \frac{c_2}{A} \quad .$$

It is worth noting that ω_0 depends only on the spring constant and the mass of the attached object. The other constants are "arbitrary" and are determined by the initial position and velocity of the attached object.

Either of the formulas from set (17.8) can be used for the position y of the box at time t. One advantage of using formula (17.8a) is that the constants c_1 and c_2 are fairly easily determined in many of initial-value problems involving this differential equation. However, formula (17.8b) gives an even simpler description of how the position varies with time. It tells us that the position is completely described by a single shifted cosine function multiplied by the positive constant A and shifted so that

$$y(0) = A \cos(\phi) \quad .$$

You should be well acquainted with such functions. The graph of one is sketched in figure 17.2b. Take a look at it, and then read on.

Formula (17.8b) tells us that the object is oscillating back and forth from $y = A$ to $y = -A$. Accordingly, we call A the *amplitude* of the oscillation. The *(natural) period* p_0 of the oscillation is the time it takes the mass to go through one complete "cycle" of oscillation. Using formula (17.8b), rewritten as

$$y(t) = A \cos(X) \qquad \text{with} \quad X = \omega_0 t - \phi \quad ,$$

we see that *our system going through one cycle as t varies from $t = t_0$ to $t = t_0 + p_0$* is the same as $\cos(X)$ *going through one complete cycle as X varies from*

$$X = X_0 = \omega_0 t_0 - \phi$$

to

$$X = \omega_0 (t_0 + p_0) - \phi = X_0 + \omega_0 p_0 \quad .$$

But, as we well know, $\cos(X)$ goes through one complete cycle as X goes from $X = X_0$ to $X = X_0 + 2\pi$. Thus,

$$\omega_0 p_0 = 2\pi \quad , \tag{17.9}$$

and the natural period of our system is

$$p_0 = \frac{2\pi}{\omega_0} \quad .$$

This is the "time per cycle" for the oscillations in the mass/spring system. Its reciprocal,

$$\nu_0 = \frac{1}{p_0} = \frac{\omega_0}{2\pi} \quad ,$$

then gives the "cycles per unit time" for the system (typically measured in terms of *hertz*, with one hertz equaling one cycle per second). We call ν_0 the *(natural) frequency* for the system. The closely related quantity originally computed, ω_0 (which can be viewed as describing "radians per unit time"), will be called the *(natural) angular frequency* for the system.[2] Because the natural frequency ν_0 is usually more easily measured than the natural circular frequency ω_0, it is sometimes more convenient to express the formulas for position (formula set 17.8) with $2\pi\nu_0$ replacing ω_0,

$$y(t) = c_1\cos(2\pi\nu_0 t) + c_2\sin(2\pi\nu_0 t) \quad , \tag{17.8a$'$}$$

and, equivalently,

$$y(t) = A\cos(2\pi\nu_0 t - \phi) \quad . \tag{17.8b$'$}$$

By the way, the angle ϕ in all the formulas above is called the *phase angle* of the oscillations, and any motion described by these formulas is referred to as *simple harmonic motion*.

!►Example 17.3: *Assume we have an undamped mass/spring system in which the spring's spring constant κ and the attached object's mass m are*

$$\kappa = 9 \quad \left(\frac{kg}{sec^2}\right) \qquad \text{and} \qquad m = 4 \quad (kg)$$

(as in example 17.1). Let us try to find and graph the position y at time t of the attached object, assuming the object's initial position and velocity are

$$y(0) = 2 \qquad \text{and} \qquad y'(0) = 3\sqrt{3} \quad .$$

With the above values for γ, m and κ, the differential equation for y, equation (17.6), becomes

$$4\frac{d^2y}{dx^2} + 9y = 0 \quad .$$

As noted in our discussion, the general solution can be given by either

$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

or

$$y(t) = A\cos(\omega_0 t - \phi)$$

where the natural angular frequency is

$$\omega_0 = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{9}{4}} = \frac{3}{2} \quad .$$

This means the the natural period p_0 and the natural frequency ν_0 of the system are

$$p_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{3/2} = \frac{4\pi}{3} \qquad \text{and} \qquad \nu_0 = \frac{1}{p_0} = \frac{3}{4\pi} \quad .$$

To determine the other constants in the above formulas for $y(t)$, we need to consider the given initial conditions. Using the first formula for y, we have

$$y(t) = c_1\cos\left(\frac{3}{2}t\right) + c_2\sin\left(\frac{3}{2}t\right)$$

[2] Many authors refer to ω_0 as a *circular frequency* instead of an angular frequency.

and

$$y'(t) \;=\; -\frac{3}{2}c_1 \sin\left(\frac{3}{2}t\right) \;+\; c_2\frac{3}{2}\cos\left(\frac{3}{2}t\right) \quad .$$

Plugging these into the initial conditions yields

$$2 \;=\; y(0) \;=\; c_1 \cos\left(\frac{3}{2}\cdot 0\right) \;+\; c_2 \sin\left(\frac{3}{2}\cdot 0\right) \;=\; c_1$$

and

$$3\sqrt{3} \;=\; y'(0) \;=\; -\frac{3}{2}c_1 \sin\left(\frac{3}{2}\cdot 0\right) \;+\; c_2\frac{3}{2}\cos\left(\frac{3}{2}\cdot 0\right) \;=\; \frac{3}{2}c_2 \quad .$$

So

$$c_1 \;=\; 2 \quad , \quad c_2 \;=\; \frac{2}{3}\cdot 3\sqrt{3} \;=\; 2\sqrt{3} \quad ,$$

and

$$A \;=\; \sqrt{(c_1)^2 + (c_2)^2} \;=\; \sqrt{(2)^2 + (2\sqrt{3})^2} \;=\; \sqrt{4+12} \;=\; 4 \quad .$$

This gives us enough information to graph $y(t)$ *. From our computations, we know this graph is a cosine shaped curve with amplitude* $A = 4$ *and period* $p_0 = {}^{4\pi}/_3$ *. It is shifted horizontally so that the initial conditions*

$$y(0) \;=\; 2 \qquad \textit{and} \qquad y'(0) \;=\; 3\sqrt{3} \;>\; 0$$

are satisfied. In other words, the graph must cross the Y*–axis at* $y = 2$ *and the graph's slope at that crossing point must be positive. That is how the graph in figure 17.2b was constructed.*

To find the phase angle ϕ *, we must solve the pair of trigonometric equations*

$$\cos(\phi) \;=\; \frac{c_1}{A} \;=\; \frac{2}{4} \;=\; \frac{1}{2} \qquad \textit{and} \qquad \sin(\phi) \;=\; \frac{c_2}{A} \;=\; \frac{2\sqrt{3}}{4} \;=\; \frac{\sqrt{3}}{2}$$

for $0 \leq \phi < 2\pi$ *. From our knowledge of trigonometry, we know the first of these equations is satisfied if and only if*

$$\phi \;=\; \frac{\pi}{3} \qquad \textit{or} \qquad \phi \;=\; \frac{5\pi}{3} \quad ,$$

while the second is satisfied satisfied if and only if

$$\phi \;=\; \frac{\pi}{3} \qquad \textit{or} \qquad \phi \;=\; \frac{2\pi}{3} \quad .$$

Hence, for both of the above trigonometric equations to hold, we must have

$$\phi \;=\; \frac{\pi}{3} \quad .$$

Finally, using the values just obtained, we can completely write out two equivalent formulas for our solution:

$$y(t) \;=\; 2\cos\left(\frac{3}{2_0}t\right) \;+\; 2\sqrt{3}\sin\left(\frac{3}{2}t\right)$$

and

$$y(t) \;=\; 4\cos\left(\frac{3}{2}t - \frac{\pi}{3}\right) \quad .$$

Damped Systems

If $\gamma > 0$, then all coefficients in our differential equation

$$m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \kappa y = 0$$

are positive. The corresponding characteristic equation is

$$mr^2 + \gamma r + \kappa = 0 \quad ,$$

and its solutions are given by

$$r = r_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\kappa m}}{2m} \quad . \tag{17.10}$$

As we saw in the last chapter, the nature of the differential equation's solution, $y = y(t)$, depends on whether $\gamma^2 - 4\kappa m$ is positive, negative or zero. And this, in turn, depends on the positive constants γ, κ and mass m as follows:

$$\gamma < 2\sqrt{\kappa m} \iff \gamma^2 - 4\kappa m < 0 \quad ,$$

$$\gamma = 2\sqrt{\kappa m} \iff \gamma^2 - 4\kappa m = 0 \quad ,$$

and

$$2\sqrt{\kappa m} < \gamma \iff \gamma^2 - 4\kappa m > 0 \quad .$$

For reasons that may (or may not) be clear by the end of this section, we say that a mass/spring system is, respectively,

underdamped , *critically damped* or *overdamped*

if and only if

$$0 < \gamma < 2\sqrt{\kappa m} \quad , \quad \gamma = 2\sqrt{\kappa m} \quad \text{or} \quad 2\sqrt{\kappa m} < \gamma \quad .$$

Since we've already considered the case where $\gamma = 0$, the first damped cases considered will be the underdamped mass/spring systems (where $0 < \gamma < 2\sqrt{\kappa m}$).

Underdamped Systems ($0 < \gamma < 2\sqrt{\kappa m}$)

In this case,

$$\sqrt{\gamma^2 - 4\kappa m} = \sqrt{-\left|\gamma^2 - 4\kappa m\right|} = i\sqrt{\left|\gamma^2 - 4\kappa m\right|} = i\sqrt{4\kappa m - \gamma^2} \quad ,$$

and formula (17.10) for the $r_{\pm}$'s can be written as

$$r_{\pm} = -\alpha \pm i\omega \quad \text{where} \quad \alpha = \frac{\gamma}{2m} \quad \text{and} \quad \omega = \frac{\sqrt{4\kappa m - \gamma^2}}{2m} \quad .$$

Both α and ω are positive real values, and, from the discussion in the previous chapter, we know a corresponding general solution to our differential equation is

$$y(t) = c_1 e^{-\alpha t}\cos(\omega t) + c_2 e^{-\alpha t}\sin(\omega t) \quad .$$

Factoring out the exponential and applying the same analysis to the linear combination of sines and cosines as was done for the undamped case, we get that the position y of the box at time t is given by any of the following:

$$y(t) = e^{-\alpha t}\left[c_1 \cos(\omega t) + c_2 \sin(\omega t)\right] \quad , \tag{17.11a}$$

$$y(t) = Ae^{-\alpha t} \cos(\omega t - \phi) \tag{17.11b}$$

and even

$$y(t) = Ae^{-\alpha t} \cos\left(\omega\left[t - \frac{\phi}{\omega}\right]\right) \quad . \tag{17.11c}$$

These three formulas are equivalent, and the arbitrary constants are related, as before, by

$$A = \sqrt{(c_1)^2 + (c_2)^2} \quad , \quad \cos(\phi) = \frac{c_1}{A} \quad \text{and} \quad \sin(\phi) = \frac{c_2}{A} \quad .$$

Note the similarities and differences in the motion of the undamped system and the underdamped system. In both cases, a shifted cosine function plays a major role in describing the position of the mass. In the underdamped system this cosine function has angular frequency ω and, hence, corresponding period and frequency

$$p = \frac{2\pi}{\omega} \quad \text{and} \quad \nu = \frac{\omega}{2\pi} \quad .$$

However, in the underdamped system, this shifted cosine function is also multiplied by a decreasing exponential, reflecting that the motion is being damped, but not so damped as to completely prevent oscillations in the box's position. (You will further analyze how p and ν vary with γ in exercise 17.9.)

Because the α in the formula set (17.11) determines the rate at which the maximum values of $y(t)$ are decreasing as t increases, let us call α the *decay coefficent* for our system. It is also tempting to call ω, p and ν the angular frequency, period and frequency of the system, but, because $y(t)$ is not truly periodic, this terminology is not be truly appropriate. Instead, let's refer to these quantities the *angular quasi-frequency*, *quasi-period* and *quasi-frequency* of the system.[3] And, if you must give them names, call A the *quasi-amplitude* and $Ae^{-\alpha t}$ the *time-varying amplitude* of the system.

And, again, it is sometimes more convenient to express our formulas in terms of the quasi-frequency ν instead of the angular quasi-frequency ω, with, for example, formulas (17.11a) and (17.11b) being rewritten as

$$y(t) = e^{-\alpha t}\left[c_1 \cos(2\pi\nu t) + c_2 \sin(2\pi\nu t)\right] \tag{17.11a$'$}$$

and

$$y(t) = Ae^{-\alpha t} \cos(2\pi\nu t - \phi) \quad . \tag{17.11b$'$}$$

!► *Example 17.4:* *Again, assume the spring constant κ and the mass m in our mass/spring system are*

$$\kappa = 9 \left(\frac{\text{kg}}{\text{sec}^2}\right) \quad \text{and} \quad m = 4 \quad (\text{kg}) \quad .$$

For the system to be underdamped, the resistance coefficient γ must satisfy

$$0 < \gamma < 2\sqrt{\kappa m} = 2\sqrt{9 \cdot 4} = 12 \quad .$$

[3] Some authors prefer using "psuedo" instead "quasi".

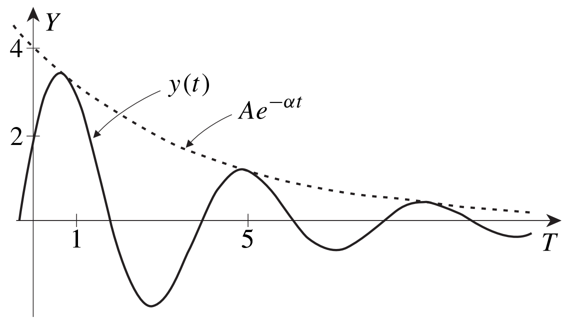

Figure 17.3: Graph of $y(t)$ for the underdamped mass/spring system of example 17.4

For this example, assume $\gamma = 2$. Then the position y at time t of the object is given by

$$y(t) = Ae^{-\alpha t}\cos(\omega t - \phi)$$

where

$$\alpha = \frac{\gamma}{2m} = \frac{2}{2 \times 4} = \frac{1}{4}$$

and

$$\omega = \frac{\sqrt{4\kappa m - \gamma^2}}{2m} = \frac{\sqrt{(4 \cdot 9 \cdot 3) - 2^2}}{2 \cdot 4} = \frac{\sqrt{35}}{4} \quad .$$

The corresponding quasi-period for the system is

$$p = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{35}/4} = \frac{8\pi}{\sqrt{35}} \approx 4.25 \quad .$$

To keep this example short, we won't solve for A and ϕ from some set of initial conditions. Instead, we'll just set

$$A = 4 \qquad \text{and} \qquad \phi = \frac{\pi}{3} \quad ,$$

and note that the resulting graph of $y(t)$ is sketched in figure 17.3.

Critically damped Systems ($\gamma = 2\sqrt{\kappa m}$)

In this case,

$$\sqrt{\gamma^2 - 4\kappa m} = 0$$

and

$$r_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\kappa m}}{2m} = \frac{-2\sqrt{\kappa m} \pm \sqrt{0}}{2m} = -\frac{\sqrt{\kappa m}}{m} = -\sqrt{\frac{\kappa}{m}} \quad .$$

So the corresponding general solution to our differential equation is

$$y(t) = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t} \qquad \text{where} \qquad \alpha = \sqrt{\frac{\kappa}{m}} \quad .$$

Factoring out the exponential yields

$$y(t) = [c_1 + c_2 t]e^{-\alpha t} \quad . \tag{17.12}$$

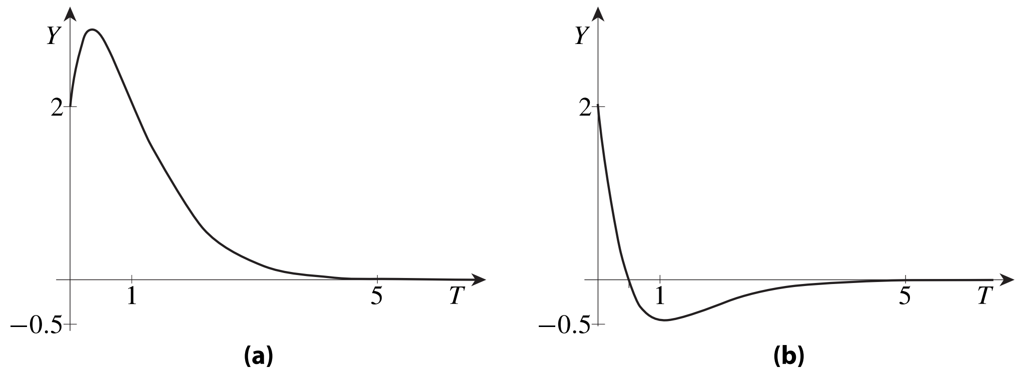

Figure 17.4: Graph of $y(t)$ for the critically damped mass/spring system of example 17.5 **(a)** with $y(0) = 2$ and $y'(0) > 0$ **(b)** with $y(0) = 2$ and $y'(0) < 0$.

The cosine factor in the underdamped case has now been replaced with a formula for a straight line, $c_1 + c_2 t$. If $y'(0)$ is positive, then $y(t)$ will initially increase as t increases. However, at some point, the decaying exponential will force the graph of $y(t)$ back down towards 0 as $t \to \infty$. This is illustrated in figure 17.4a.

If, on the other hand, $y(0)$ is positive and $y'(0)$ is negative, then the slope of the straight line is negative, and the graph will initially head downward as t increases. Eventually, $c_1 + c_2 t$ will be negative. And, again, the decaying exponential will eventually force $y(t)$ back (up) towards 0 as $t \to \infty$. This is illustrated in figure 17.4b.

!►*Example 17.5:* *Once again, assume the spring constant κ and the mass m in our mass/spring system are*

$$\kappa = 9 \left(\frac{kg}{sec^2}\right) \qquad \text{and} \qquad m = 4 \quad (kg) \quad .$$

For the system to be critically damped, the resistance coefficient γ must satisfy

$$\gamma = 2\sqrt{\kappa m} = 2\sqrt{9 \cdot 4} = 12 \quad .$$

Assuming this, the position y at time t of the object in this mass/spring system is given by

$$y(t) = [c_1 + c_2 t]e^{-\alpha t} \qquad \text{where} \quad \alpha = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{9}{4}} = \frac{3}{2} \quad .$$

The graph of this with $(c_1, c_2) = (2, 8)$ is sketched in figure 17.4a; the graph of this with $(c_1, c_2) = (2, -4)$ is sketched in figure 17.4b.

Overdamped Systems ($2\sqrt{\kappa m} < \gamma$)

In this case, it is first worth observing that

$$\gamma > \sqrt{\gamma^2 - 4\kappa m} > 0 \quad .$$

Consequently, the formula for the $r_\pm$'s (equation (17.10)),

$$r_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\kappa m}}{2m}$$

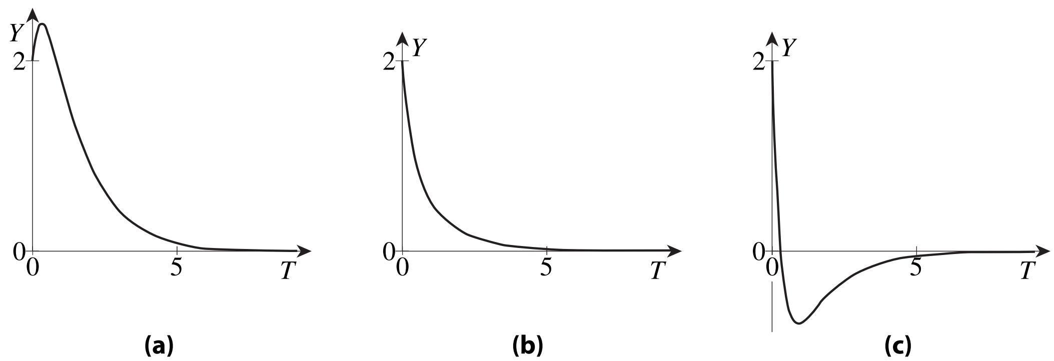

Figure 17.5: Graphs of $y(t)$ (with $y(0) = 2$) for the overdamped mass/spring system of example 17.6. In **(a)** $y'(0) > 0$. In **(b)** and **(c)** $y'(0) < 0$ with the magnitude of $y'(0)$ in **(c)** being significantly larger than in **(b)**.

can be written as

$$r_+ = \alpha \qquad \text{and} \qquad r_- = \beta$$

where α and β are the *positive* values

$$\alpha = \frac{\gamma - \sqrt{\gamma^2 - 4\kappa m}}{2m} \qquad \text{and} \qquad \beta = \frac{\gamma + \sqrt{\gamma^2 - 4\kappa m}}{2m} \quad .$$

Hence, the corresponding general solution to the differential equation is

$$y(t) = c_1 e^{-\alpha t} + c_2 e^{-\beta t} \quad ,$$

a linear combination of two decaying exponentials.

Some of the possible graphs for y are illustrated in figure 17.5.

!►***Example 17.6:*** *Once again, assume the spring constant and the mass in our mass/spring system are, respectively,*

$$\kappa = 9 \left(\frac{kg}{sec^2}\right) \qquad \textit{and} \qquad m = 4 \quad (kg) \quad .$$

For the system to be overdamped, the resistance coefficient γ must satisfy

$$\gamma > 2\sqrt{\kappa m} = 2\sqrt{9 \cdot 4} = 12 \quad .$$

In particular, the system is overdamped if the resistance coefficient γ is 15. Assuming this, the general position y at time t of the object in our system is given by

$$y(t) = c_1 e^{-\alpha t} + c_2 e^{-\beta t}$$

where

$$\alpha = \frac{\gamma - \sqrt{\gamma^2 - 4\kappa m}}{2m} = \frac{15 - \sqrt{15^2 - 4 \cdot 9 \cdot 4}}{2 \cdot 4} = \cdots = \frac{3}{4}$$

and

$$\beta = \frac{\gamma + \sqrt{\gamma^2 - 4\kappa m}}{2m} = \frac{15 + \sqrt{15^2 - 4 \cdot 9 \cdot 4}}{2 \cdot 4} = \cdots = 3 \quad .$$

Figures 17.5a, 17.5b, and 17.5c were drawn using this formula with, respectively,

$$(c_1, c_2) = (4, -2) \quad , \quad (c_1, c_2) = (1, 1) \quad \textit{and} \quad (c_1, c_2) = (-2, 4) \quad .$$

Additional Exercises

17.1. *Find the spring constant κ for a spring that, when pulled out 1 meter beyond its natural length, pulls back with a force F_0 of magnitude*

$$|F_0| = 2 \quad \left(\frac{kg{\cdot}meter}{sec^2}\right) \quad .$$

17.2. *Assume we have a single undamped mass/spring system, and do the following:*

a. *Find the spring constant κ for the spring given that, when pulled out $1/2$ meter beyond its natural length, the spring pulls back with a force F_0 of magnitude*

$$|F_0| = 2 \quad \left(\frac{kg{\cdot}meter}{sec^2}\right) \quad .$$

b. *Find the the natural angular frequency ω_0, the natural frequency ν_0, and the natural period p_0 of this system assuming the mass of the attached object is 16 kilograms.*

c. *Four different sets of initial conditions are given below for this mass/spring system. For each, determine the corresponding amplitude A and phase angle ϕ for the system, and sketch the graph of the position over time $y = y(t)$. (Use the values of κ and ω_0 derived above, and assume position and time are given in meters and seconds, respectively.):*

i. $y(0) = 2$ *and* $y'(0) = 0$ **ii.** $y(0) = 0$ *and* $y'(0) = 2$

iii. $y(0) = 0$ *and* $y'(0) = -2$ **iv.** $y(0) = 2$ *and* $y'(0) = \sqrt{3}$

17.3. *Suppose we have an undamped mass/spring system with natural angular frequency ω_0. Let y_0 and v_0 be, respectively, the position and velocity of the object at $t = 0$. Show that the corresponding amplitude A is given by*

$$A = \sqrt{{y_0}^2 + \left[\frac{v_0}{\omega_0}\right]^2} \quad ,$$

and that the phase angle satisfies

$$\tan(\phi) = \frac{v_0}{y_0\omega_0} \quad .$$

17.4. *Assume we have a single undamped mass/spring system, and do the following:*

a. *Find the spring constant κ for the spring given that, when pulled out $1/4$ meter beyond its natural length, the spring pulls back with a force F_0 of magnitude*

$$|F_0| = 72 \quad \left(\frac{kg{\cdot}meter}{sec^2}\right) \quad .$$

b. *Find the the natural angular frequency ω_0, the natural frequency ν_0, and the natural period p_0 of this system when the mass of the attached object is 2 kilograms.*

c. *Three different sets of initial conditions are given below for this mass/spring system. For each, determine the corresponding amplitude* A *, and sketch the graph of the position over time* $y = y(t)$ *:*

i. $y(0) = 1$ *and* $y'(0) = 0$ **ii.** $y(0) = 0$ *and* $y'(0) = 1$

iii. $y(0) = 1$ *and* $y'(0) = 3$

17.5. *Suppose that a particular undamped mass/spring system has natural period* $p_0 = 3$ *seconds. What is the spring constant* κ *of the spring if the mass* m *of the object is (in kilograms)*

a. $m = 1$ **b.** $m = 2$ **c.** $m = \frac{1}{2}$

17.6. *Suppose we have an underdamped mass/spring system with decay coefficient* α *and angular quasi-frequency* ω *. Let* y_0 *and* v_0 *be, respectively, the position and velocity of the object at* $t = 0$ *. Show that the corresponding amplitude* A *is given by*

$$A = \sqrt{{y_0}^2 + \left[\frac{v_0 + \alpha y_0}{\omega}\right]^2} \quad ,$$

while the phase angle satisfies

$$\tan(\phi) = \frac{v_0 + \alpha y_0}{y_0 \omega} \quad .$$

17.7. *Consider a damped mass/spring system with spring constant, mass and damping coefficient being, respectively,*

$$\kappa = 37 \quad , \quad m = 4 \quad \text{and} \quad \gamma = 4 \quad .$$

a. *Verify that this is an underdamped system.*

b. *Find the decay coefficient* α *, the angular quasi-frequency* ω *, the quasi-period* p *and the quasi-frequency* ν *of this system.*

c. *Three different sets of initial conditions are given below for this mass/spring system. For each, determine the corresponding quasi-amplitude* A *for the system, and sketch the graph of the position over time* $y = y(t)$ *.*

i. $y(0) = 1$ *and* $y'(0) = 0$ **ii.** $y(0) = 0$ *and* $y'(0) = 1$

iii. $y(0) = 2$ *and* $y'(0) = 2$

17.8. *Let* ω *be the angular quasi-frequency of some underdamped mass/spring system. Show that*

$$\omega = \sqrt{(\omega_0)^2 - \left(\frac{\gamma}{2m}\right)^2}$$

where m *is the mass of the object in the system,* γ *is the damping constant, and* ω_0 *is the natural frequency of the corresponding undamped system.*

17.9. *Suppose we have a mass/spring system in which we can adjust the damping coefficient* γ *(the mass* m *and the spring constant* κ *remain constant). How does the quasi-frequency* ν *and the quasi-period* p *vary as* γ *varies from* $\gamma = 0$ *up to* $\gamma = 2\sqrt{\kappa m}$ *? (Compare* ν *and* p *to the natural frequency and period of the corresponding undamped system,* ν_0 *and* p_0 *.)*

17.10. *Consider a damped mass/spring system with spring constant, mass and damping coefficient being, respectively,*

$$\kappa = 4 \quad , \quad m = 1 \quad \text{and} \quad \gamma = 4 \quad .$$

a. *Verify that this system is critically damped.*

b. *Find and graph the position of the object over time,* $y(t)$ *, assuming that*

$$y(0) = 2 \quad \text{and} \quad y'(0) = 0 \quad .$$

c. *Find and graph the position of the object over time,* $y(t)$ *, assuming that*

$$y(0) = 0 \quad \text{and} \quad y'(0) = 2 \quad .$$

17.11. *Consider a damped mass/spring system with spring constant, mass and damping coefficient being, respectively,*

$$\kappa = 4 \quad , \quad m = 1 \quad \text{and} \quad \gamma = 5 \quad .$$

a. *Verify that this system is overdamped.*

b. *Find and graph the position of the object over time,* $y(t)$ *, assuming that*

$$y(0) = 2 \quad \text{and} \quad y'(0) = 0 \quad .$$

c. *Find and graph the position of the object over time,* $y(t)$ *, assuming that*

$$y(0) = 0 \quad \text{and} \quad y'(0) = 2 \quad .$$