16

Second-Order Homogeneous Linear Equations with Constant Coefficients

A very important class of second-order homogeneous linear equations consists of those with constant coefficients; that is, those that can be written as

$$ay'' + by' + cy = 0$$

where a , b and c are real-valued constants (with $a \neq 0$). Some examples are

$$y'' - 5y' + 6y = 0$$
$$y'' - 6y' + 9y = 0$$

and

$$y'' - 6y' + 13y = 0 \quad .$$

There are two reasons these sorts of differential equations are important: First of all, they often arise in applications. Secondly, as we will see, it is relatively easy to find fundamental sets of solutions for these equations.

Do note that, because the coefficients are constants, they are, trivially, continuous functions on the entire real line. Consequently, we can take the entire real line as the interval of interest, and be confident that any solutions derived will be valid on all of $(-\infty, \infty)$.

IMPORTANT: What we will derive and define here (e.g., "the characteristic equation") is based on the assumption that the coefficients in our differential equation— the a , b and c above — are constants. Some of the results will even require that these constants be real valued. Do not, later, try to blindly apply what we develop here to differential equations in which the coefficients are not real-valued constants.

16.1 Deriving the Basic Approach

Seeking Inspiration

Let us look for clues on how to solve our second-order equations by first looking at solving a first-order, homogeneous linear differential equation with constant coefficients, say,

$$2\frac{dy}{dx} + 6y = 0 \quad .$$

Since we are considering 'linear' equations, let's solve it using the method developed for first-order linear equations: First divide through by the first coefficient, 2, to get

$$\frac{dy}{dx} + 3y = 0 \quad .$$

The integrating factor is then

$$\mu = e^{\int 3\,dx} = e^{3x} \quad .$$

Multiplying through and proceeding as usual with first-order linear equations:

$$e^{3x}\left[\frac{dy}{dx} + 3y\right] = e^{3x}\cdot 0$$

$$\hookrightarrow \qquad e^{3x}\frac{dy}{dx} + 3e^{3x}y = 0$$

$$\hookrightarrow \qquad \frac{d}{dx}\left[e^{3x}y\right] = 0$$

$$\hookrightarrow \qquad e^{3x}y = c$$

$$\hookrightarrow \qquad y = ce^{-3x} \quad .$$

So a general solution to

$$2\frac{dy}{dx} + 6y = 0$$

is

$$y = ce^{-3x} \quad .$$

Clearly, there is nothing special about the numbers used here. Replacing 2 and 6 with constants a and b in the above would just as easily have given us the fact that a general solution to

$$a\frac{dy}{dx} + by = 0$$

is

$$y = ce^{rx} \qquad \text{where} \quad r = -\frac{b}{a} \quad .$$

Thus we see that all solutions to first-order homogeneous linear equations with constant coefficients are given by constant multiples of exponential functions.

Exponential Solutions with Second-Order Equations

Now consider the second-order case. For convenience, we will use

$$y'' - 5y' + 6y = 0$$

as an example, keeping in mind that our main interest is in finding all possible solutions to an arbitrary second-order homogeneous differential equation

$$ay'' + by' + cy = 0$$

where a, b and c are constants.

From our experience with the first-order case, it seems reasonable to expect at least some of the solutions to be exponentials. So let us find all such solutions by setting

$$y = e^{rx}$$

where r is a constant to be determined, plugging this formula into our differential equation, and seeing if a constant r can be determined.

For our example,

$$y'' - 5y' + 6y = 0 \quad .$$

Letting $y = e^{rx}$ yields

$$\frac{d^2}{dx^2}\left[e^{rx}\right] - 5\frac{d}{dx}\left[e^{rx}\right] + 6\left[e^{rx}\right] = 0$$

$$\hookrightarrow \qquad r^2 e^{rx} - 5re^{rx} + 6e^{rx} = 0$$

$$\hookrightarrow \qquad e^{rx}\left[r^2 - 5r + 6\right] = 0 \quad .$$

Since e^{rx} can never be zero, we can divide it out, leaving the algebraic equation

$$r^2 - 5r + 6 = 0 \quad .$$

Before solving this for r, let us pause and consider the more general case.

More generally, letting $y = e^{rx}$ in

$$ay'' + by' + cy = 0 \tag{16.1}$$

yields

$$a\frac{d^2}{dx^2}\left[e^{rx}\right] + b\frac{d}{dx}\left[e^{rx}\right] + c\left[e^{rx}\right] = 0$$

$$\hookrightarrow \qquad ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$\hookrightarrow \qquad e^{rx}\left[ar^2 + br + c\right] = 0 \quad .$$

Since e^{rx} can never be zero, we can divide it out, leaving us with the algebraic equation

$$ar^2 + br + c = 0 \tag{16.2}$$

(remember: a, b and c are constants). Equation (16.2) is called called the *characteristic equation* for differential equation (16.1). Note the similarity between the original differential equation and its characteristic equation. The characteristic equation is nothing more that the algebraic equation obtained by replacing the various derivatives of y with corresponding powers of r (treating y as being the zeroth derivative of y):

$$ay'' + by' + cy = 0 \qquad \text{(original differential equation)}$$

$$\hookrightarrow \qquad ar^2 + br + c = 0 \qquad \text{(characteristic equation)}$$

The nice thing is that the characteristic equation is easily solved for r by either factoring the polynomial or using the quadratic formula. These values for r must then be the values of r for which $y = e^{rx}$ are (particular) solutions to our original differential equation. Using what we developed in previous chapters, we may then be able to construct a general solution to the differential equation.

In our example, letting $y = e^{rx}$ in

$$y'' - 5y' + 6y = 0$$

lead to the characteristic equation

$$r^2 - 5r + 6 = 0 \quad ,$$

which factors to

$$(r-2)(r-3) = 0 \quad .$$

Hence,

$$r - 2 = 0 \qquad \text{or} \qquad r - 3 = 0 \quad .$$

So the possible values of r are

$$r = 2 \qquad \text{and} \qquad r = 3 \quad ,$$

which, in turn, means

$$y_1 = e^{2x} \qquad \text{and} \qquad y_2 = e^{3x}$$

are solutions to our original differential equation. Clearly, neither of these functions is a constant multiple of the other; so, after recalling the big theorem on solutions to second-order, homogeneous linear differential equations, theorem 14.1 on page 302, we know that

$$\left\{ e^{2x}, e^{3x} \right\}$$

is a fundamental set of solutions and

$$y(x) = c_1 e^{2x} + c_2 e^{3x}$$

is a general solution to our differential equation.

We will discover that we can always construct a general solution to any given homogeneous linear differential equation with constant coefficients using the solutions to its characteristic equation. But first, let us restate what we have just derived in a somewhat more concise and authoritative form, and briefly consider the nature of the possible solutions to the characteristic equation.

16.2 The Basic Approach, Summarized

To solve a second-order homogeneous linear differential equation

$$ay'' + by' + cy = 0$$

in which a, b and c are constants, start with the assumption that

$$y(x) = e^{rx}$$

where r is a constant to be determined. Plugging this formula for y into the differential equation yields, after a little computation and simplification, the differential equation's *characteristic equation* for r,

$$ar^2 + br + c = 0 \quad .$$

Alternatively, the characteristic equation can simply be constructed by replacing the derivatives of y in the original differential equation with the corresponding powers of r.

(By the way, the polynomial on the left side of the characteristic equation,

$$ar^2 + br + c \quad ,$$

is called the *characteristic polynomial* for the differential equation. Recall from algebra that a "root" of a polynomial $p(r)$ is the same as a solution to $p(r) = 0$. So we can — and will — use the terms "solution to the characteristic equation" and "root of the characteristic polynomial" interchangeably.)

Since the characteristic polynomial is only of degree two, solving the characteristic equation for r should present no problem. If this equation is simple enough, we can factor the polynomial and find the values of r by inspection. At worst, we must recall that the solution to the polynomial equation

$$ar^2 + br + c = 0$$

can always be obtained via the quadratic formula,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

Notice how the nature of the value r depends strongly on the value under the square root, $b^2 - 4ac$. There are three possibilities:

1. If $b^2 - 4ac > 0$, then $\sqrt{b^2 - 4ac}$ is some positive value, and we have two distinct real values for r,

$$r_- = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad r_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad .$$

(In practice, we may denote these solutions by r_1 and r_2, instead.)

2. If $b^2 - 4ac = 0$, then

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} \quad ,$$

and we only have one real root for our characteristic equation, namely,

$$r = -\frac{b}{2a} \quad .$$

3. If $b^2 - 4ac < 0$, then the quantity under the square root is negative, and, thus, this square root gives rise to an *imaginary number*. To be explicit,

$$\sqrt{b^2 - 4ac} = \sqrt{-1 \cdot \left|b^2 - 4ac\right|} = i\sqrt{\left|b^2 - 4ac\right|}$$

where "$i = \sqrt{-1}$".[1] Thus, in this case, we will get two distinct complex roots, r_+ and r_- with

$$r_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{|b^2 - 4ac|}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{|b^2 - 4ac|}}{2a} \quad .$$

[1] Some people prefer to use j instead of i for $\sqrt{-1}$. Clearly, these people don't know how to spell 'imaginary'.

Whatever the case, if we find r_0 to be a root of the characteristic polynomial, then, by the very steps leading to the characteristic equation, it follows that

$$y_0(x) = e^{r_0 x}$$

is a solution to our original differential equation. As you can imagine, though, the nature of the corresponding general solution to this differential equation depends strongly on which of the above three cases we are dealing with. Let us consider each case.

16.3 Case 1: Two Distinct Real Roots

Suppose the characteristic equation for

$$ay'' + by' + cy = 0$$

has two distinct (i.e., different) real solutions r_1 and r_2. Then we have that both

$$y_1 = e^{r_1 x} \quad \text{and} \quad y_2 = e^{r_2 x}$$

are solutions to the differential equation. Since we are assuming r_1 and r_2 are not the same, it should be clear that neither y_1 nor y_2 is a constant multiple of the other. Hence

$$\left\{ e^{r_1 x}, e^{r_2 x} \right\}$$

is a linearly independent set of solutions to our second-order, homogeneous linear differential equation. The big theorem on solutions to second-order, homogenous linear differential equations, theorem 14.1 on page 302, then tells us that

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution to our differential equation.

We will later find it convenient to have a way to refer back to the results of the above observations. That is why those results are now restated in the following lemma:

Lemma 16.1
Let a, b and c be constants with $a \neq 0$. If the characteristic equation for

$$ay'' + by' + cy = 0$$

has two distinct real solutions r_1 and r_2, then

$$y_1(x) = e^{r_1 x} \quad \text{and} \quad y_2(x) = e^{r_2 x} \quad .$$

are two solutions to this differential equation. Moreover, $\{e^{r_1 x}, e^{r_2 x}\}$ is a fundamental set for the differential equation, and

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution.

The example done while deriving the basic approach illustrated this case. Another example, however, may not hurt.

!►*Example 16.1:* *Consider the differential equation*

$$y'' + 7y' = 0 \quad .$$

Assuming $y = e^{rx}$ *in this equation gives*

$$\frac{d^2}{dx^2}\left[e^{rx}\right] + \frac{d}{dx}\left[e^{rx}\right] = 0$$

$$\hookrightarrow \qquad r^2 e^{rx} + 7re^{rx} = 0 \quad .$$

Dividing out e^{rx} *gives us the characteristic equation*

$$r^2 + 7r = 0 \quad .$$

In factored form, this is

$$r(r+7) = 0 \quad ,$$

which means that

$$r = 0 \qquad \text{or} \qquad r + 7 = 0 \quad .$$

Consequently, the solutions to the characteristic equation are

$$r = 0 \qquad \text{and} \qquad r = -7 \quad .$$

The two corresponding solutions to the differential equation are

$$y_1 = e^{0\cdot x} = 1 \qquad \text{and} \qquad y_2 = e^{-7x} \quad .$$

Thus, our fundamental set of solutions is

$$\left\{ 1,\ e^{-7x} \right\}$$

and the corresponding general solution to our differential equation is

$$y(x) = c_1 \cdot 1 + c_2 e^{-7x} \quad ,$$

which is slightly more simply written as

$$y(x) = c_1 + c_2 e^{-7x} \quad .$$

16.4 Case 2: Only One Root Using Reduction of Order

If the characteristic polynomial only has one root r , then

$$y_1(x) = e^{rx}$$

is one solution to our differential equation. This, alone, is not enough for a general solution, but we can use this one solution with the reduction of order method to get the full general solution. Let us do one example this way.

!►***Example 16.2:*** *Consider the differential equation*

$$y'' - 6y' + 9y = 0 \quad .$$

The characteristic equation is

$$r^2 - 6r + 9 = 0 \quad ,$$

which factors nicely to

$$(r-3)^2 = 0 \quad ,$$

giving us $r = 3$ *as the only root. Consequently, we have*

$$y_1(x) = e^{3x}$$

as one solution to our differential equation.

To find the general solution, we start the reduction of order method as usual by letting

$$y(x) = y_1(x)u(x) = e^{3x}u(x) \quad .$$

The derivatives are then computed,

$$y'(x) = \left[e^{3x}u\right]' = 3e^{3x}u + e^{3x}u'$$

and

$$\begin{aligned} y''(x) &= \left[3e^{3x}u + e^{3x}u'\right]' \\ &= 3 \cdot 3e^{3x}u + 3e^{3x}u' + 3e^{3x}u' + e^{3x}u'' \\ &= 9e^{3x}u + 6e^{3x}u' + e^{3x}u'' \quad , \end{aligned}$$

and plugged into the differential equation,

$$\begin{aligned} 0 &= y'' - 6y' + 9y \\ &= \left[9e^{3x}u + 6e^{3x}u' + e^{3x}u''\right] - 6\left[3e^{3x}u + e^{3x}u'\right] + 9\left[e^{3x}u\right] \\ &= e^{3x}\left\{9u + 6u' + u'' - 18u - 6u' + 9u\right\} \quad . \end{aligned}$$

Dividing out the exponential and grouping together the coefficients for u *,* u' *and* u'' *, yields*

$$0 = u'' + [6-6]u' + [9-18+9]u = u'' \quad .$$

As expected, the "u *term" drops out. Even nicer, though, is that the "*u' *term" also drops out, leaving us with* $u'' = 0$ *; that is, to be a little more explicit,*

$$\frac{d^2u}{dx^2} = 0 \quad .$$

No need to do anything fancy here — just integrate twice. The first time yields

$$\frac{du}{dx} = \int \frac{d^2u}{dx^2}\,dx = \int 0\,dx = A \quad .$$

Integrating again,

$$u(x) = \int \frac{du}{dx}\,dx = \int A\,dx = Ax + B \quad .$$

Thus,

$$y(x) = e^{3x}u(x) = e^{3x}[Ax+B] = Axe^{3x} + Be^{3x}$$

is the general solution to the differential equation being considered here.

Most of the labor in the last example was in carrying out the reduction of order method. That labor was greatly simplified by the fact that the differential equation for u simplified to

$$u'' = 0 \quad ,$$

which, in turn, meant that

$$u(x) = Ax + B \quad ,$$

and so

$$y(x) = e^{3x}u(x) = Axe^{3x} + Be^{3x} \quad .$$

Will we always be this lucky? To see, let us consider the most general case where the characteristic equation

$$ar^2 + br + c = 0$$

has only one root. As noted when we discussed the possible of solutions to the characteristic polynomial (see page 341), this means

$$r = -\frac{b}{2a} \quad .$$

Let us go through the reduction of order method, keeping this fact in mind.

Start with the one known solution,

$$y_1(x) = e^{rx} \qquad \text{where} \quad r = -\frac{b}{2a} \quad .$$

Set

$$y(x) = y_1(x)u(x) = e^{rx}u(x) \quad ,$$

compute the derivatives,

$$y'(x) = \left[e^{rx}u\right]' = re^{rx}u + e^{rx}u'$$

and

$$\begin{aligned} y''(x) &= \left[re^{rx}u + e^{rx}u'\right]' \\ &= r \cdot re^{rx}u + re^{rx}u' + re^{rx}u' + e^{rx}u'' \\ &= r^2e^{rx}u + 2re^{rx}u' + e^{rx}u'' \quad , \end{aligned}$$

and plug these into the differential equation,

$$\begin{aligned} 0 &= ay'' + by' + cy \\ &= a\left[r^2e^{rx}u + 2re^{rx}u' + e^{rx}u''\right] + b\left[re^{rx}u + e^{rx}u'\right] + c\left[e^{rx}u\right] \\ &= e^{rx}\left\{ar^2u + 2aru' + au'' + bru + bu' + cu\right\} \quad . \end{aligned}$$

Dividing out the exponential and grouping together the coefficients for u, u' and u'', we get

$$0 = au'' + [2ar + b]u' + [ar^2 + br + c]u \quad .$$

Since r satisfies the characteristic equation,

$$ar^2 + br + c = 0 \quad ,$$

the " u term" drops out, as it should. Moreover, because $r = -b/2a$,

$$2ar + b = 2a\left[-\frac{b}{2a}\right] + b = -b + b = 0 \quad ,$$

and the " u' term" also drops out, just as in the example. Dividing out the a (which, remember, is a nonzero constant), the differential equation for u simplifies to

$$u'' = 0 \quad .$$

Integrating twice yields

$$u(x) = Ax + B \quad ,$$

and, thus,

$$y(x) = y_1(x)u(x) = e^{rx}[Ax + B] = Axe^{rx} + Be^{rx} \quad . \tag{16.3}$$

Skipping Reduction of Order

Let us stop and reflect on what the last formula, equation (16.3), tells us. It tells us that, whenever the characteristic polynomial has only one root r , then the general solution of the differential equation is a linear combination of the two functions

$$e^{rx} \qquad \text{and} \qquad xe^{rx} \quad .$$

If we remember this, we don't need to go through the reduction of order method when solving these sorts of equations. This is a nice shortcut for solving these differential equations. And since these equations arise relatively often in applications, it is a shortcut worth remembering. To aid remembrance, here is the summary of what we have derived:

Lemma 16.2
Let a , b and c be constants with $a \neq 0$. If the characteristic equation for

$$ay'' + by' + cy = 0$$

has only one solution r , then

$$y_1(x) = e^{rx} \qquad \text{and} \qquad y_2(x) = xe^{rx} \quad .$$

are two solutions to this differential equation. Moreover, $\{e^{rx}, xe^{rx}\}$ is a fundamental set for the differential equation, and

$$y(x) = c_1e^{rx} + c_2xe^{rx}$$

is a general solution.

Let's redo example 16.2 using this lemma:

!►*Example 16.3:* *Consider the differential equation*

$$y'' - 6y' + 9y = 0 \quad .$$

The characteristic equation is

$$r^2 - 6r + 9 = 0 \quad ,$$

which factors nicely to

$$(r-3)^2 = 0 \quad ,$$

giving us $r = 3$ *as the only root. Consequently,*

$$y_1(x) = e^{3x}$$

is one solution to our differential equation. By our work above, summarized in lemma 16.2, we know a second solution is

$$y_2(x) = xe^{3x} \quad ,$$

and a general solution is

$$y(x) = c_1 e^{3x} + c_2 x e^{3x} \quad .$$

16.5 Case 3: Complex Roots

Blindly Using Complex Roots

Let us start with an example.

!►*Example 16.4:* *Consider solving*

$$y'' - 6y' + 13y = 0 \quad .$$

The characteristic equation is

$$r^2 - 6r + 13 = 0 \quad .$$

Factoring this is not easy for most, so we will resort to the quadratic formula for finding the possible values of r *:*

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} \\ &= \frac{6 \pm \sqrt{-16}}{2} \\ &= \frac{6 \pm i4}{2} = 3 \pm i2 \quad . \end{aligned}$$

So we get

$$y_+(x) = e^{(3+i2)x} \qquad \text{and} \qquad y_-(x) = e^{(3-i2)x}$$

as two solutions to the differential equation. Since the factors in the exponents are different, we can reasonably conclude that neither $e^{(3+i2)x}$ *nor* $e^{(3-i2)x}$ *is a constant multiple of the other, and so*

$$y(x) = c_+ e^{(3+i2)x} + c_- e^{(3-i2)x}$$

should be a general solution to our differential equation.

In general, if the roots to the characteristic equation for

$$ay'' + by' + cy = 0$$

are not real valued, then, as noted at the beginning of these notes, we will get a pair of complex-valued roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{|b^2 - 4ac|}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{|b^2 - 4ac|}}{2a} \quad .$$

For convenience, let's write these roots generically as

$$r_+ = \lambda + i\omega \qquad \text{and} \qquad r_- = \lambda - i\omega$$

where λ and ω are the real numbers

$$\lambda = -\frac{b}{2a} \qquad \text{and} \qquad \omega = \frac{\sqrt{|b^2 - 4ac|}}{2a} \quad .$$

Don't bother memorizing these formulas for λ and ω, but do observe that these two values for r, $\lambda + i\omega$ and $\lambda - i\omega$, form a "conjugate pair", that is, they differ only by the sign on the imaginary part.

It should also be noted that ω cannot be zero, otherwise we would be back in case 2 (one real root). So $r_- = \lambda + i\omega$ and $r_+ = \lambda - i\omega$ are two distinct roots. As we will verify, this means

$$y_+(x) = e^{r_+ x} = e^{(\lambda + i\omega)x} \qquad \text{and} \qquad y_-(x) = e^{r_- x} = e^{(\lambda - i\omega)x}$$

are two independent solutions, and, from a purely mathematical point of view, there is nothing wrong with using

$$y(x) = c_+ e^{(\lambda + i\omega)x} + c_- e^{(\lambda - i\omega)x}$$

as a general solution to our differential equation.

There are problems, however, with using e^{rx} when r is a complex number. For one thing, it introduces complex numbers into problems that, most likely, should be entirely describable using just real-valued functions. More importantly, you might not yet know what e^{rx} means when r is a complex number! (Quick, graph $e^{(3+i2)x}$!) To deal with these problems, you have several choices:

1. Pray you never have to deal with complex roots when solving differential equations. (A very bad choice since such equations turn out to be among the most important ones in applications.)

2. Blindly use $e^{(\lambda \pm i\omega)x}$, hoping that no one questions your results and that your results are never used in a system the failure of which could result in death, injury, or the loss of substantial sums of money and/or prestige.

3. Derive an alternative formula for the general solution, and hope that no one else ever springs a $e^{(\lambda \pm i\omega)x}$ on you. (Another bad choice since these exponentials are used throughout the sciences and engineering.)

4. Spend a little time learning what $e^{(\lambda \pm i\omega)x}$ means.

Guess which choice we take.

The Complex Exponential Function

Let us now consider the quantity e^z where z is any complex value. We, like everyone else, will call this the *complex exponential function*. In our work,

$$z = rx = (\lambda \pm i\omega)x = \lambda x \pm i\omega x \quad .$$

where λ and ω are real constants. Keep in mind that our r value came from a characteristic equation that, in turn, came from plugging $y(x) = e^{rx}$ into a differential equation. In getting the characteristic equation, we did our computations assuming the e^{rx} behaved just as it would behave if r were a real value. So, in determining just what e^{rx} means when r is complex, we should assume the complex exponential satisfies the rules satisfied by the exponential we already know. In particular, we must have, for any constant r (real or complex),

$$\frac{d}{dx}\left[e^{rx}\right] = re^{rx} \quad ,$$

and, for any two constants A and B,

$$e^{A+B} = e^A e^B \quad .$$

Thus,

$$e^{rx} = e^{(\lambda \pm i\omega)x} = e^{\lambda x \pm i\omega x} = e^{\lambda x} e^{\pm i\omega x} \quad .$$

The first factor, $e^{\lambda x}$ is just the ordinary, 'real' exponential — a function you should be able to graph in your sleep. It is the second factor, $e^{\pm i\omega x}$, that we need to come to grips with.

To better understand $e^{\pm i\omega x}$, let us first examine

$$\phi(t) = e^{it} \quad .$$

Later, we'll replace t with $\pm\omega x$.

One thing we can do with this $\phi(t)$ is to compute it at $t = 0$,

$$\phi(0) = e^{i \cdot 0} = e^0 = 1 \quad .$$

We can also differentiate ϕ, getting

$$\phi'(t) = \frac{d}{dt} e^{it} = ie^{it} \quad .$$

Letting $t = 0$ here gives

$$\phi'(0) = ie^{i \cdot 0} = ie^0 = i \quad .$$

Notice that the right side of the formula for $\phi'(t)$ is just i times ϕ. Differentiating $\phi'(t)$ again, we get

$$\phi''(t) = \frac{d}{dt}\left[ie^{it}\right] = i^2 e^{it} = -1 \cdot \phi(t) \quad ,$$

giving us the differential equation

$$\phi'' = -\phi \quad ,$$

which is better written as

$$\phi'' + \phi = 0$$

simply because we've seen this equation before (using y instead of ϕ for the unknown function).

What we have just derived is that $\phi(t) = e^{it}$ satisfies the initial-value problem

$$\phi'' + \phi = 0 \qquad \text{with} \quad \phi(0) = 1 \quad \text{and} \quad \phi'(0) = i \quad .$$

We can solve this! From example 14.2 on page 302, we know

$$\phi(t) = A\cos(t) + B\sin(t)$$

is a general solution to the differential equation. Using this formula for ϕ we also have

$$\phi'(t) = \frac{d}{dt}[A\cos(t) + B\sin(t)] = -A\sin(t) + B\cos(t) \quad .$$

Applying the initial conditions, gives

$$1 = \phi(0) = A\cos(0) + B\sin(0) = A \cdot 1 + B \cdot 0 = A$$

and

$$i = \phi'(0) = -A\sin(0) + B\cos(0) = -a \cdot 0 + B \cdot 1 = B \quad .$$

Thus,

$$e^{it} = \phi(t) = 1 \cdot \cos(t) + i \cdot \sin(t) = \cos(t) + i\sin(t) \quad .$$

The last formula is so important, we should write it again with the middle cut out and give it a reference number:

$$e^{it} = \cos(t) + i\sin(t) \quad . \tag{16.4}$$

Before replacing t with $\pm\omega x$, it is worth noting what happens when t is replaced by $-t$ in the above formula. In doing this, remember that the cosine is an even function and that the sine is an odd function, that is,

$$\cos(-t) = \cos(t) \qquad \text{while} \qquad \sin(-t) = -\sin(t) \quad .$$

So

$$e^{-it} = e^{i(-t)} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t) \quad .$$

Cutting out the middle leaves us with

$$e^{-it} = \cos(t) - i\sin(t) \quad . \tag{16.5}$$

Formulas (16.4) and (16.5) are the (famous) Euler formulas for e^{it} and e^{-it}. They really should be written as a pair:

$$e^{it} = \cos(t) + i\sin(t) \tag{16.6a}$$

$$e^{-it} = \cos(t) - i\sin(t) \tag{16.6b}$$

That way, it is clear that when you add these two equations together, the sines cancel out and we get

$$e^{it} + e^{-it} = 2\cos(t) \quad .$$

On the other hand, subtracting equation (16.6b) from equation (16.6a) yields

$$e^{it} - e^{-it} = 2i\sin(t) \quad .$$

So, the sine and cosine functions can be rewritten in terms of complex exponentials,

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \qquad \text{and} \qquad \sin(t) = \frac{e^{it} - e^{-it}}{2i} \quad . \tag{16.7}$$

This is nice. This means that many computations involving trigonometric functions can be done using exponentials instead, and this can greatly simplify those computations.

!▶*Example 16.5:* *Consider deriving the trigonometric identity involving the product of the sine and the cosine functions. Using equation set (16.7) and basic algebra,*

$$\begin{aligned}
\sin(t)\cos(t) &= \frac{e^{it}+e^{-it}}{2}\cdot\frac{e^{it}-e^{-it}}{2i} \\
&= \frac{\left(e^{it}\right)^2-\left(e^{-it}\right)^2}{2\cdot 2i} \\
&= \frac{e^{i2t}-e^{-i2t}}{2\cdot 2i} \\
&= \frac{1}{2}\cdot\frac{e^{i(2t)}-e^{-i(2t)}}{2i} \;=\; \frac{1}{2}\sin(2t) \quad .
\end{aligned}$$

Thus, we have (re)derived the trigonometric identity

$$2\sin(t)\cos(t) \;=\; \sin(2t) \quad .$$

(Compare this to the derivation you did years ago without complex exponentials!)

But we digress — our interest is not in rederiving trigonometric identities, but in figuring out what to do when we get $e^{(\lambda\pm i\omega)x}$ as solutions to a differential equation. Using the law of exponents and formulas (16.6a) and (16.6b) (with ωx replacing t), we see that

$$e^{(\lambda+i\omega)x} \;=\; e^{\lambda x+i\omega x} \;=\; e^{\lambda x}e^{i\omega x} \;=\; e^{\lambda x}[\cos(\omega x) \;+\; i\sin(\omega x)]$$

and

$$e^{(\lambda-i\omega)x} \;=\; e^{\lambda x-i\omega x} \;=\; e^{\lambda x}e^{-i\omega x} \;=\; e^{\lambda x}[\cos(\omega x) \;-\; i\sin(\omega x)] \quad .$$

So now we know how to interpret $e^{(\lambda\pm i\omega)x}$, and now we can get back to solving our differential equations.

Intelligently Using Complex Roots

Recall, we are interested in solving

$$ay'' \;+\; by' \;+\; cy \;=\; 0$$

when a , b and c are real-valued constants, and the solutions to the corresponding characteristic equation are complex. Remember, also, that these complex roots will form a conjugate pair,

$$r_+ \;=\; \lambda \;+\; i\omega \qquad \text{and} \qquad r_- \;=\; \lambda - i\omega$$

where λ and ω are real numbers with $\omega \neq 0$. This gave us

$$y_+(x) \;=\; e^{r_+x} \;=\; e^{(\lambda+i\omega)x} \qquad \text{and} \qquad y_-(x) \;=\; e^{r_-x} \;=\; e^{(\lambda-i\omega)x}$$

as two solutions to our differential equation. From our discussion of the complex exponential, we now know that

$$y_+(x) \;=\; e^{(\lambda+i\omega)x} \;=\; e^{\lambda x}[\cos(\omega x) \;+\; i\sin(\omega x)] \tag{16.8a}$$

and

$$y_-(x) \;=\; e^{(\lambda-i\omega)x} \;=\; e^{\lambda x}[\cos(\omega x) \;-\; i\sin(\omega x)] \quad . \tag{16.8b}$$

Clearly, neither y_+ nor y_- is a constant multiple of each other. So each of the equivalent formulas

$$y(x) = c_+ y_+(x) + c_- y_-(x) \quad ,$$

$$y(x) = c_+ e^{(\lambda + i\omega)x} + c_- e^{(\lambda - i\omega)x}$$

and

$$y(x) = c_+ e^{\lambda x}[\cos(\omega x) + i \sin(\omega x)] + c_- e^{\lambda x}[\cos(\omega x) - i \sin(\omega x)] \quad ,$$

can, legitimately, be used as a general solution to our differential equation.

Still, however, these solutions introduce complex numbers into formulas that, in applications, should probably just involve real numbers. To avoid that, let us derive an alternative fundamental pair of solutions by choosing the constants c_+ and c_- appropriately. The basic idea is the same as used to derive the formulas (16.7) for $\sin(t)$ and $\cos(t)$ in terms complex exponentials. First add equations (16.8a) and (16.8b) together. The sine terms cancel out leaving

$$y_+(x) + y_-(x) = 2e^{\lambda x} \cos(\omega x) \quad .$$

So

$$y_1(x) = \frac{1}{2} y_+(x) + \frac{1}{2} y_-(x) = e^{\lambda x} \cos(\omega x)$$

is a solution to our differential equation. On the other hand, the cosine terms cancel out when we subtract equation (16.8b) from equation (16.8a), leaving us with

$$y_+(x) - y_-(x) = 2i e^{\lambda x} \sin(\omega x) \quad .$$

So

$$y_2(x) = \frac{1}{2i} y_+(x) - \frac{1}{2i} y_-(x) = e^{\lambda x} \sin(\omega x)$$

is another solution to our differential equation. Again, it should be clear that our latest solutions are not constant multiples of each other. Consequently,

$$y_1(x) = e^{\lambda x} \cos(\omega x) \qquad \text{and} \qquad y_2(x) = e^{\lambda x} \sin(\omega x)$$

form a fundamental set, and

$$y(x) = c_1 e^{\lambda x} \cos(\omega x) + c_2 e^{\lambda x} \sin(\omega x)$$

is a general solution to our differential equation.

All this work should be summarized so we won't forget:

Lemma 16.3

Let a, b and c be real-valued constants with $a \neq 0$. If the characteristic equation for

$$ay'' + by' + cy = 0$$

does not have one or two distinct real solutions, then it will have a conjugate pair of solutions

$$r_+ = \lambda + i\omega \qquad \text{and} \qquad r_- = \lambda - i\omega$$

where λ and ω are real numbers with $\omega \neq 0$.

Moreover, both

$$\left\{ e^{(\lambda+i\omega)x} \, , \, e^{(\lambda-i\omega)x} \right\} \qquad \text{and} \qquad \left\{ e^{\lambda x}\cos(\omega x) \, , \, e^{\lambda x}\sin(\omega x) \right\}$$

are fundamental sets of solutions for the differential equation, and the general solution can be written as either

$$y(x) \;=\; c_+ e^{(\lambda+i\omega)x} \;+\; c_- e^{(\lambda-i\omega)x} \tag{16.9a}$$

or

$$y(x) \;=\; c_1 e^{\lambda x}\cos(\omega x) \;+\; c_2 e^{\lambda x}\sin(\omega x) \tag{16.9b}$$

as desired.

In practice, formula (16.9b) is often preferred since it is a formula entirely in terms of real-valued functions. On the other hand, if you are going to do a bunch of calculations with $y(x)$ involving differentiation and/or integration, you may prefer using formula (16.9a), since calculus with exponentials — even complex exponentials — is much easier that calculus with products of exponentials and trigonometric functions.

By the way, instead of "memorizing" the above theorem, it may be better to just remember that you can get the $e^{\lambda x}\cos(\omega x)$ and $e^{\lambda x}\sin(\omega x)$ solutions from the real and imaginary parts of

$$\begin{aligned} e^{(\lambda\pm i\omega)x} &= e^{\lambda x}e^{\pm i\omega x} \\ &= e^{\lambda x}\left[\cos(\omega x) \pm i\sin(\omega x)\right] \;=\; e^{\lambda x}\cos(\omega x) \;\pm\; ie^{\lambda x}\sin(\omega x) \quad . \end{aligned}$$

!►*Example 16.6:* *Again, consider solving*

$$y'' - 6y' + 13y = 0 \quad .$$

From example 16.4, we already know

$$r_+ = 3 + i2 \qquad \text{and} \qquad r_- = 3 - i2$$

are the two solutions to the characteristic equation,

$$r^2 - 6r + 13 = 0 \quad .$$

Thus, one fundamental set of solutions consists of

$$e^{(3+i2)x} \qquad \text{and} \qquad e^{(3-i2)x}$$

with

$$\begin{aligned} e^{(3\pm i2)x} &= e^{3x}e^{\pm i2x} \\ &= e^{3x}\left[\cos(2x) \pm i\sin(2x)\right] \;=\; e^{3x}\cos(2x) \;\pm\; ie^{3x}\sin(2x) \quad . \end{aligned}$$

Alternatively, we can use the pair of real-valued functions

$$e^{3x}\cos(2x) \qquad \text{and} \qquad e^{3x}\sin(2x)$$

as a fundamental set. Using this, we have

$$y(x) = c_1 e^{3x}\cos(2x) + c_2 e^{3x}\sin(2x)$$

as the general solution to the differential equation in terms of just real-valued functions.

16.6 Summary

Combining lemma 16.1 on page 342, lemma 16.2 on page 346, and lemma 16.3 on page 352, we get the big theorem on solving second-order homogeneous linear differential equations with constant coefficients:

Theorem 16.4

Let a , b and c be real-valued constants with $a \neq 0$. Then the characteristic polynomial for

$$ay'' + by' + cy = 0$$

will have either one or two has two distinct real roots or will have two complex roots that are complex conjugates of each other. Moreover:

1. *If there are two distinct real roots r_1 and r_2 , then*

$$\{ e^{r_1 x} , e^{r_2 x} \}$$

is a fundamental set of solutions to the differential equation, and

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution.

2. *If there is only one real root r , then*

$$\{ e^{rx} , xe^{rx} \}$$

is a fundamental set of solutions to the differential equation, and

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$

is a general solution.

3. *If there is there is a conjugate pair of roots $r = \lambda \pm i\omega$, then both*

$$\{ e^{(\lambda+i\omega)x} , e^{(\lambda-i\omega)x} \} \qquad \text{and} \qquad \{ e^{\lambda x} \cos(\omega x) , e^{\lambda x} \sin(\omega x) \}$$

are fundamental sets of solutions to the differential equation, and either

$$y(x) = c_+ e^{(\lambda+i\omega)x} + c_- e^{(\lambda-i\omega)x}$$

or

$$y(x) = c_1 e^{\lambda x} \cos(\omega x) + c_2 e^{\lambda x} \sin(\omega x)$$

can be used as a general solution.

Additional Exercises

16.1. *Below are several second-order, homogeneous linear differential equations with constant coefficients, along with one to three sets of initial conditions for each. Find the general solution to each differential equation, and then, for each set of initial data, solve the corresponding initial-value problem.*

a. $y'' - 7y' + 10y = 0$

i. with $y(0) = 1$ and $y'(0) = 0$

ii. with $y(0) = 0$ and $y'(0) = 1$

iii. with $y(0) = 5$ and $y'(0) = 16$

b. $y'' + 2y' - 24y = 0$

i. with $y(0) = 1$ and $y'(0) = 0$

ii. with $y(0) = 10$ and $y'(0) = 10$

c. $y'' - 25y = 0$

i. with $y(0) = 1$ and $y'(0) = 0$

ii. with $y(0) = 0$ and $y'(0) = 1$

iii. with $y(0) = 3$ and $y'(0) = -5$

d. $y'' + 3y' = 0$

i. with $y(0) = 1$ and $y'(0) = 0$

ii. with $y(0) = 0$ and $y'(0) = 1$

iii. with $y(0) = 10$ and $y'(0) = -12$

16.2. *Below are some more second-order, homogeneous linear differential equations with constant coefficients, along with one to three sets of initial conditions for each. Find the general solution to each differential equation, and then, for each set of initial data, solve the corresponding initial-value problem.*

a. $y'' - 10y' + 25y = 0$

i. with $y(0) = 1$ and $y'(0) = 0$

ii. with $y(0) = 0$ and $y'(0) = 1$

iii. with $y(0) = 3$ and $y'(0) = 17$

b. $y'' + 2y' + y = 0$

i. with $y(0) = 1$ and $y'(0) = 0$

ii. with $y(0) = 0$ and $y'(0) = 1$

iii. with $y(0) = 18$ and $y'(0) = 2$

c. $4y'' - 4y' + y = 0$

i. *with* $y(0) = 1$ *and* $y'(0) = 0$

ii. *with* $y(0) = 0$ *and* $y'(0) = 1$

iii. *with* $y(0) = 6$ *and* $y'(0) = 1$

16.3. *Below are even more second-order, homogeneous linear differential equations with constant coefficients, along with one to three sets of initial conditions for each. Find the general solution to each differential equation, and then, for each set of initial data, solve the corresponding initial-value problem.*

a. $y'' + 25y = 0$

i. *with* $y(0) = 1$ *and* $y'(0) = 0$

ii. *with* $y(0) = 0$ *and* $y'(0) = 1$

iii. *with* $y(0) = 4$ *and* $y'(0) = -15$

b. $y'' + 2y' + 5y = 0$

i. *with* $y(0) = 1$ *and* $y'(0) = 0$

ii. *with* $y(0) = 0$ *and* $y'(0) = 1$

iii. *with* $y(0) = 5$ *and* $y'(0) = 9$

16.4. *Find and graph (by hand) the solution to each of the following initial-value problems:*

a. $y'' - 4y' + 5y = 0$ *with* $y(0) = 1$ *and* $y'(0) = 2$

b. $y'' + 4y' + 5y = 0$ *with* $y(0) = 1$ *and* $y'(0) = -2$

16.5. *Find the general solution to each of the following. Express your solution in terms of real-valued functions only.*

a. $y'' - 9y = 0$

b. $y'' + 9y = 0$

c. $y'' + 6y' + 9y = 0$

d. $y'' + 6y' - 9y = 0$

e. $9y'' - 6y' + y = 0$

f. $y'' + 6y' + 10y = 0$

g. $y'' - 4y' + 40y = 0$

h. $2y'' - 5y' + 2y = 0$

i. $y'' + 10y' + 25y = 0$

j. $9y'' - y = 0$

k. $9y'' + y = 0$

l. $9y'' + y' = 0$

m. $y'' + 4y' + 7y = 0$

n. $y'' + 4y' + 5y = 0$

o. $y'' + 4y' + 4y = 0$

p. $y'' - 2y' - 15y = 0$

q. $y'' - 4y' = 0$

r. $y'' - 8y' + 16y = 0$

s. $4y'' + 3y = 0$

t. $4y'' - 4y' + 5y = 0$