# 20

# *Nonhomogeneous Equations in General*

Now that we are proficient at solving many homogeneous linear differential equations, including

$$y'' - 4y = 0 \quad ,$$

it is time to expand our skills to solving nonhomogeneous linear equations, such as

$$y'' - 4y = 5e^{3x} \quad .$$

## 20.1 Basic Theory

### Recollections about Linearity*

Let us go back to our generic, $N^{\text{th}}$-order, linear differential equation,

$$a_0 \frac{d^N y}{dx^N} + a_1 \frac{d^{N-1} y}{dx^{N-1}} + \cdots + a_{N-2} \frac{d^2 y}{dx^2} + a_{N-1} \frac{dy}{dx} + a_N y = g \quad .$$

Remember, $g$ and the $a_k$'s denote known functions of $x$ over some interval of interest, $\mathcal{I}$. As usual, we will assume these functions are continuous and that $a_0$ is never zero on this interval.

As before, it is convenient to let $L$ denote the corresponding differential operator from the left side of the equation,

$$L = a_0 \frac{d^N}{dx^N} + a_1 \frac{d^{N-1}}{dx^{N-1}} + \cdots + a_{N-2} \frac{d^2}{dx^2} + a_{N-1} \frac{d}{dx} + a_N \quad .$$

That is, given any sufficiently differentiable function $\phi(x)$ on $\mathcal{I}$,

$$L[\phi] = a_0 \frac{d^N \phi}{dx^N} + a_1 \frac{d^{N-1} \phi}{dx^{N-1}} + \cdots + a_{N-2} \frac{d^2 \phi}{dx^2} + a_{N-1} \frac{d\phi}{dx} + a_N \phi \quad .$$

Using this operator, we can write our generic differential equation as

$$L[y] = g \quad .$$

* You may want to briefly review the material in chapter 12.

Remember, this equation is said to be homogeneous if $g$ is always zero on our interval, and nonhomogeneous otherwise. Since we have already discussed the homogeneous case, let us now assume $g(x)$ is a function that is nonzero over at least a portion of our interval of interest.

Don't forget, however, that, for each nonhomogeneous equation

$$L[y(x)] = g(x) \quad ,$$

we still have the *corresponding homogeneous equation*

$$L[y(x)] = 0$$

where we simply replace $g(x)$ with $0$. This equation will play a significant role in solving the nonhomogeneous equation.

**!►Example 20.1:** *Using the linear differential operator*

$$L = \frac{d^2}{dx^2} - 4 \quad ,$$

*we can rewrite the nonhomogeneous equation*

$$y'' - 4y = 5e^{3x}$$

*and the corresponding homogeneous equation*

$$y'' - 4y = 0$$

*as*

$$L[y(x)] = 5e^{3x} \qquad \text{and} \qquad L[y(x)] = 0 \quad ,$$

*respectively*

## General Solutions to Nonhomogeneous Equations

Now assume we have some linear differential operator $L$ (such as in the last example), and let us recall that, given any pair of sufficiently differentiable functions $\phi_1(x)$ and $\phi_2(x)$ along with any pair of constants $c_1$ and $c_2$,

$$L[c_1\phi_1(x) + c_2\phi_2(x)] = c_1 L[\phi_1(x)] + c_2 L[\phi_2(x)] \quad .$$

This is the "linearity" of our operator, and was used to construct general solutions to homogeneous equations as linear combinations of different solutions. With *non*homogeneous equations we must be a little more careful. After all, if $y_p$ and $y_q$ are two particular solutions to a nonhomogeneous equation

$$L[y] = g$$

$$(\text{i.e.,} \quad L[y_p(x)] = g(x) \qquad \text{and} \qquad L[y_q(x)] = g(x) \quad ),$$

and $c_1$ and $c_2$ are any two constants that do not add up to $1$, then, since $g$ is a nonzero function,

$$\begin{aligned} L[c_1 y_p(x) + c_2 y_q(x)] &= c_1 L[y_p(x)] + c_2 L[y_q] \\ &= c_1 g(x) + c_2 g(x) \end{aligned}$$

$$
\begin{aligned}
&= [c_1 + c_2]g(x) \\
&\neq g(x) \quad .
\end{aligned}
$$

Thus, *a linear combination of solutions to a NONhomogeneous linear differential equation is usually NOT a solution to that differential equation.*

Notice, however, what happens when we consider the difference

$$y_q(x) - y_p(x)$$

between these two particular solutions to our nonhomogeneous equation. Plugging this into $L$ gives us

$$L[y_q(x) - y_p(x)] = L[y_q(x)] - L[y_p(x)] = g(x) - g(x) = 0 \quad .$$

So

$$y_q(x) - y_p(x) = \text{a solution to the corresponding homogeneous equation} \quad .$$

Let me rephrase this:

> *If $y_p$ and $y_q$ are any two solutions to a given nonhomogeneous linear differential equation, then*
>
> $$y_q(x) = y_p(x) + \textit{a solution to the corresponding homogeneous equation} \quad .$$

On the other hand, if we start with

$$y_q(x) = y_p(x) + y_0(x)$$

where $y_p$ is any particular solution to the nonhomogeneous equation and $y_0$ is any solution to the corresponding homogeneous equation

$$\Big(\text{i.e.,} \quad L[y_p(x)] = g(x) \qquad \text{and} \qquad L[y_0(x)] = 0 \quad\Big),$$

then

$$L[y_q(x)] = L[y_p(x) + y_0(x)] = L[y_p(x)] + L[y_0] = g(x) + 0 = g(x) \quad .$$

Thus:

> *If $y_p$ is a particular solution to a given nonhomogeneous linear differential equation, and*
>
> $$y_q(x) = y_p(x) + \textit{any solution to the corresponding homogeneous equation} \quad ,$$
>
> *then $y_q$ is also a solution to the nonhomogeneous differential equation.*

If you think about it, you will realize that we've just derived the form for a general solution to any nonhomogeneous linear differential equation. We only need one particular solution to that nonhomogeneous differential equation and the general formula describing all solutions to the corresponding homogeneous linear differential equation. To be precise, the two results derived above, combined, yield the following theorem.

***Theorem 20.1 (general solutions to nonhomogeneous equations)***
*A general solution to a given nonhomogeneous linear differential equation is given by*

$$y(x) = y_p(x) + y_h(x)$$

*where $y_p$ is any particular solution to the nonhomogeneous equation, and $y_h$ is a general solution to the corresponding homogeneous differential equation.*[1]

**!►*Example 20.2:*** *Consider the nonhomogeneous differential equation*

$$y'' - 4y = 5e^{3x} \quad . \tag{20.1}$$

*Observe that*

$$\left[e^{3x}\right]'' - 4\left[e^{3x}\right] = 3^2 e^{3x} - 4e^{3x} = 5e^{3x} \quad .$$

*So one particular solution to our nonhomogeneous equation is*

$$y_p(x) = e^{3x} \quad .$$

*The corresponding homogeneous equation is*

$$y'' - 4y = 0 \quad ,$$

*a linear equation with constant coefficients. Its characteristic equation,*

$$r^2 - 4 = 0 \quad ,$$

*has solutions $r = 2$ and $r = -2$. So this homogeneous equation has*

$$\left\{ y_1(x) \, , \, y_2(x) \right\} = \left\{ e^{2x} \, , \, e^{-2x} \right\}$$

*as a fundamental set of solutions, and*

$$y_h(x) = c_1 e^{2x} + c_2 e^{-2x}$$

*as a general solution.*

*Thus, (according to our work above as summarized in theorem 20.1), the general solution to the nonhomogeneous differential equation (20.1) is*

$$\begin{aligned} y(x) &= y_p(x) + y_h(x) \\ &= e^{3x} + c_1 e^{2x} + c_2 e^{-2x} \quad . \end{aligned}$$

*(Note that there are only two arbitrary constants, and that they are only in the formula for $y_h$. There is no arbitrary constant corresponding to $y_p$ !)*

This last example illustrates what happens when we limit ourselves to second-order equations. More generally, if we recall how we construct general solutions to the corresponding homogeneous equations, then we get the following corollary of theorem 20.1:

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[1] Many texts refer to the general solution of the corresponding homogeneous differential equation as “the complementary solution” and denote it by $y_c$ instead of $y_h$. We are using $y_h$ to help remind us that this is the general solution to the corresponding *h*omogeneous differential equation.

***Corollary 20.2 (general solutions to nonhomogeneous second-order equations)***
*A general solution to a second-order, nonhomogeneous linear differential equation*

$$ay'' + by' + cy = g$$

*is given by*

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) \tag{20.2}$$

*where* $y_p$ *is any particular solution to the nonhomogeneous equation, and* $\{y_1, y_2\}$ *is any fundamental set of solutions for the corresponding homogeneous equation*

$$ay'' + by' + cy = 0 \quad .$$

Do note that there are only two arbitrary constants $c_1$ and $c_2$ in formula (20.2), and that they are multiplying only particular solutions to the corresponding homogeneous equation. The particular solution to the nonhomogeneous equation, $y_p$, is NOT multiplied by an arbitrary constant!

Of course, if we don't limit ourselves to second-order equations, but still recall how to construct general solutions to homogeneous equations from a fundamental set of solutions to that homogeneous equation, then we get the $N^{\text{th}}$-order analog of the last corollary:

***Corollary 20.3 (general solutions to nonhomogeneous $N^{th}$-order equations)***
*A general solution to an* $N^{th}$*-order, nonhomogeneous linear differential equation*

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = g$$

*is given by*

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x) \tag{20.3}$$

*where* $y_p$ *is any particular solution to the nonhomogeneous equation, and* $\{y_1, y_2, \ldots, y_N\}$ *is any fundamental set of solutions for the corresponding homogeneous equation*

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0 \quad .$$

## 20.2 Superposition for Nonhomogeneous Equations

Before discussing methods for finding particular solutions, we should use linearity of a linear differential operator $L$ to make one more observation: If $y_1$, $y_2$, $g_1$ and $g_2$ are two functions satisfying

$$L[y_1(x)] = g_1(x) \qquad \text{and} \qquad L[y_2(x)] = g_2(x) \quad ,$$

then, for any two constants $a_1$ and $a_2$,

$$L[a_1 y_1(x) + a_2 y_2(x)] = a_1 L[y_1(x)] + a_2 L[y_2(x)] = a_1 g_1(x) + a_2 g_2(x) \quad .$$

Obviously, similar computations will yield similar results involving any number of " $(y_j, g_j)$ pairs", giving us a *principle of superposition for nonhomogeneous equations*:[2]

[2] You might want to compare this principle of superposition to the principle of superposition for homogeneous equations was described in theorem 12.2 on page 266.

***Theorem 20.4 (principle of superposition for nonhomogeneous equations)***
*Let $L$ be a linear differential operator and $K$ a positive integer. Assume $\{y_1, y_2, \dots, y_K\}$ and $\{g_1, g_2, \dots, g_K\}$ are two sets of $K$ functions related over some interval of interest by*

$$L[y_1] = g_1 \quad , \quad L[y_2] = g_2 \quad , \quad \dots \quad \text{and} \quad L[y_K] = g_K \quad .$$

*Then, for any set of $K$ constants $\{a_1, a_2, \dots, a_K\}$, a particular solution to*

$$L[y(x)] = a_1 g_1(x) + a_2 g_2(x) + \cdots + a_K g_K(x)$$

*is given by*

$$y_p(x) = a_1 y_1(x) + a_2 y_2(x) + \cdots + a_K y_K(x) \quad .$$

This principle gives us a means for constructing solutions to certain nonhomogeneous equations as linear combinations of solutions to simpler nonhomogeneous equations, provided, of course, we have the solutions to those simpler equations.

**!►*Example 20.3:*** *From our last example, we know that*

$$y_1(x) = e^{3x} \quad \text{satisfies} \quad y_1'' - 4y_1 = 5e^{3x} \quad .$$

*The principle of superposition (with $K = 1$) then assures us that, for any constant $a_1$,*

$$y_p(x) = a_1 y_1(x) = a_1 e^{3x} \quad \text{satisfies} \quad y_1'' - 4y_1 = a_1 \left[5e^{3x}\right] \quad .$$

*For example, a particular solution to*

$$y'' - 4y = e^{3x} \quad ,$$

*which we will rewrite as*

$$y'' - 4y = \frac{1}{5}\left[5e^{3x}\right]$$

*is given by*

$$y_p(x) = \frac{1}{5} y_1(x) = \frac{1}{5} e^{3x} \quad .$$

*And for the general solution, we simply add the general solution to the corresponding homogeneous equation found in the previous example:*

$$y(x) = y_p(x) + y_h(x) = \frac{1}{5} e^{3x} + c_1 e^{2x} + c_2 e^{-2x} \quad .$$

The basic use of superposition requires that we already know the appropriate "$y_k$'s ". At times, we may not already know them, but, with luck, we can make good "guesses" as to appropriate $y_k$'s and then, after computing the corresponding $g_k$'s, use the principle of superposition.

**!►*Example 20.4:*** *Consider solving*

$$y'' - 4y = 2x^2 - 8x + 3 \quad . \tag{20.4}$$

*Let us "guess" that a particular solution can be given by a linear combination of*

$$y_1(x) = x^2 \quad , \quad y_2(x) = x \quad \text{and} \quad y_3(x) = 1 \quad .$$

*Plugging these into the lefthand side of equation (20.4), we get*

$$g_1(x) = {y_1}'' - 4y_1 = \frac{d^2}{dx^2}\left[x^2\right] - 4\left[x^2\right] = 2 - 4x^2 \quad ,$$

$$g_2(x) = {y_2}'' - 4y_2 = \frac{d^2}{dx^2}[x] - 4[x] = -4x \quad ,$$

*and*

$$g_3(x) = {y_3}'' - 4y_3 = \frac{d^2}{dx^2}[1] - 4[1] = -4 \quad .$$

*Now set*

$$y_p(x) = a_1 y_1(x) + a_2 y_2(x) + a_3 y_3(x) \quad .$$

*By the principle of superposition,*

$$\begin{aligned} {y_p}'' - 4y_p &= a_1 g_1(x) + a_2 g_2(x) + a_3 g_3(x) \\ &= a_1\left[2 - 4x^2\right] + a_2[-4x] + a_3[-4] \\ &= -4a_1 x^2 - 4a_2 x + [2a_1 - 4a_3] \quad . \end{aligned}$$

*This means* $y_p$ *is a solution to our differential equation,*

$$y'' - 4y = 2x^2 - 8x + 3 \quad ,$$

*if and only if*

$$-4a_1 = 2 \quad , \quad -4a_2 = -8 \quad \text{and} \quad 2a_1 - 4a_3 = 3 \quad .$$

*Solving for the* $a_k$*'s yields*

$$a_1 = -\frac{1}{2} \quad , \quad a_2 = 2 \quad \text{and} \quad a_1 = -1 \quad .$$

*Thus, a particular solution to our differential equation is given by*

$$y_p(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) = -\frac{1}{2}x^2 + 2x - 1 \quad ,$$

*and a general solution is*

$$y(x) = y_p(x) + y_h(x) = -\frac{1}{2}x^2 + 2x - 1 + c_1 e^{2x} + c_2 e^{-2x} \quad .$$

By the way, we'll further discuss the "art of making good guesses" in the next chapter, and develop a somewhat more systematic method that uses superposition is a slightly more subtle way. Unfortunately, as we will see, "guessing" is only suitable for relatively simple problems.

## 20.3 Reduction of Order

In practice, finding a particular solution to a nonhomogeneous linear differential equation can be a challenge. One method, the basic reduction of order method for second-order, nonhomogeneous linear differential equations, was briefly discussed in section 13.3. If you haven't already looked at that section, or don't remember the basic ideas discussed there, you can go back and skim that section. Or not. Truth is, better methods will be developed in the next few sections.

## Additional Exercises

**20.1.** *What should* $g(x)$ *be so that* $y(x) = e^{3x}$ *is a solution to*

**a.** $y'' + y = g(x)$ ?

**b.** $x^2y'' - 4y = g(x)$ ?

**c.** $y^{(3)} - 4y' + 5y = g(x)$ ?

**20.2.** *What should* $g(x)$ *be so that* $y(x) = x^3$ *is a solution to*

**a.** $y'' + 4y' + 4y = g(x)$ ?

**b.** $x^2y'' + 4xy' + 4y = g(x)$ ?

**c.** $y^{(4)} + xy^{(3)} + 4y'' - \frac{3}{x}y' = g(x)$ ?

**20.3 a.** *Can* $y(x) = \sin(x)$ *be a solution to*

$$y'' + y = g(x)$$

*for some nonzero function* $g$ *? (Give a reason for your answer.)*

**b.** *What should* $g(x)$ *be so that* $y(x) = x\sin(x)$ *is a solution to*

$$y'' + y = g(x) \quad ?$$

**20.4.** *Consider the nonhomogeneous linear differential equation*

$$y'' + 4y = 24e^{2x} \quad .$$

**a.** *Verify that one particular solution to this nonhomogeneous differential equation is*

$$y_p(x) = 3e^{2x} \quad .$$

**b.** *What is* $y_h$ *, the general solution to the corresponding homogeneous equation?*

**c.** *What is the general solution to the above nonhomogeneous equation?*

**d.** *Find the solution to the above nonhomogeneous equation that also satisfies each of the following sets of initial conditions:*

**i.** $y(0) = 6$ *and* $y'(0) = 6$ **ii.** $y(0) = -2$ *and* $y'(0) = 2$

**20.5.** *Consider the nonhomogeneous linear differential equation*

$$y'' + 2y' - 8y = 8x^2 - 3 \quad .$$

**a.** *Verify that one particular solution to this equation is*

$$y_p(x) = -x^2 - \frac{1}{2}x \quad .$$

**b.** *What is* $y_h$ *, the general solution to the corresponding homogeneous equation?*

**c.** *What is the general solution to the above nonhomogeneous equation?*

**d.** *Find the solution to the above nonhomogeneous equation that also satisfies each of the following sets of initial conditions:*

**i.** $y(0) = 0$ *and* $y'(0) = 0$ **ii.** $y(0) = 1$ *and* $y'(0) = -3$

**20.6.** *Consider the nonhomogeneous linear differential equation*

$$y'' - 9y = 36 \quad .$$

**a.** *Verify that one particular solution to this equation is*

$$y_p(x) = -4 \quad .$$

**b.** *Find the general solution to this nonhomogeneous equation.*

**c.** *Find the solution to the above nonhomogeneous equation that also satisfies*

$$y(0) = 8 \qquad \text{and} \qquad y'(0) = 6 \quad .$$

**20.7.** *Consider the nonhomogeneous linear differential equation*

$$y'' - 3y' - 10y = -6e^{4x} \quad .$$

**a.** *Verify that one particular solution to this equation is*

$$y_p(x) = e^{4x} \quad .$$

**b.** *Find the general solution to this nonhomogeneous equation.*

**c.** *Find the solution to the above nonhomogeneous equation that also satisfies*

$$y(0) = 6 \qquad \text{and} \qquad y'(0) = 8 \quad .$$

**20.8.** *Consider the nonhomogeneous linear differential equation*

$$y'' - 3y' - 10y = 7e^{5x} \quad .$$

**a.** *Verify that one particular solution to this equation is*

$$y_p(x) = xe^{5x} \quad .$$

**b.** *Find the general solution to this nonhomogeneous equation.*

**c.** *Find the solution to the above nonhomogeneous equation that also satisfies*

$$y(0) = 12 \qquad \text{and} \qquad y'(0) = -2 \quad .$$

***20.9.*** *Consider the nonhomogeneous linear differential equation*

$$y'' + 6y' + 9y = 169\sin(2x) \quad .$$

**a.** *Verify that one particular solution to this equation is*

$$y_p(x) = 5\sin(2x) - 12\cos(2x) \quad .$$

**b.** *Find the general solution to this nonhomogeneous equation.*

**c.** *Find the solution to the above nonhomogeneous equation that also satisfies*

$$y(0) = -10 \qquad \text{and} \qquad y'(0) = 9 \quad .$$

***20.10.*** *Consider the nonhomogeneous linear differential equation*

$$x^2y'' - 4xy' + 6y = 10x + 12 \qquad \text{for} \quad x > 0 \quad .$$

**a.** *Verify that one particular solution to this equation is*

$$y_p(x) = 5x + 2 \quad .$$

**b.** *Find the general solution to this nonhomogeneous equation.*

**c.** *Find the solution to the above nonhomogeneous equation that also satisfies*

$$y(1) = 6 \qquad \text{and} \qquad y'(1) = 8 \quad .$$

***20.11.*** *Consider the nonhomogeneous linear differential equation*

$$y^{(4)} + y'' = 1 \quad .$$

**a.** *Verify that one particular solution to this equation is*

$$y_p(x) = \frac{1}{2}x^2 \quad .$$

**b.** *Find the general solution to this nonhomogeneous equation.*

**c.** *Find the solution to the above nonhomogeneous equation that also satisfies*

$$y(0) = 4 \quad , \quad y'(0) = 3 \quad , \quad y''(0) = 0 \qquad \text{and} \qquad y^{(3)}(0) = 2 \quad .$$

**20.12.** *In exercises 20.7 and 20.8 you saw that* $y_p(x) = e^{4x}$ *is a particular solution to*

$$y'' - 3y' - 10y = -6e^{4x} \quad ,$$

*and that* $y_p(x) = xe^{5x}$ *is a particular solution to*

$$y'' - 3y' - 10y = 7e^{5x} \quad .$$

*Using this and superposition, find a particular solution* $y_p$ *and a general solution* $y$ *to each of the following:*

**a.** $y'' - 3y' - 10y = e^{5x}$

**b.** $y'' - 3y' - 10y = 7e^{5x} - 6e^{4x}$

**c.** $y'' - 3y' - 10y = 35e^{5x} + 12e^{4x}$

**20.13 a.** *What should* $g(x)$ *be so that* $y(x)$ *is a solution to*

$$x^2y'' - 7xy' + 15y = g(x) \qquad \text{for} \quad x > 0$$

**i.** *when* $y(x) = x^2$ ? **ii.** *when* $y(x) = x$ ? **iii.** *when* $y(x) = 1$ ?

**b.** *Using the results just derived, find a particular solution* $y_p$ *and a general solution* $y$ *to each of the following (for* $x > 0$ *):*

**i.** $x^2y'' - 5xy' + 9y = 4x^2 + 2x + 3$

**ii.** $x^2y'' - 5xy' + 9y = x^2$

**20.14 a.** *What should* $g(x)$ *be so that* $y(x)$ *is a solution to*

$$y'' - 2y' + y = g(x)$$

**i.** *when* $y(x) = \cos(2x)$ ? **ii.** *when* $y(x) = \sin(2x)$ ?

**b.** *Using the results just derived, find a particular solution* $y_p$ *and a general solution* $y$ *to each of the following:*

**i.** $y'' - 2y' + y = \cos(2x)$

**ii.** $y'' - 2y' + y = \sin(2x)$