12

Higher-Order Linear Equations: Introduction and Basic Theory

We have just seen that some higher-order differential equations can be solved using methods for first-order equations after applying the substitution $v = {}^{dy}/_{dx}$. Unfortunately, this approach has its limitations. Moreover, as we will later see, many of those differential equations that can be so solved can also be solved much more easily using the theory and methods that will be developed in the next few chapters. This theory and methodology apply to the class of "linear" differential equations. This is a rather large class that includes a great many differential equations arising in applications. In fact, so important is this class of equations and so extensive is the theory for dealing with these equations, that we will not seriously consider higher-order nonlinear differential equations (excluding those in the previous chapter) for many, many chapters.

12.1 Basic Terminology

Recall that a first-order differential equation is said to be linear if and only it can be written as

$$\frac{dy}{dx} + py = f \tag{12.1}$$

where $p = p(x)$ and $f = f(x)$ are known functions. Observe that this is the same as saying that a first-order differential equation is linear if and only if it can be written as

$$a\frac{dy}{dx} + by = g \tag{12.2}$$

where a, b, and g are known functions of x. After all, the first equation is equation (12.2) with $a = 1$, $b = p$ and $f = g$, and any equation in the form of equation (12.2) can be converted to one looking like equation (12.1) by simply dividing through by a (so $p = {}^{b}/_{a}$ and $f = {}^{g}/_{a}$).

Higher order analogs of either equation (12.1) or equation (12.2) can be used to define when a higher-order differential equation is "linear". We will find it slightly more convenient to use analogs of equation (12.2) (which was the reason for the above observations). Second- and third-order linear equations will first be described so you can start seeing the pattern. Then the general definition will be given. For convenience (and because there are only so many letters in the alphabet), we may start denoting different functions with subscripts.

A second-order differential equation is said to be *linear* if and only if it can be written as

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = g \tag{12.3}$$

where a_0, a_1, a_2, and g are known functions of x. (In practice, generic second-order differential equations are often denoted by

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = g \quad ,$$

instead.) For example,

$$\frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} - 6x^4 y = \sqrt{x+1} \qquad \text{and} \qquad 3\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 6y = 0$$

are second-order linear differential equations, while

$$\frac{d^2y}{dx^2} + y^2 \frac{dy}{dx} = \sqrt{x+1} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

are not.

A third-order differential equation is said to be *linear* if and only if it can be written as

$$a_0 \frac{d^3y}{dx^3} + a_1 \frac{d^2y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = g$$

where a_0, a_1, a_2, a_3, and g are known functions of x. For example,

$$x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 6y = e^x \qquad \text{and} \qquad \frac{d^3y}{dx^3} - y = 0$$

are third-order linear differential equations, while

$$\frac{d^3y}{dx^3} - y^2 = 0 \qquad \text{and} \qquad \frac{d^3y}{dx^3} + y\frac{dy}{dx} = 0$$

are not.

Getting the idea?

In general, for any positive integer N, we refer to a N^{th}-order differential equation as being *linear* if and only if it can be written as

$$a_0 \frac{d^N y}{dx^N} + a_1 \frac{d^{N-1}y}{dx^{N-1}} + \cdots + a_{N-2} \frac{d^2y}{dx^2} + a_{N-1} \frac{dy}{dx} + a_N y = g \tag{12.4}$$

where a_0, a_1, $\ldots$, a_N, and g are known functions of x. For convenience, this equation will often be written using the prime notation for derivatives,

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = g \quad .$$

The function g on the right side of the above equation is often called the *forcing function* for the differential equation (because it often describes a force affecting whatever phenomenon the equation is modeling). If $g = 0$ (i.e., $g(x) = 0$ for every x in the interval of interest), then the equation is said to be *homogeneous*.[1] Conversely, if g is nonzero somewhere on the interval of interest, then we say the differential equation is *nonhomogeneous*.

As we will later see, solving a nonhomogeneous equation

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = g$$

is usually best done after first solving the homogeneous equation generated from the original equation by simply replacing g with 0,

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0 \quad .$$

This corresponding homogeneous equation is officially called either the *corresponding homogeneous equation* or the *associated homogeneous equation*, depending on the author (we will use whichever phrase we feel like at the time). Do observe that the zero function,

$$y(x) = 0 \qquad \text{for all} \quad x \quad ,$$

is always a solution to a homogeneous linear differential equation (verify this for yourself). This is called the *trivial solution* and is not a very exciting solution. Invariably, the interest is in finding the *non*trivial solutions.

The rest of this chapter will mainly focus on developing some simple but very useful theory regarding linear differential equations. Since solving a nonhomogeneous equation usually first involves solving the associated homogeneous equation, we will concentrate on homogeneous equations for now, and extend our discussions to nonhomogeneous equations later (in chapter 20).

By the way, many texts state that a second-order differential equation is linear if it can be written as

$$y'' + py' + qy = f$$

(where p, q and f are known functions of x), and state that an N^{th}-order differential equation is linear if it can be written as

$$y^{(N)} + p_1 y^{(N-1)} + \cdots + p_{N-2} y'' + p_{N-1} y' + p_N y = f \tag{12.5}$$

(where f and the p_k's are known functions of x). These equations are the higher order analogs of first-order equation (12.1) on page 259, and they are completely equivalent to the equations given earlier for higher order linear differential equations (equations (12.3) and (12.4) — just divide those equations by $a_0(x)$). There are three reasons for using the forms immediately above:

1. It saves a little space. If you count, there are $N + 1$ a_k's in equation (12.4) and only N p_k's in equation (12.5).

2. It is easier to state a few theorems. This is because the conditions normally imposed when using the form given in equation (12.4) are

   All the a_k's are continuous functions on the interval of interest, with a_0 never being 0 on that interval.

[1] You may recall the term "homogeneous" from chapter 6. If you compare what "homogeneous' meant there with what it means here, you will find absolutely no connection. The same term is being used for two completely different concepts.

Since each p_k is a_k/a_0, the equivalent conditions when using form (12.5) are

All the p_k's are continuous functions on the interval of interest.

3. A few formulas (chiefly, the formulas for the "variation of parameters" method for solving nonhomogeneous equations) are best written assuming form (12.5)

In practice, at least until we get to "variation of parameters" (chapter 23), there is little advantage to "dividing through by a_0". In fact, sometimes it just complicates computations.

12.2 Basic Useful Theory about 'Linearity'

The Operator Associated with a Linear Differential Equation

Some shorthand will simplify our discussions: Given any N^{th}-order linear differential equation

$$a_0 \frac{d^N y}{dx^N} + a_1 \frac{d^{N-1} y}{dx^{N-1}} + \cdots + a_{N-2} \frac{d^2 y}{dx^2} + a_{N-1} \frac{dy}{dx} + a_N y = g \quad ,$$

we will let $L[y]$ denote the expression on the left side, whether or not y is a solution to the differential equation. That is, for any sufficiently differentiable function y,

$$L[y] = a_0 \frac{d^N y}{dx^N} + a_1 \frac{d^{N-1} y}{dx^{N-1}} + \cdots + a_{N-2} \frac{d^2 y}{dx^2} + a_{N-1} \frac{dy}{dx} + a_N y \quad .$$

To emphasize that y is a function of x, we may also use $L[y(x)]$ instead of $L[y]$. For much of what follows, y need not be a solution to the given differential equation, but it does need to be sufficiently differentiable on the interval of interest for all the derivatives in the formula for $L[y]$ to make sense.

While we defined $L[y]$ as the left side of the above differential equation, the expression for $L[y]$ is completely independent of the equation's right side. Because of this and the fact that the choice of y is largely irrelevant to the basic definition, we will often just define "L" by stating

$$L = a_0 \frac{d^N}{dx^N} + a_1 \frac{d^{N-1}}{dx^{N-1}} + \cdots + a_{N-2} \frac{d^2}{dx^2} + a_{N-1} \frac{d}{dx} + a_N$$

where the a_k's are functions of x on the interval of interest.[2]

!►*Example 12.1:* *If our differential equation is*

$$\frac{d^2 y}{dx^2} + x^2 \frac{dy}{dx} - 6y = \sqrt{x+1} \quad ,$$

then

$$L = \frac{d^2}{dx^2} + x^2 \frac{d}{dx} - 6 \quad ,$$

[2] If using "L" is just too much shorthand for you, observe that the formulas for L can be written in summation form:

$$L[y] = \sum_{k=0}^{N} a_k \frac{d^{N-k} y}{dx^{N-k}} \qquad \text{and} \qquad L = \sum_{k=0}^{N} a_k \frac{d^{N-k}}{dx^{N-k}} \quad .$$

You can use these summation formulas instead of "L" if you wish.

and, for any twice-differentiable function $y = y(x)$ *,*

$$L[y(x)] = L[y] = \frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - 6y \quad .$$

In particular, if $y = \sin(2x)$ *, then*

$$\begin{aligned} L[y] = L\big[\sin(2x)\big] &= \frac{d^2}{dx^2}\big[\sin(2x)\big] + x^2\frac{d}{dx}\big[\sin(2x)\big] - 6\big[\sin(2x)\big] \\ &= -4\sin(2x) + x^2 \cdot 2\cos(2x) - 6\sin(2x) \\ &= 2x^2\cos(2x) - 10\sin(2x) \quad . \end{aligned}$$

Observe that L is something into which we plug a function (such as the $\sin(2x)$ in the above example) and out of which pops another function (which, in the above example, ended up being $2x^2\cos(2x) - 10\sin(2x)$). Anything that so converts one function into another is often called an *operator* (on functions), and since the formula for computing the output of $L[y]$ involves computing derivatives of y , it is standard to refer to L as a (*linear*) *differential operator*.

There are two good reasons for using this notation. First of all, it is very convenient shorthand — using L , we can write our differential equation as

$$L[y] = g$$

and the corresponding homogeneous equation as

$$L[y] = 0 \quad .$$

More importantly, it makes it easier to describe certain "linearity properties" upon which the fundamental theory of linear differential equations is based. To uncover the most basic of these properties, let us first assume (for simplicity) that L is a second-order operator

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$$

where a , b and c are known functions of x on some interval of interest $\mathcal{I}$. So, if y is any sufficiently differentiable function,

$$L[y] = ay'' + by' + cy \quad .$$

(Using the prime notation will make it a little easier to follow our derivations.)

Uncovering the most basic linearity property begins with two simple observations:

1. Let ϕ and ψ be any two sufficiently differentiable functions on the interval $\mathcal{I}$. Keeping in mind that "the derivative of a sum is the sum of the derivatives", we see that

$$\begin{aligned} L[\phi + \psi] &= a[\phi + \psi]'' + b[\phi + \psi]' + c[\phi + \psi] \\ &= a[\phi'' + \psi''] + b[\phi' + \psi'] + c[\phi + \psi] \\ &= \big\{a\phi'' + b\phi' + c\phi\big\} + \big\{a\psi'' + b\psi' + c\psi\big\} \\ &= L[\phi] + L[\psi] \quad . \end{aligned}$$

Cutting out the middle, this gives

$$L[\phi + \psi] = L[\phi] + L[\psi] \quad .$$

That is, " L of a sum of functions is the sum of L's of the individual functions".

2. Next, let y be any sufficiently differentiable function, and observe that, because "constants factor out of derivatives",

$$\begin{aligned} L[3y] &= a[3y]'' + b[3y]' + c[3y] \\ &= a3y'' + b3y' + c3y \\ &= 3\left[ay'' + by' + cy\right] = 3L[y] \quad . \end{aligned}$$

Of course, there was nothing special about the constant 3 — the above computations hold replacing 3 with any constant c. That is, if c is any constant and y is any sufficiently differentiable function on the interval, then

$$L[cy(x)] = cL[y(x)] \quad .$$

In other words, "constants factor out of L".

Now, suppose $y_1(x)$ and $y_2(x)$ are any two sufficiently differentiable functions on our interval, and c_1 and c_2 are any two constants. From the first observation (with $\phi = c_1 y_1$ and $\psi = c_2 y_2$), we know that

$$L[c_1 y_2(x) + c_2 y_2(x)] = L[c_1 y_2(x)] + L[c_2 y_2(x)] \quad .$$

Combined with the second observation (that "constants factor out"), this then yields

$$L[c_1 y_2(x) + c_2 y_2(x)] = c_1 L[y_2(x)] + c_2 L[y_2(x)] \quad . \tag{12.6}$$

(If you've had linear algebra, you will recognize that this means L is a *linear* operator. That is the real reason these differential equations and operators are said to be 'linear'.)

Equation (12.6) describes the basic "linearity property" of L. Much of the general theory used to construct solutions to linear differential equations will follow from this property. We derived it assuming

$$L[y] = ay'' + by' + cy \quad ,$$

but, if you think about it, you will realize that equation (12.6) could have been derived almost as easily had we assumed

$$L = a_0 \frac{d^N}{dx^N} + a_1 \frac{d^{N-1}}{dx^{N-1}} + \cdots + a_{N-2} \frac{d^2}{dx^2} + a_{N-1} \frac{d}{dx} + a_N \quad .$$

The only change in our derivation would have been to account for the additional terms in the operator. Moreover, there was no real need to limit ourselves to two functions and two constants in deriving equation (12.6). In our first observation, we could have easily replaced the sum of two functions $\phi + \psi$ with a sum of three functions $\phi + \psi + \chi$, obtaining

$$L[\phi + \psi + \chi] = L[\phi] + L[\psi] + L[\chi] \quad .$$

This, with the second observation, would then have led to

$$L[c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x)] = c_1 L[y_1(x)] + c_2 L[y_2(x)] + c_3 L[y_3(x)]$$

for any three sufficiently differentiable functions y_1, y_2 and y_3, and any three constants c_1, c_2 and c_3.

Continuing along these lines quickly leads to the following basic theorem on linearity for linear differential equations:

Theorem 12.1 (basic linearity property for differential operators)
Assume

$$L = a_0\frac{d^N}{dx^N} + a_1\frac{d^{N-1}}{dx^{N-1}} + \cdots + a_{N-2}\frac{d^2}{dx^2} + a_{N-1}\frac{d}{dx} + a_N$$

where the a_k*'s are known functions on some interval of interest* $\mathcal{I}$. *Let* M *be some finite positive integer, and assume*

$$\{c_1, c_2, \dots, c_M\} \qquad \text{and} \qquad \{y_1(x), y_2(x), \dots, y_M(x)\}$$

are sets, respectively, of constants and sufficiently differentiable functions on $\mathcal{I}$. *Then*

$$L[c_1y_1(x) + c_2y_2(x) + \cdots + c_My_M(x)] \\ = c_1L[y_1(x)] + c_2L[y_2(x)] + \cdots + c_ML[y_M(x)] \quad .$$

This leads to another bit of terminology which will simplify future discussions: Given a finite set of functions — y_1, y_2, … and y_M — a *linear combination* of these y_k's is any expression of the form

$$c_1y_1 + c_2y_2 + \cdots + c_My_M$$

where the c_k's are constants. To emphasize the fact that the c_k's are constants and the y_k's are functions, we may (as we did in the above theorem) write the linear combination as

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_My_M(x) \quad .$$

This also points out the fact that a linear combination of functions on some interval is, itself, a function on that interval.

The Principle of Superposition

Now suppose y_1, y_2, … and y_M are all solutions to the *homogeneous* differential equation

$$L[y] = 0 \quad .$$

That is, y_1, y_2, … and y_M are all functions satisfying

$$L[y_1] = 0 \quad , \quad L[y_2] = 0 \quad , \quad \dots \quad \text{and} \quad L[y_M] = 0 \quad .$$

Now let y be any linear combination of these y_k's,

$$y = c_1y_1 + c_2y_2 + \cdots + c_My_M \quad .$$

Applying the above theorem, we get

$$\begin{aligned} L[y] &= L[c_1y_1 + c_2y_2 + \cdots + c_My_k] \\ &= c_1L[y_1] + c_2L[y_2] + \cdots + c_ML[y_k] \\ &= c_1\cdot 0 + c_2\cdot 0 + \cdots + c_M\cdot 0 \\ &= 0 \quad . \end{aligned}$$

So y is also a solution to the homogeneous equation. This, too, is a major result and is often called the "principle of superposition".[3] Being a major result, it naturally deserves its own theorem:

Theorem 12.2 (principle of superposition)
Any linear combination of solutions to a homogeneous linear differential equation is, itself, a solution to that homogeneous linear equation.

This, combined with a few results derived in the next few chapters, will tell us that general solutions to homogeneous linear differential equations can be easily constructed as linear combinations of appropriately chosen particular solutions to those differential equations. It also means that, after finding those appropriately chosen particular solutions — y_1, y_2, ... and y_M — solving an initial-value problem is reduced to finding the constants c_1, c_2, ... and c_M such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x)$$

satisfies all the given initial values. Of course, we will still have the problem of finding those "appropriately chosen particular solutions — y_1, y_2, ..., y_M".

!►*Example 12.2:* *Consider the homogeneous second-order linear differential equation*

$$\frac{d^2y}{dx^2} + y = 0 \quad .$$

We can find at least two solutions by rewriting this as

$$\frac{d^2y}{dx^2} = -y(x) \quad ,$$

and then asking ourselves if we know of any basic functions (powers, exponentials, trigonometric functions, etc.) that satisfy this equation. It should not take long to recall that

$$y_1(x) = \cos(x) \qquad \textit{and} \qquad y_2(x) = \sin(x)$$

are two such functions:

$$\frac{d^2y_1}{dx^2} = \frac{d^2}{dx^2}[\cos(x)] = \frac{d}{dx}[-\sin(x)] = -\cos(x) = -y_1(x)$$

and

$$\frac{d^2y_2}{dx^2} = \frac{d^2}{dx^2}[\sin(x)] = \frac{d}{dx}[\cos(x)] = -\sin(x) = -y_2(x) \quad .$$

The theorem on superposition then assures us that, for any pair of constants c_1 and c_2,

$$y(x) = c_1\cos(x) + c_2\sin(x) \tag{12.7}$$

is a also a solution to the differential equation.

In particular, taking $c_1 = 4$ and $c_2 = 2$ gives us another particular solution,

$$y_3(x) = 4\cos(x) + 2\sin(x) \quad .$$

Thus, we now have three particular solutions,

$$y_1(x) = \cos(x) \quad , \quad y_2(x) = \sin(x) \quad \textit{and} \quad y_3(x) = 4\cos(x) + 2\sin(x) \quad .$$

[3] The name comes from the fact that, geometrically, the graph of a linear combination of functions can be viewed as a "superposition" of the graphs of the individual functions.

Linear Independence
The Basic Ideas

As commented above, we will be constructing general solutions in the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x)$$

where the c's are constants and

$$\{ y_1(x), y_2(x), \cdots . y_M(x) \}$$

is a set of "appropriately chosen" particular solutions. Naturally, we will want the smallest necessary sets of "appropriately chosen" solutions. This, in turn, means that none of our chosen solutions should be a linear combination of the others. After all, if, say, we have a set of three functions $\{y_1, y_2, y_3\}$ with

$$y_3(x) = 4y_1(x) + 2y_2(x)$$

(as in the last example), and we have another function y given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x)$$

for some constants c_1, c_2 and c_3, then we can simplify our expression for y by noting that

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) \\ &= c_1 y_1(x) + c_2 y_2(x) + c_3[4y_1(x) + 2y_2(x)] \\ &= [c_1 + 4c_3]y_1(x) + [c_2 + 2c_3]y_2(x) \quad . \end{aligned}$$

Since $c_1 + 4c_3$ and $c_2 + 2c_3$ are, themselves, just constants — call them a_1 and a_2 — our formula reduces for y reduces to

$$y(x) = a_1 y_1(x) + a_2 y_2(x) \quad .$$

Thus, our original formula for y did not require y_3 at all. In fact, including this redundant function gives us a formula with more constants than necessary. Not only is this a waste of ink, it will cause difficulties when we use these formulas in solving initial-value problems.

This prompts even more terminology to simplify future discussion. Suppose

$$\{ y_1(x), y_2(x), \cdots . y_M(x) \}$$

is a set of functions defined on some interval. This set is said to be *linearly independent* (over the given interval) if none of the y_k's can be written as a linear combination of any of the others (over the given interval). If this is not the case and at least one y_k in the set can be written as a linear combination of some of the others, then the set is said to be *linearly dependent* (over the given interval).

!►*Example 12.3:* *The set of functions*

$$\{ y_1(x), y_2(x), y_3(x) \} = \{ \cos(x), \sin(x), 4\cos(x) + 2\sin(x) \} \quad .$$

is linearly dependent (over any interval) since the last function is clearly a linear combination of the first two.

By the way, we should observe the almost trivial fact that, whatever functions y_1, y_2, ... and y_M may be,

$$0 = 0 \cdot y_1(x) + 0 \cdot y_2(x) + \cdots + 0 \cdot y_M(x) \quad .$$

So the zero function can always be treated as a linear combination of other functions, and, hence, cannot be one of the functions chosen for a linearly *in*dependent set.

Linear Independence for Function Pairs

Matters simplify greatly when our set is just a pair of functions

$$\{ y_1(x), y_2(x) \} \quad .$$

In this case, the statement that one of these y_k's is a linear combination of the other over some interval $\mathcal{I}$ is just the statement that either, for some constant c_2,

$$y_1(x) = c_2 y_2(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad ,$$

or else, for some constant c_1,

$$y_2(x) = c_1 y_1(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

Either way, one function is simply a constant multiple of the other over the interval of interest. (In fact, unless $c_1 = 0$ or $c_2 = 0$, then each function will clearly be a constant multiple of the other with $c_1 \cdot c_2 = 1$.) Thus, for a pair of functions, the concepts of linear independence and dependence reduce to the following:

The set $\{ y_1(x), y_2(x) \}$ is linearly *in*dependent.

$\iff$ Neither y_1 nor y_2 is a constant multiple of the other.

and

The set $\{ y_1(x), y_2(x) \}$ is linearly dependent.

$\iff$ Either y_1 or y_2 is a constant multiple of the other.

In practice, this makes it relatively easy to determine when two functions form a linearly independent set.

!►*Example 12.4:* *In example 12.3 we obtained*

$$\{ y_1(x), y_2(x) \} = \{ \cos(x), \sin(x) \} \quad .$$

as a pair of solutions the homogeneous second-order linear differential equation

$$y'' + y = 0 \quad .$$

Clearly, neither $\sin(x)$ *nor* $\cos(x)$ *is a constant multiple of the other over the real line. So this set is a linearly independent set (over the entire real line), and solution formula (12.7),*

$$y(x) = c_1 \sin(x) + c_2 \sin(x)$$

not only describes many possible solutions to our differential equation, it contains no "redundant" solutions.

Linear Independence for Larger Function Sets

If $M > 2$, then the basic approach to determining whether a set of functions

$$\{ y_1(x),\ y_2(x),\ \cdots .\ y_M(x) \}$$

is linearly dependent or independent (over some interval) requires recognizing whether one of the y_k's is a linear combination of the others. This may — or may not — be easily done. Fortunately, there is a test involving something called "the Wronskian for the set" which greatly simplifies determining the linear dependence or independence of a set of solutions to a given homogeneous differential equation. However, the definition of the Wronskian and a discussion of this test will have to wait until chapter 14, after we've further developed the theory for linear differential equations.

12.3 Fundamental Sets of Solutions and Some Suspicions

Let N and M be any two positive integers, and suppose

$$\{ y_1,\ y_2,\ \ldots,\ y_M \}$$

is a linearly independent set of M particular solutions (over some interval) to some homogeneous N^{th}-order differential equation

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0 \quad .$$

From the principle of superposition (theorem 12.2) we know that, if $\{c_1,\ c_2,\ \ldots, c_M\}$ is any set of M constants, then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x) \tag{12.8}$$

is a solution to the given homogeneous differential equation. The obvious question now is whether *every* solution to this differential equation can be so written. If so, then

1. the set $\{y_1,\ y_2,\ \ldots,\ y_M\}$ is called a *fundamental set* of solutions to the differential equation,

and, more importantly,

2. formula (12.8) is a general solution to the differential equation with the c_k's being the arbitrary constants.

!►Example 12.5: *In example 12.4, we saw that*

$$\{ \cos(x),\ \sin(x) \}$$

is a linearly independent set of two solutions to

$$y'' + y = 0 \quad .$$

If (as you may suspect) every other solution y *to this differential equation can be written as a linear combination of* $\cos(x)$ *and* $\sin(x)$ *,*

$$y(x) = c_1 \sin(x) + c_2 \cos(x) \quad ,$$

then $\{\cos(x), \sin(x)\}$ *is a fundamental set of solutions for the above differential equation, and the above expression (with* c_1 *and* c_2 *being arbitrary constants) is a general solution for this differential equation.*

At this point, though, we cannot be absolutely sure there is not another solution to the above differential equation (but see exercises 12.9 and 12.10).

At this point, we don't know for certain that fundamental sets exist (though you probably suspect they do, since we are discussing them). Worse yet, even if we know they exist, we are still left with the problem of determining when a linearly independent set of particular solutions

$$\{ y_1, y_2, \ldots, y_M \}$$

to a given homogeneous N^{th}-order differential equation

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0 \quad ,$$

is big enough so that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x)$$

describes all possible solutions to the given differential equation over the interval of interest. How can we know if there are solutions not given by this formula?

We will deal with these issues in the next few chapters. However, you should have some suspicions as to the final outcomes. After all:

- The general solution to a first-order linear differential equation contains exactly one arbitrary constant.
- The general solutions to the few second-order differential equations we saw in the previous chapter all contain exactly two arbitrary constants.
- It has already be stated that an N^{th}-order set of initial values

  $$y(x_0) \quad , \quad y'(x_0) \quad , \quad y''(x_0) \quad , \quad y'''(x_0) \quad , \quad \ldots \quad \text{and} \quad y^{(N-1)}(x_0)$$

  is especially appropriate for an N^{th}-order differential equation (in particular, it was stated in the theorems of section 11.4 starting on page 253).

All this should lead you to suspect that the general solution to an N^{th}-order differential equation should contain N arbitrary constants. In particular, you should suspect that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x)$$

really will be the general solution to some given N^{th}-order homogeneous linear differential equation whenever both of the following hold:

1. $\{y_1, y_2, \ldots, y_M\}$ is a linearly independent set of particular solutions to that equation

and

2. $M = N$.

Let us hope we can confirm this suspicion. It could prove invaluable.

12.4 "Multiplying" and "Factoring" Operators*

Occasionally, a high-order differential operator can be expressed as a "product" of lower-order (preferably first-order) operators. When we can do this, then at least some of the solutions to corresponding differential equations can be found with relative ease.

Actually, what we will be calling a "product of operators" is more closely related to the composition of two functions

$$f \circ g(x) = f(g(x))$$

than to the classical product of two functions

$$fg(x) = f(x)g(x) \quad .$$

Our terminology is standard, but, to reduce the possibility of confusion, we will initially use the term "composition product" rather than simply "product".

The Composition Product

Definition and Notation

The *(composition) product* L_2L_1 of two linear differential operators L_1 and L_2 is the differential operator given by

$$L_2L_1[\phi] = L_2\big[L_1[\phi]\big]$$

for every sufficiently differentiable function $\phi = \phi(x)$.[4]

!►*Example 12.6:* *Let*

$$L_1 = \frac{d}{dx} + x^2 \qquad \text{and} \qquad L_2 = \frac{d}{dx} + 4 \quad .$$

For any twice-differentiable function $\phi = \phi(x)$, we have

$$\begin{aligned}
L_2L_1[\phi] = L_2\left[L_1[\phi]\right] &= L_2\left[\frac{d\phi}{dx} + x^2\phi\right] \\
&= \frac{d}{dx}\left[\frac{d\phi}{dx} + x^2\phi\right] + 4\left[\frac{d\phi}{dx} + x^2\phi\right] \\
&= \frac{d^2\phi}{dx^2} + \frac{d}{dx}\left[x^2\phi\right] + 4\frac{d\phi}{dx} + 4x^2\phi \\
&= \frac{d^2\phi}{dx^2} + 2x\phi + x^2\frac{d\phi}{dx} + 4\frac{d\phi}{dx} + 4x^2\phi \\
&= \frac{d^2\phi}{dx^2} + \left[4 + x^2\right]\frac{d\phi}{dx} + \left[2x + 4x^2\right]\phi \quad .
\end{aligned}$$

---

* The material in this section, though of some interest in itself, will mainly be used later in proving theorems.

[4] The notation $L_2 \circ L_1$, instead of L_2L_1 would also be correct.

Cutting out the middle yields

$$L_2L_1[\phi] \;=\; \frac{d^2\phi}{dx^2} \;+\; \left[4 + x^2\right]\frac{d\phi}{dx} \;+\; \left[2x + 4x^2\right]\phi$$

for every sufficiently differentiable function ϕ . *Thus*

$$L_2L_1 \;=\; \frac{d^2}{dx^2} \;+\; \left[4 + x^2\right]\frac{d}{dx} \;+\; \left[2x + 4x^2\right] \quad .$$

When we have formulas for our operators L_1 and L_2 , it will often be convenient to replace the symbols " L_1 " and " L_2 " with their formulas enclosed in parentheses. We will also enclose any function ϕ being "plugged into" the operators with square brackets, " $[\phi]$ ". This will be called the *product notation*.[5]

!►Example 12.7: *Using the product notation, let us recompute* L_2L_1 *for*

$$L_1 \;=\; \frac{d}{dx} \;+\; x^2 \qquad \text{and} \qquad L_2 \;=\; \frac{d}{dx} \;+\; 4 \quad .$$

Letting $\phi = \phi(x)$ *be any twice-differentiable function,*

$$\begin{aligned}
\left(\frac{d}{dx} \;+\; 4\right)\left(\frac{d}{dx} \;+\; x^2\right)[\phi] \;&=\; \left(\frac{d}{dx} \;+\; 4\right)\left[\frac{d\phi}{dx} \;+\; x^2\phi\right] \\
&=\; \frac{d}{dx}\left[\frac{d\phi}{dx} \;+\; x^2\phi\right] \;+\; 4\left[\frac{d\phi}{dx} \;+\; x^2\phi\right] \\
&=\; \frac{d^2\phi}{dx^2} \;+\; \frac{d}{dx}\left[x^2\phi\right] \;+\; 4\frac{d\phi}{dx} \;+\; 4x^2\phi \\
&=\; \frac{d^2\phi}{dx^2} \;+\; 2x\phi \;+\; x^2\frac{d\phi}{dx} \;+\; 4\frac{d\phi}{dx} \;+\; 4x^2\phi \\
&=\; \frac{d^2\phi}{dx^2} \;+\; \left[4 + x^2\right]\frac{d\phi}{dx} \;+\; \left[2x + 4x^2\right]\phi \quad .
\end{aligned}$$

So,

$$L_2L_1 \;=\; \left(\frac{d}{dx} \;+\; 4\right)\left(\frac{d}{dx} \;+\; x^2\right) \;=\; \frac{d^2}{dx^2} \;+\; \left[4 + x^2\right]\frac{d}{dx} \;+\; \left[2x + 4x^2\right] \quad ,$$

just as derived in the previous example.

[5] Many authors do not enclose "the function being plugged in" in square brackets, and just write $L_2L_1\phi$. We are avoiding that because it does not explicitly distinguish between " ϕ as a function being plugged in" and " ϕ as an operator, itself". For the first, $L_2L_1\phi$ means the function you get from computing $L_2\left[L_1[\phi]\right]$. For the second, $L_2L_1\phi$ means the operator such that, for any sufficiently differentiable function ψ ,

$$L_2\left[L_1\left[\phi[\psi]\right]\right] \;=\; L_2\left[L_1[\phi\psi]\right] \quad ,$$

The two possible interpretations for $L_2L_1\phi$ are not the same.

Algebra of the Composite Product

The notation $L_2 L_1[\phi]$ is convenient, but it is important to remember that it is shorthand for

compute $L_1[\phi]$ *and plug the result into* L_2 .

The result of this can be quite different from

compute $L_2[\phi]$ *and plug the result into* L_1 ,

which is what $L_1 L_2[\phi]$ means. Thus, in general,

$$L_2 L_1 \neq L_1 L_2 \quad .$$

In other words, the composition product of differential operators is generally *not* commutative.

!►Example 12.8: *In the previous two examples, we saw that*

$$\left(\frac{d}{dx} + 4\right)\left(\frac{d}{dx} + x^2\right) = \frac{d^2}{dx^2} + \left[4 + x^2\right]\frac{d}{dx} + \left[2x + 4x^2\right] \quad .$$

On the other hand, switching the order of the two operators, and letting ϕ *be any sufficiently differentiable function gives*

$$\begin{aligned}\left(\frac{d}{dx} + x^2\right)\left(\frac{d}{dx} + 4\right)[\phi] &= \left(\frac{d}{dx} + x^2\right)\left[\frac{d\phi}{dx} + 4\phi\right] \\ &= \frac{d}{dx}\left[\frac{d\phi}{dx} + 4\phi\right] + x^2\left[\frac{d\phi}{dx} + 4\phi\right] \\ &= \frac{d^2\phi}{dx^2} + 4\frac{d\phi}{dx} + x^2\frac{d\phi}{dx} + 4x^2\phi \\ &= \frac{d^2\phi}{dx^2} + \left[4 + x^2\right]\frac{d\phi}{dx} + 4x^2\phi \quad .\end{aligned}$$

Thus,

$$\left(\frac{d}{dx} + x^2\right)\left(\frac{d}{dx} + 4\right) = \frac{d^2}{dx^2} + \left[4 + x^2\right]\frac{d}{dx} + 4x^2 \quad .$$

After comparing this with the first equation in this example, we clearly see that

$$\left(\frac{d}{dx} + x^2\right)\left(\frac{d}{dx} + 4\right) \neq \left(\frac{d}{dx} + 4\right)\left(\frac{d}{dx} + x^2\right) \quad .$$

?►Exercise 12.1: *Let*

$$L_1 = \frac{d}{dx} \qquad \text{and} \qquad L_2 = x \quad ,$$

and verify that

$$L_2 L_1 = x\frac{d}{dx} \qquad \text{while} \qquad L_1 L_2 = x\frac{d}{dx} + 1 \quad .$$

Later (in chapters 18 and 21) we will be dealing with special situations in which the composition product is commutative. In fact, the material we are now developing will be most useful verifying certain theorems involving those situations. In the meantime, just remember that, in general,

$$L_2 L_1 \neq L_1 L_2 \quad .$$

Here are a few other short and easily verified notes about the composition product:

1. In the above examples, the operators L_2 and L_1 were all first order differential operators. This was not necessary. We could have used, say,

$$L_2 = x^3 \frac{d^3}{dx^3} + \sin(x) \frac{d^2}{dx^2} - xe^{3x} \frac{d}{dx} + 87\sqrt{x}$$

and

$$L_1 = \frac{d^{26}}{dx^{26}} - x^3 \frac{d^3}{dx^3} \quad ,$$

though we would have certainly needed many more pages for the calculations.

2. There is no need to limit our selves to composition products of just two operators. Given any number of linear differential operators — L_1, L_2, L_3, $\ldots$ — the composition products $L_3 L_2 L_1$, $L_4 L_3 L_2 L_1$, etc. are defined to be the differential operators satisfying, for each and every sufficiently differentiable function ϕ,

$$L_3 L_2 L_1[\phi] = L_3\big[L_2\big[L_1[\phi]\big]\big] \quad ,$$

$$L_4 L_3 L_2 L_1[\phi] = L_4\big[L_3\big[L_2\big[L_1[\phi]\big]\big]\big] \quad ,$$

$$\vdots$$

Naturally, the order of the operators is still important.

3. Any composition product of linear differential operators is, itself, a linear differential operator. Moreover, the order of the product

$$L_K \cdots L_2 L_1$$

is the sum

$$(\text{the order of } L_K) + \cdots + (\text{the order of } L_2) + (\text{the order of } L_1) \quad .$$

4. Though not commutative, the composition product is associative. That is, if L_1, L_2 and L_3 are three linear differential operators, and we 'precompute' the products $L_2 L_1$ and $L_3 L_2$, and then compute

$$(L_3 L_2) L_1 \quad , \quad L_3(L_2 L_1) \quad \text{and} \quad L_3 L_2 L_1 \quad ,$$

we will discover that

$$(L_3 L_2) L_1 = L_3(L_2 L_1) = L_3 L_2 L_1 \quad .$$

5. Keep in mind that we are dealing with linear differential operators and that their products are linear differential operators. In particular, if α is some constant and ϕ is any sufficiently differentiable function, then

$$L_K \cdots L_2 L_1[\alpha\phi] = \alpha L_K \cdots L_2 L_1[\phi] \quad .$$

And, of course,

$$L_K \cdots L_2 L_1[0] = 0 \quad .$$

Factoring

Now suppose we have some linear differential operator L. If we can find other linear differential operators L_1, L_2, L_3, ..., and L_K such that

$$L = L_K \cdots L_2 L_1 \quad ,$$

then, in analogy with the classical concept of factoring, we will say that we have *factored* the operator L. The product $L_N \cdots L_2 L_1$ will be called a *factoring* of L, and we may even refer to the individual operators L_1, L_2, L_3, ... and L_N as *factors* of L. Keep in mind that, since composition multiplication is order dependent, it is not usually enough to simply specify the factors. The order must also be given.

!►Example 12.9: *In example 12.7, we saw that*

$$\frac{d^2}{dx^2} + \left[4 + x^2\right]\frac{d}{dx} + \left[2x + 4x^2\right] = \left(\frac{d}{dx} + 4\right)\left(\frac{d}{dx} + x^2\right) \quad .$$

So

$$\left(\frac{d}{dx} + 4\right)\left(\frac{d}{dx} + x^2\right)$$

is a factoring of

$$\frac{d^2}{dx^2} + \left[4 + x^2\right]\frac{d}{dx} + \left[2x + 4x^2\right]$$

with factors

$$\frac{d}{dx} + 4 \qquad \text{and} \qquad \frac{d}{dx} + x^2 \quad .$$

In addition, from example 12.8 we know

$$\frac{d^2}{dx^2} + \left[4 + x^2\right]\frac{d}{dx} + 4x^2 = \left(\frac{d}{dx} + x^2\right)\left(\frac{d}{dx} + 4\right) \quad .$$

Thus

$$\frac{d}{dx} + x^2 \qquad \text{and} \qquad \frac{d}{dx} + 4 \quad .$$

are also factors for

$$\frac{d^2}{dx^2} + \left[4 + x^2\right]\frac{d}{dx} + 4x^2 \quad ,$$

but the factoring here is

$$\left(\frac{d}{dx} + x^2\right)\left(\frac{d}{dx} + 4\right) \quad .$$

Let's make a simple observation. Assume a given linear differential operator L can be factored as $L = L_K \cdots L_2 L_1$. Assume, also, that $y_1 = y_1(x)$ is a function satisfying

$$L_1[y_1] = 0 \quad .$$

Then

$$L[y_1] = L_K \cdots L_2 L_1[y_1] = L_K \cdots L_2\big[L_1[y_1]\big] = L_K \cdots L_2[0] = 0 \quad .$$

This proves the following theorem:

Theorem 12.3
Let L be a linear differential operator with factoring $L = L_K \cdots L_2 L_1$. Then any solution to

$$L_1[y] = 0$$

is also a solution to

$$L[y] = 0 \quad .$$

Warning: On the other hand, if, say, $L = L_2 L_1$, then solutions to $L_2[y] = 0$ will usually *not* be solutions to $L[y] = 0$.

!►*Example 12.10:* *Consider*

$$\frac{d^2y}{dx^2} + \left[4 + x^2\right]\frac{dy}{dx} + 4x^2 y = 0 \quad .$$

As derived in example 12.8,

$$\frac{d^2}{dx^2} + \left[4 + x^2\right]\frac{d}{dx} + 4x^2 = \left(\frac{d}{dx} + x^2\right)\left(\frac{d}{dx} + 4\right) \quad .$$

So our differential equation can be written as

$$\left(\frac{d}{dx} + x^2\right)\left(\frac{d}{dx} + 4\right)[y] = 0 \quad .$$

That is,

$$\left(\frac{d}{dx} + x^2\right)\left[\frac{dy}{dx} + 4y\right] = 0 \quad . \tag{12.9}$$

Now consider

$$\frac{dy}{dx} + 4y = 0 \quad .$$

This is a simple first-order linear and separable differential equation, whose general solution is easily found to be $y = c_1 e^{-4x}$. In particular, e^{-4x} is a solution. According to the above theorem, e^{-4x} is also a solution to our original differential equation. Let's check to be sure:

$$\begin{aligned}
\frac{d^2}{dx^2}\left[e^{-4x}\right] + \left[4 + x^2\right]\frac{d}{dx}\left[e^{-4x}\right] + 4x^2 e^{-4x} &= \left(\frac{d}{dx} + x^2\right)\left(\frac{d}{dx} + 4\right)\left[e^{-4x}\right] \\
&= \left(\frac{d}{dx} + x^2\right)\left[\frac{d}{dx}\left[e^{-4x}\right] + 4e^{-4x}\right] \\
&= \left(\frac{d}{dx} + x^2\right)\left[-4e^{-4x} + 4e^{-4x}\right] \\
&= \left(\frac{d}{dx} + x^2\right)[0] \\
&= 0 \quad .
\end{aligned}$$

Keep in mind, though, that e^{-4x} is simply one of the possible solutions, and that there will be solutions not given by $c_1 e^{-4x}$.

Unfortunately, unless it is of an exceptionally simple type (such as considered in chapter 18), factoring a linear differential operator is a very nontrivial problem. And even with those simple types that we will be able to factor, we will find the main value of the above to be in deriving even simpler methods for finding solutions. Consequently, in practice, you should not expect to be solving many differential equations via "factoring".

Additional Exercises

12.2. *For each of the following differential equations, identify*

i. *the order of the equation,*

ii. *whether the equation is linear or not, and,*

iii. *if it is linear, whether the equation is homogeneous or not.*

a. $y'' + x^2y' - 4y = x^3$

b. $y'' + x^2y' - 4y = 0$

c. $y'' + x^2y' = 4y$

d. $y'' + x^2y' + 4y = y^3$

e. $xy' + 3y = e^{2x}$

f. $y''' + y = 0$

g. $(y+1)y'' = (y')^3$

h. $y'' = 2y' - 5y + 30e^{3x}$

i. $y^{(iv)} + 6y'' + 3y' - 83y - 25 = 0$

j. $yy''' + 6y'' + 3y' = y$

k. $y''' + 3y' = x^2y$

l. $y^{(55)} = \sin(x)$

12.3 a. *State the linear differential operator* L *corresponding to the left side of*

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \quad .$$

b. *Using this* L*, compute each of the following:*

i. $L[\sin(x)]$ **ii.** $L[e^{4x}]$ **iii.** $L[e^{-3x}]$ **iv.** $L[x^2]$

c. *Based on the answers to the last part, what is one solution to the* homogeneous *linear equation corresponding to the nonhomogeneous equation in part* ***a?***

12.4 a. *State the linear differential operator* L *corresponding to the left side of*

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 9y = 0 \quad .$$

b. *Using this* L*, compute each of the following:*

i. $L[\sin(x)]$ **ii.** $L[\sin(3x)]$ **iii.** $L[e^{2x}]$ **iv.** $L[e^{2x}\sin(x)]$

12.5 a. *State the linear differential operator* L *corresponding to the left side of*

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 6y = 0 \quad .$$

b. *Using this* L *, compute each of the following:*

i. $L[\sin(x)]$ **ii.** $L[e^{4x}]$ **iii.** $L[x^3]$

12.6 a. *State the linear differential operator* L *corresponding to the left side of*

$$\frac{d^3y}{dx^3} - \sin(x)\frac{dy}{dx} + \cos(x)\,y = x^2 + 1 \quad ,$$

b. *and then, using this* L *, compute each of the following:*

i. $L[\sin(x)]$ **ii.** $L[\cos(x)]$ **iii.** $L[x^2]$

12.7. *Several initial-value problems are given below, each involving a second-order homogeneous linear differential equation, and each with a pair of functions* $y_1(x)$ *and* $y_2(x)$ *. Verify that these two functions are particular solutions to the given differential equation, and then find a linear combination of these solutions that satisfies the given initial-value problem.*

a. *I.v. problem:* $y'' + 4y = 0$ *with* $y(0) = 2$ *and* $y'(0) = 6$.

Functions: $y_1(x) = \cos(2x)$ *and* $y_2(x) = \sin(2x)$.

b. *I.v. problem:* $y'' - 4y = 0$ *with* $y(0) = 0$ *and* $y'(0) = 12$.

Functions: $y_1(x) = e^{2x}$ *and* $y_2(x) = e^{-2x}$.

c. *I.v. problem:* $y'' + y' - 6y = 0$ *with* $y(0) = 8$ *and* $y'(0) = -9$.

Functions: $y_1(x) = e^{2x}$ *and* $y_2(x) = e^{-3x}$.

d. *I.v. problem:* $y'' - 4y' + 4y = 0$ *with* $y(0) = 1$ *and* $y'(0) = 6$.

Functions: $y_1(x) = e^{2x}$ *and* $y_2(x) = xe^{2x}$.

e. *I.v. problem:* $4x^2y'' + 4xy' - y = 0$ *with* $y(1) = 8$ *and* $y'(1) = 1$.

Functions: $y_1(x) = \sqrt{x}$ *and* $y_2(x) = \frac{1}{\sqrt{x}}$.

f. *I.v. problem:* $x^2y'' - xy' + y = 0$ *with* $y(1) = 5$ *and* $y'(1) = 3$.

Functions: $y_1(x) = x$ *and* $y_2(x) = x\ln|x|$.

g. *I.v. problem:* $xy'' - y' + 4x^3y = 0$
with $y(\sqrt{\pi}) = 3$ *and* $y'(\sqrt{\pi}) = 4$.

Functions: $y_1(x) = \cos\left(x^2\right)$ *and* $y_2(x) = \sin\left(x^2\right)$.

h. *I.v. problem:* $(x+1)^2y'' - 2(x+1)y' + 2y = 0$
with $y(0) = 0$ *and* $y'(0) = 4$.

Functions: $y_1(x) = x^2 - 1$ *and* $y_2(x) = x + 1$.

12.8. *Some third- and fourth-order initial-value problems are given below, each involving a homogeneous linear differential equation, and each with a set of three or four functions* $y_1(x)$ *,* $y_2(x)$ *, ... Verify that these functions are particular solutions to the given differential equation, and then find a linear combination of these solutions that satisfies the given initial-value problem.*

a. *I.v. problem:* $y''' + 4y' = 0$

with $y(0) = 3$, $y'(0) = 8$ and $y''(0) = 4$.

Functions: $y_1(x) = 1$, $y_2(x) = \cos(2x)$ and $y_3(x) = \sin(2x)$.

b. *I.v. problem:* $y''' + 4y' = 0$

with $y(0) = 3$, $y'(0) = 8$ and $y''(0) = 4$.

Functions: $y_1(x) = 1$, $y_2(x) = \sin^2(x)$ and $y_3(x) = \sin(x)\cos(x)$.

c. *I.v. problem:* $y^{(4)} - y = 0$

with $y(0) = 0$, $y'(0) = 4$, $y'''(0) = 0$ and $y''(0) = 0$.

Functions: $y_1(x) = \cos(x)$, $y_2(x) = \sin(x)$, $y_3(x) = \cosh(x)$ and $y_4(x) = \sinh(x)$.

12.9. *In chapter 11, it was shown that every solution to*

$$y'' + y = 0$$

can be written as

$$y(x) = a\sin(x + b)$$

using suitable constants a *and* b *. Now, using a trigonometric identity, show that, for every pair of constants* a *and* b *, there is a corresponding pair* c_1 *and* c_2 *such that*

$$a\sin(x + b) = c_1\sin(x) + c_2\cos(x) \quad .$$

What does this say about

$$y(x) = c_1\sin(x) + c_2\cos(x)$$

being a general solution to

$$\frac{d^2y}{dx^2} + y = 0 \quad ?$$

12.10. *For the following, assume* $Y(x)$ *is a particular function on* $(-\infty, \infty)$ *satisfying*

$$Y'' + Y = 0 \quad ,$$

and let $A = Y(0)$ *and* $B = Y'(0)$ *.*

a. *Verify that*

$$y(x) = A\cos(x) + B\sin(x)$$

is a solution (on $(-\infty, \infty)$ *) to the initial-value problem*

$$y'' + y = 0 \quad \text{with} \quad y(0) = A \quad \text{and} \quad y'(0) = B \quad .$$

b. *Verify that*

i. *theorem 11.2 on page 253 applies to the initial-value problem just above, and*

ii. *it assures us that*

$$Y(x) = A\cos(x) + B\sin(x) \qquad \text{for} \quad -\infty < x < \infty \quad .$$

c. *What does all this tell us about* $\{\cos(x), \sin(x)\}$ *being a fundamental set of solutions for*

$$y'' + y = 0 \quad ?$$

12.11. *Several choices for linear differential operators* L_1 *and* L_2 *are given below. For each choice, compute* L_2L_1 *and* L_1L_2 .

a. $L_1 = \frac{d}{dx} + x$ and $L_2 = \frac{d}{dx} - x$

b. $L_1 = \frac{d}{dx} + x^2$ and $L_2 = \frac{d}{dx} + x^3$

c. $L_1 = x\frac{d}{dx} + 3$ and $L_2 = \frac{d}{dx} + 2x$

d. $L_1 = \frac{d^2}{dx^2}$ and $L_2 = x$

e. $L_1 = \frac{d^2}{dx^2}$ and $L_2 = x^3$

f. $L_1 = \frac{d^2}{dx^2}$ and $L_2 = \sin(x)$

12.12. *Compute the following composition products:*

a. $\left(\frac{d}{dx} + 2\right)\left(\frac{d}{dx} + 3\right)$

b. $\left(x\frac{d}{dx} + 2\right)\left(x\frac{d}{dx} + 3\right)$

c. $\left(x\frac{d}{dx} + 4\right)\left(\frac{d}{dx} + \frac{1}{x}\right)$

d. $\left(\frac{d}{dx} + 4x\right)\left(\frac{d}{dx} + \frac{1}{x}\right)$

e. $\left(\frac{d}{dx} + \frac{1}{x}\right)\left(\frac{d}{dx} + 4x\right)$

f. $\left(\frac{d}{dx} + 5x^2\right)^2$

g. $\left(\frac{d}{dx} + x^2\right)\left(\frac{d^2}{dx^2} + \frac{d}{dx}\right)$

h. $\left(\frac{d^2}{dx^2} + \frac{d}{dx}\right)\left(\frac{d}{dx} + x^2\right)$

12.13. *Verify that*

$$\frac{d^2}{dx^2} + [\sin(x) - 3]\frac{d}{dx} - 3\sin(x) = \left(\frac{d}{dx} + \sin(x)\right)\left(\frac{d}{dx} - 3\right) \quad ,$$

and, using this factorization, find one solution to

$$\frac{d^2y}{dx^2} + [\sin(x) - 3]\frac{dy}{dx} - 3\sin(x)y = 0 \quad .$$

12.14. *Verify that*

$$\frac{d^2}{dx^2} + x\frac{d}{dx} + \left[2 - 2x^2\right] = \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + 2x\right) \quad ,$$

and, using this factorization, find one solution to

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left[2 - 2x^2\right]y = 0 \quad .$$

12.15. *Verify that*

$$x^2 \frac{d^2}{dx^2} - 7x \frac{d}{dx} + 16 = \left(x \frac{d}{dx} - 4 \right)^2 \quad ,$$

and, using this factorization, find one solution to

$$x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} + 16y = 0 \quad .$$