15

Homogeneous Linear Equations — Verifying the Big Theorems

As promised, here we rigorously verify the claims made in the previous chapter. In a sense, there are two parts to this chapter. The first is mainly concerned with proving the claims when the differential equation in question is second order, and it occupies the first two sections. The arguments in these sections are fairly elementary, though, perhaps, a bit lengthy. The rest of the chapter deals with differential equations of arbitrary order, and uses more advanced ideas from linear algebra.

If you've had an introductory course in linear algebra, just skip ahead to section 15.3 starting on page 323. After all, the set of differential equations of arbitrary order includes the second-order equations.

If you've not had an introductory course in linear algebra, then you may have trouble following some of the discussion in section 15.3. Concentrate, instead, on the development for second-order equations given in sections 15.1 and 15.2. You may even want to try to extend the arguments given in those sections to deal with higher-order differential equations. It is "do-able", but will probably take a good deal more space and work than we will spend in section 15.3 using the more advanced notions from linear algebra.

And if you don't care about 'why' the results in the previous chapter are true, and are blindly willing to accept the claims made there, then you can skip this chapter entirely.

15.1 First-Order Equations

While our main interest is with higher-order homogeneous differential equations, it is worth spending a little time looking at the general solutions to the corresponding first-order equations. After all, via reduction of order, we can reduce the solving of second-order linear equations to that of solving first-order linear equations. Naturally, we will confirm that our general suspicions hold at least for first-order equations. More importantly, though, we will discover a property of these solutions that, perhaps surprisingly, will play a major role is discussing linear independence for sets of solutions to higher-order differential equations.

With $N = 1$ the generic equation describing any N^{th}-order homogeneous linear differential equation reduces to

$$A\frac{dy}{dx} + By = 0$$

where A and B are functions of x on some open interval of interest $\mathcal{I}$ (using A and B instead of a_0 and a_1 will prevent confusion later). We will assume A and B are continuous functions on $\mathcal{I}$, and that A is never zero on that interval. Since the order is one, we suspect that the general solution (on $\mathcal{I}$) is given by

$$y(x) = c_1 y_1(x)$$

where y_1 is any one particular solution and c_1 is an arbitrary constant. This, in turn, corresponds to a fundamental set of solutions — a linearly independent set of particular solutions whose linear combinations generate all other solutions — being just the singleton set $\{y_1\}$.

Though this is a linear differential equation, it is also a relatively simple separable first-order differential equation, and easily solved as such. Algebraically solving for the derivative, we get

$$\frac{dy}{dx} = -\frac{B}{A}y \quad .$$

Obviously, the only constant solution is the trivial solution, $y = 0$. To find the other solutions, we will need to compute the indefinite integral of B/A. That indefinite integral implicitly contains an arbitrary constant. To make that arbitrary constant explicit, choose any x_0 in $\mathcal{I}$ and define the function β by

$$\beta(x) = \int_{x_0}^{x} \frac{B(s)}{A(s)}\,ds \quad .$$

Then

$$-\int \frac{B(x)}{A(x)}\,dx = -\beta(x) + c_0$$

where c_0 is an arbitrary constant. Observe that the conditions assumed about the functions A and B ensure that B/A is continuous on the interval $\mathcal{I}$ (see why we insist on A never being zero?). Consequently, $\beta(x)$, being a definite integral of a continuous function, is also a continuous function on $\mathcal{I}$.

Now, to finish solving this differential equation:

$$\frac{dy}{dx} = -\frac{B}{A}y$$

$$\hookrightarrow \qquad \frac{1}{y}\frac{dy}{dx} = -\frac{B}{A}$$

$$\hookrightarrow \qquad \int \frac{1}{y}\frac{dy}{dx}\,dx = -\int \frac{B(x)}{A(x)}\,dx$$

$$\hookrightarrow \qquad \ln|y| = -\beta(x) + c_0$$

$$\hookrightarrow \qquad y(x) = \pm e^{-\beta(x)+c_0}$$

$$\hookrightarrow \qquad y(x) = \pm e^{c_0}e^{-\beta(x)} \quad .$$

So the general solution is

$$y(x) = ce^{-\beta(x)}$$

where c is an arbitrary constant (this accounts for the constant solution as well).

There are two observations to make here:

1. If y_1 is any nontrivial solution to the differential equation, then the above formula tells us that there must be a nonzero constant κ such that

$$y_1(x) = \kappa e^{\beta(x)} \quad .$$

Consequently, the above general solution then can be written as

$$y(x) = \frac{c}{\kappa}\kappa e^{\beta(x)} = c_1 y_1(x)$$

where $c_1 = {}^{c}/_{\kappa}$ is an arbitrary constant. This pretty well confirms our suspicions regarding the general solutions to first-order homogeneous differential equations.

2. Since β is a continuous function on $\mathcal{I}$, $\beta(x)$ is a finite real number for each x in the interval $\mathcal{I}$. This, in turn, means that

$$e^{\beta(x)} > 0 \qquad \text{for each } x \text{ in } \mathcal{I} \quad .$$

Thus,

$$y(x) = ce^{\beta(x)}$$

can never be zero on this interval unless $c = 0$, in which case $y(x) = 0$ for every x in the interval.

For future reference, let us summarize our findings:

Lemma 15.1

Assume A and B are continuous functions on an open interval $\mathcal{I}$ with A never being zero on this interval. Then nontrivial solutions to

$$Ay' + By = 0$$

on $\mathcal{I}$ exist, and the general solution is given by

$$y(x) = c_1 y_1(x)$$

where c_1 is an arbitrary constant and y_1 is any particular nontrivial solution. Moreover,

1. $y_1(x) = \kappa e^{-\beta(x)}$ *for some nonzero constant κ and a function β satisfying*

$$\frac{d\beta}{dx} = \frac{B}{A} \quad .$$

2. *Any single solution is either nonzero everywhere on the interval $\mathcal{I}$ or is zero everywhere on the interval $\mathcal{I}$.*

15.2 The Second-Order Case

The Equation and Basic Assumptions

Now, let us discuss the possible solutions to a second-order homogeneous linear differential equation

$$ay'' + by' + cy = 0 \quad . \tag{15.1}$$

We will assume the coefficients a, b and c are continuous functions over some open interval of interest $\mathcal{I}$ with a never being zero on that interval. Our goal is to verify, as completely as possible, the big theorem on second-order homogeneous equations (theorem 14.1 on page 302), which stated that a general solution exists and can be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where $\{y_1, y_2\}$ is any linearly independent pair of particular solutions, and c_1 and c_2 are arbitrary constants. Along the way, we will also verify the second-order version of the big theorem on Wronskians, theorem 14.3 on page 305.

Basic Existence and Uniqueness

One step towards verifying theorem 14.1 is to prove the following theorem.

Theorem 15.2 (existence and uniqueness of solutions)

Let $\mathcal{I}$ be an open interval, and assume a, b and c are continuous functions on $\mathcal{I}$ with a never being zero on this interval. Then, for any point x_0 in $\mathcal{I}$ and any pair of values A and B, there is exactly one solution y (over $\mathcal{I}$) to the initial-value problem

$$ay'' + by' + cy = 0 \quad \textit{with} \quad y(x_0) = A \quad \textit{and} \quad y'(x_0) = B \quad .$$

PROOF: Rewriting the differential equation in second-derivative form,

$$y'' = -\frac{c}{a}y - \frac{b}{a}y' \quad ,$$

we see that our initial-value problem can be rewritten as

$$y'' = F(x, y, y') \quad \text{with} \quad y(x_0) = A \quad \text{and} \quad y'(x_0) = B \quad ,$$

where

$$F(x, y, z) = -\frac{c(x)}{a(x)}y - \frac{b(x)}{a(x)}z \quad .$$

Observe that

$$\frac{\partial F}{\partial y} = -\frac{c(x)}{a(x)} \quad \text{and} \quad \frac{\partial F}{\partial z} = -\frac{b(x)}{a(x)} \quad .$$

Also observe that, because a, b and c are continuous functions with a never being zero on $\mathcal{I}$, the function $F(x, y, z)$ and the above partial derivatives are all continuous functions on $\mathcal{I}$. Moreover, the above partial derivatives depend only on x, not on y or z. Because of this, theorem 11.2 on page 253 applies and tells us that our initial-value problem has exactly one solution valid on all of $\mathcal{I}$. ∎

This theorem assures us that any reasonable second-order initial-value problem has exactly one solution. Let us note two fairly immediate consequences that we will find useful.

Lemma 15.3 (solution to the null initial-value problem)
Let $\mathcal{I}$ be an open interval containing a point x_0, and assume a, b and c are continuous functions on $\mathcal{I}$ with a never being zero on this interval. Then the only solution to

$$ay'' + by' + cy = 0 \qquad \text{with} \quad y(x_0) = 0 \quad \text{and} \quad y'(x_0) = 0$$

is

$$y(x) = 0 \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

Lemma 15.4
Let y_1 and y_2 be two solutions over an open interval $\mathcal{I}$ to

$$ay'' + by' + cy = 0$$

where a, b and c are continuous functions on $\mathcal{I}$ with a never being zero on this interval. If, for some point x_0 in $\mathcal{I}$,

$$y_1(x_0) = 0 \qquad \text{and} \qquad y_2(x_0) = 0$$

then either one of these functions is zero over the entire interval, or there is a nonzero constant κ such that

$$y_2(x) = \kappa y_1(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

PROOF (lemma 15.3): Clearly, the constant function $y = 0$ is a solution to the given initial-value problem. Theorem 15.2 then tells us that it is the only solution. ∎

PROOF (lemma 15.4): First, note that, if $y_1{}'(x_0) = 0$, then y_1 would be a solution to

$$ay'' + by' + cy = 0 \qquad \text{with} \quad y(x_0) = 0 \quad \text{and} \quad y'(x_0) = 0 \quad ,$$

which, as we just saw, means that $y_1(x) = 0$ for every x in $\mathcal{I}$.

Likewise, if $y_2{}'(x_0) = 0$, then we must have that $y_2(x) = 0$ for every x in $\mathcal{I}$.

Thus, if neither $y_1(x)$ nor $y_2(x)$ is zero for every x in $\mathcal{I}$, then

$$y_1{}'(x_0) \neq 0 \qquad \text{and} \qquad y_2{}'(x_0) \neq 0 \quad ,$$

and

$$\kappa = \frac{y_2{}'(x_0)}{y_1{}'(x_0)}$$

is a finite, nonzero number. Now consider the function

$$y_3(x) = y_2(x) - \kappa y_1(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

Being a linear combination of solutions to our homogeneous differential equation, y_3 is also a solution to our differential equation. Also,

$$y_3(x_0) = y_2(x_0) - \kappa y_1(x_0) = 0 - \kappa \cdot 0 = 0$$

and

$$y_3{}'(x_0) = y_2{}'(x_0) - \kappa y_1{}'(x_0) = y_2{}'(x_0) - \frac{y_2{}'(x_0)}{y_1{}'(x_0)} y_1{}'(x_0) = 0 \quad .$$

So $y = y_3$ satisfies

$$ay'' + by' + cy = 0 \qquad \text{with} \quad y(x_0) = 0 \quad \text{and} \quad y'(x_0) = 0 \quad .$$

Again, as we just saw, this means $y_3 = 0$ on $\mathcal{I}$. And since $y_2 - \kappa y_1 = y_3 = 0$ on $\mathcal{I}$, we must have

$$y_2(x) = \kappa y_1(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

■

What Wronskians Tell Us

Much of the analysis leading to our goal (theorem 14.1) is based on an examination of properties of Wronskians of pairs of solutions.

Arbitrary Pairs of Functions

Recall that the *Wronskian* of any pair of functions y_1 and y_2 is the function

$$W = W[y_1, y_2] = y_1 y_2{}' - y_1{}' y_2 \quad .$$

As noted in the previous chapter, this formula arises naturally in solving second-order initial-value problems. To see this more clearly, let's look closely at the problem of finding constants c_1 and c_2 such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

satisfies

$$y(x_0) = A \qquad \text{and} \qquad y'(x_0) = B$$

for any constant values A and B , and any given point x_0 in our interval of interest. We start by replacing y in the last pair of equations with its formula $c_1 y_1 + c_2 y_2$, giving us the system

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= A \\ c_1 y_1{}'(x_0) + c_2 y_2{}'(x_0) &= B \end{aligned}$$

to be solved for c_1 and c_2 . But this is easy. Start by multiplying each equation by $y_2{}'(x_0)$ or $y_2(x_0)$, as appropriate:

$$\begin{matrix} \big[c_1 y_1(x_0) + c_2 y_2(x_0) = A\big] y_2{}'(x_0) \\ \big[c_1 y_1{}'(x_0) + c_2 y_2{}'(x_0) = B\big] y_2(x_0) \end{matrix} \quad \Longrightarrow \quad \begin{matrix} c_1 y_1(x_0) y_2{}'(x_0) + c_2 y_2(x_0) y_2{}'(x_0) = A y_2{}'(x_0) \\ c_1 y_1{}'(x_0) y_2(x_0) + c_2 y_2{}'(x_0) y_2(x_0) = B y_2(x_0) \end{matrix}$$

Subtracting the second equation from the first (and looking carefully at the results) yields

$$c_1\big[\underbrace{y_1(x_0) y_2{}'(x_0) - y_1{}'(x_0) y_2(x_0)}_{W(x_0)}\big] + c_2\big[\underbrace{y_2(x_0) y_2{}'(x_0) - y_2{}'(x_0) y_2(x_0)}_{0}\big]$$

$$= A y_2{}'(x_0) - B y_2(x_0) \quad .$$

That is,

$$c_1 W(x_0) = A y_2{}'(x_0) - B y_2(x_0) \quad . \tag{15.2a}$$

Similar computations yield

$$c_2 W(x_0) = B y_1(x_0)) - A y_1{}'(x_0) \quad . \tag{15.2b}$$

Thus, if $W(x_0) \neq 0$, then there is exactly one possible value for c_1 and one possible value for c_2, namely,

$$c_1 = \frac{A y_2{}'(x_0) - B y_2(x_0)}{W(x_0)} \qquad \text{and} \qquad c_2 = \frac{B y_1(x_0) - A y_1{}'(x_0)}{W(x_0)} \quad .$$

However, if $W(x_0) = 0$, then system (15.2) reduces to

$$0 = A y_2{}'(x_0) - B y_2(x_0) \qquad \text{and} \qquad 0 = B y_1(x_0) - A y_1{}'(x_0)$$

which cannot be solved for c_1 and c_2. If the right sides of these last two equations just happen to be 0, then the values of c_1 and c_2 are irrelevant, any values work. And if either right-hand side is nonzero, then no values for c_1 and c_2 will work.

The fact that the solvability of the above initial-value problem depends entirely on whether $W(x_0)$ is zero or not is an important fact that we will soon expand upon. So let's enshrine this fact in a lemma.

Lemma 15.5 (Wronskians and initial values)
$\{y_1, y_2\}$ is a pair of differentiable functions on some interval, and let W be the corresponding Wronskian. Let x_0 be a point in the interval; let A and B be any two fixed values, and consider the problem of finding a pair of constants $\{c_1, c_2\}$ such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

satisfies both

$$y(x_0) = A \qquad \text{and} \qquad y'(x_0) = B \quad .$$

Then this problem has exactly one solution (i.e., exactly choice for c_1 and exactly one choice for c_2) if and only if $W(x_0) \neq 0$.

The role played by the Wronskian in determining whether a system such as

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= 4 \\ c_1 y_1{}'(x_0) + c_2 y_2{}'(x_0) &= 7 \end{aligned}$$

can be solved for c_1 and c_2 should suggest that the Wronskian can play a role in determining whether the function pair $\{y_1, y_2\}$ is linearly dependent over a given interval $\mathcal{I}$. To see this better, observe what happens to W when $\{y_1, y_2\}$ is linearly dependent over some interval. Then, over this interval, either $y_1 = 0$ or $y_2 = \kappa y_1$ for some constant κ. If $y_1 = 0$ on the interval, then

$$W = y_1 y_2{}' - y_1{}' y_2 = 0 \cdot y_2{}' - 0 \cdot y_2 = 0 \quad ,$$

and if $y_2 = \kappa y_1$, then

$$\begin{aligned} W &= y_1 y_2{}' - y_1{}' y_2 \\ &= y_1 [\kappa y_1]' - y_1{}' \kappa y_1 \\ &= y_1 [\kappa y_1{}'] - y_1{}' \kappa y_1 \\ &= \kappa y_1 y_1{}' - \kappa y_1 y_1{}'(x) = 0 \quad . \end{aligned}$$

So

$$\{y_1,\ y_2\} \text{ is linearly dependent over an interval } \mathcal{I} \quad \Longrightarrow \quad W = 0 \text{ everywhere on } \mathcal{I} .$$

Conversely then,

$$\begin{aligned} &W \neq 0 \text{ somewhere on an interval } \mathcal{I} \\ &\qquad\qquad \Longrightarrow \quad \{y_1,\ y_2\} \text{ is linearly independent over } \mathcal{I} . \end{aligned} \tag{15.3}$$

On the other hand, suppose $W = 0$ over some interval $\mathcal{I}$; that is,

$$y_1 y_2{}' - y_1{}' y_2 = 0 \quad \text{on } \mathcal{I} \quad .$$

Observe that this is equivalent to y_2 being a solution on $\mathcal{I}$ to

$$Ay' + By = 0 \qquad \text{where} \quad A = y_1 \quad \text{and} \quad B = -y_1{}' \quad . \tag{15.4}$$

Also observe that

$$Ay_1{}' + By_1 = y_1 y_1{}' - y_1{}' y_1 = 0 \quad .$$

So y_1 and y_2 are both solutions to first-order differential equation (15.4). This is the same type of equation considered in lemma 15.1. Applying that lemma, we see that, if A is never zero on $\mathcal{I}$ (i.e., y_1 is never zero on $\mathcal{I}$), then

$$y_2(x) = c_1 y_1(x)$$

for some constant c_1. Hence, $\{y_1,\ y_2\}$ are linearly dependent on $\mathcal{I}$ (provided y_1 is never zero on $\mathcal{I}$).

Clearly, we could have switched the roles of y_1 and y_2 in the last paragraph. That gives us

$$\begin{aligned} &W = 0 \text{ on an interval where either } y_1 \text{ is never } 0 \text{ or } y_2 \text{ is never } 0 \\ &\qquad\qquad \Longrightarrow \quad \{y_1,\ y_2\} \text{ is linearly dependent over that interval.} \end{aligned} \tag{15.5}$$

In the above computations, no reference was made to y_1 and y_2 being solutions to a differential equation. Those observations hold in general for any pair of differentiable functions.

Just for easy reference, let us summarize the above results in a lemma.

Lemma 15.6

Assuming W is the Wronskian for two functions y_1 and y_2 over some interval $\mathcal{I}_0$:

1. *If $W \neq 0$ somewhere on $\mathcal{I}_0$, then $\{y_1,\ y_2\}$ is linearly independent over $\mathcal{I}_0$.*
2. *If $W = 0$ on $\mathcal{I}_0$ and either y_1 is never 0 or y_2 is never 0 on $\mathcal{I}_0$, then $\{y_1,\ y_2\}$ is linearly dependent over $\mathcal{I}_0$.*

(To see what can happen if $W = 0$ on an interval and y_2 or y_2 vanishes at a point on that interval, see exercise 15.1 on page 334.)

Pairs of Solutions

If y_1 and y_2 are solutions to a second-order differential equation, then even more can be derived if we look at the derivative of W,

$$W' = \left[y_1 y_2{}' - y_1{}' y_2\right]' = \left[y_1{}' y_2{}' + y_1 y_2{}''\right] - \left[y_1{}'' y_2 + y_1{}' y_2{}'\right]$$

The $y_1{}' y_2{}'$ terms cancel out leaving

$$W' = y_1 y_2{}'' - y_1{}'' y_2 \quad .$$

Now, suppose y_1 and y_2 each satisfies

$$ay'' + by' + cy = 0$$

where, as usual, a, b and c are continuous functions on some interval $\mathcal{I}$, with a never being zero on $\mathcal{I}$. Solving for the second derivative yields

$$y'' = -\frac{b}{a}y' - \frac{c}{a}y$$

for both $y = y_1$ and $y = y_2$. Combining this with the last formula for W', and then recalling the original formula for W, we get

$$\begin{aligned} W' &= y_1 y_2{}'' - y_1{}'' y_2 \\ &= y_1\left[-\frac{b}{a}y_2{}' - \frac{c}{a}y_2\right] - \left[-\frac{b}{a}y_1{}' - \frac{c}{a}y_1\right]y_2 \\ &= -\frac{b}{a}\big[\underbrace{y_1 y_2{}' - y_1{}' y_2}_{W}\big] - \frac{c}{a}\big[\underbrace{y_1 y_2 - y_1 y_2}_{0}\big] = -\frac{b}{a}W \quad . \end{aligned}$$

Thus, on the entire interval of interest, W satisfies the first-order differential equation

$$W' = -\frac{b}{a}W \quad ,$$

which we will rewrite as

$$aW' + bW = 0$$

so that lemma 15.1 can again be invoked. Take a look back at that lemma. One thing it says about solutions to this first-order equation is that

> *any single solution is either nonzero everywhere on the interval $\mathcal{I}$ or is zero everywhere on the interval $\mathcal{I}$.*

So, there are exactly two possibilities for $W = y_1 y_2{}' - y_1{}' y_2$:

1. Either $W \neq 0$ everywhere on the interval $\mathcal{I}$.
2. Or $W = 0$ everywhere on the interval $\mathcal{I}$.

The one possibility not available for W is that it be zero at some points and nonzero at others. Thus, to determine whether $W(x)$ is zero or nonzero everywhere in the interval, it suffices to check the value of W at any one convenient point. If it is zero there, it is zero everywhere. If it is nonzero there, it is nonzero everywhere.

Combining these observations with some of the lemmas verified earlier yields our test for linear independence, we have:

Theorem 15.7 (second-order test for linear independence)
Assume y_1 and y_2 are two solutions on an interval $\mathcal{I}$ to

$$ay'' + by' + cy = 0$$

where the coefficients a, b and c are continuous functions over $\mathcal{I}$, and a is never zero on $\mathcal{I}$. Let W be the Wronskian for $\{y_1, y_2\}$ and let x_0 be any conveniently chosen point in $\mathcal{I}$. Then:

1. *If $W(x_0) \neq 0$, then $W(x) \neq 0$ for every x in $\mathcal{I}$, and $\{y_1, y_2\}$ is a linearly independent pair of solutions on $\mathcal{I}$.*
2. *If $W(x_0) = 0$, then $W(x) = 0$ for every x in $\mathcal{I}$, and $\{y_1, y_2\}$ is not a linearly independent pair of solutions on $\mathcal{I}$.*

PROOF: First suppose $W(x_0) \neq 0$. As recently noted, this means $W(x) \neq 0$ for every x in $\mathcal{I}$, and lemma 15.6 then tells us that $\{y_1, y_2\}$ is a linearly independent pair of solutions on $\mathcal{I}$ and every subinterval. This verifies the first claim in this theorem.

Now assume $W(x_0) = 0$. From our discussions above, we know this means $W(x) = 0$ for every x in $\mathcal{I}$. If y_1 is never zero on $\mathcal{I}$, then lemma 15.6 tells us that $\{y_1, y_2\}$ is not linearly independent on $\mathcal{I}$ or on any subinterval of $\mathcal{I}$. If, however, y_1 is zero at a point x_1 of $\mathcal{I}$, then we have

$$\begin{aligned} 0 = W(x_1) &= y_1(x_1){y_2}'(x_1) - {y_1}'(x_1)y_2(x_1) \\ &= 0 \cdot {y_2}'(x_1) - {y_1}'(x_1)y_2(x_1) = -{y_1}'(x_1)y_2(x_1) \end{aligned}$$

Thus, either

$${y_1}'(x_1) = 0 \qquad \text{or} \qquad y_2(x_1) = 0 \quad .$$

Now, if ${y_1}'(x_1) = 0$, then $y = y_1$ satisfies

$$ay'' + by' + cy = 0 \qquad \text{with} \quad y(x_1) = 0 \quad \text{and} \quad y'(x_1) = 0 \quad .$$

From lemma 15.3, we then know $y_1 = 0$ on all of $\mathcal{I}$, and, thus, $\{y_1, y_2\}$ is not linearly independent on any subinterval of $\mathcal{I}$.

On the other hand, if $y_2(x_1) = 0$, then y_1 and y_2 satisfy the conditions in lemma 15.4, and that lemma tells us that either one of these two solutions is the zero solution, or that there is a constant κ such that

$$y_2(x) = \kappa y_1(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

Either way, $\{y_1, y_2\}$ is not linearly independent on any subinterval of $\mathcal{I}$, finishing the proof of the second claim. ∎

The Big Theorem on Second-Order Homogeneous Linear Differential Equations

Finally, we have all the tools needed to confirm the validity of the big theorem on general solutions to second-order homogeneous linear differential equations (theorem 14.1 on page 302). Rather than insist that you go back an look at it, I'll reproduce it here.

Theorem 15.8 (same as theorem 14.1)

Let $\mathcal{I}$ be some open interval, and suppose we have a second-order homogeneous linear differential equation

$$ay'' + by' + cy = 0$$

where, on $\mathcal{I}$, the functions a, b and c are continuous, and a is never zero. Then the following statements all hold:

1. *Fundamental sets of solutions for this differential equation (over $\mathcal{I}$) exist.*
2. *Every fundamental solution set consists of a pair of solutions.*
3. *If $\{y_1, y_2\}$ is any linearly independent pair of particular solutions over $\mathcal{I}$, then:*
   (a) *$\{y_1, y_2\}$ is a fundamental set of solutions.*
   (b) *A general solution to the differential equation is given by*
   $$y(x) = c_1 y_1(x) + c_2 y_2(x)$$
   where c_1 and c_2 are arbitrary constants.
   (c) *Given any point x_0 in $\mathcal{I}$ and any two fixed values A and B, there is exactly one ordered pair of constants $\{c_1, c_2\}$ such that*
   $$y(x) = c_1 y_1(x) + c_2 y_2(x)$$
   also satisfies the initial conditions
   $$y(x_0) = A \quad \text{and} \quad y'(x_0) = B \quad .$$

PROOF: Let us start with the third claim, and assume $\{y_1, y_2\}$ is any linearly independent pair of particular solutions over $\mathcal{I}$ to the above differential equation.

From theorem 15.7 on page 320 we know the Wronskian $W[y_1, y_2]$ is nonzero at every point in $\mathcal{I}$. Lemma 15.5 on page 317 then assures us that, given any point x_0 in $\mathcal{I}$, and any two fixed values A and B, there is exactly one value for c_1 and one value for c_2 such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

satisfies

$$y(x_0) = A \quad \text{and} \quad y'(x_0) = B \quad .$$

This takes care of part (c).

Now let y_0 be any solution to the differential equation in the theorem. Pick any point x_0 in $\mathcal{I}$ and let $A = y(x_0)$ and $B = y'(x_0)$. Then y_0 is automatically one solution to the initial-value problem

$$ay'' + by' + cy = 0 \quad \text{with} \quad y(x_0) = A \quad \text{and} \quad y'(x_0) = B \quad .$$

Since $W[y_1, y_2]$ is nonzero at every point in $\mathcal{I}$, it is nonzero at x_0, and lemma 15.5 on page 317 again applies and tells us that there are constants c_1 and c_2 such that

$$c_1 y_1(x) + c_2 y_2(x)$$

satisfies the above initial conditions. By the principle of superposition, this linear combination also satisfies the differential equation. Thus, both y_0 and the above linear combination satisfy the initial-value problem derived above from y_0. But from theorem 15.2 on page 314 we know that there is only one solution. Hence these two solutions must be the same,

$$y_0(x) = c_1 y_1(x) + c_2 y_2(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad ,$$

confirming that every solution to the differential equation can be written as a linear combination of y_1 and y_2. This (with the principle of superposition) confirms that a general solution to the differential equation is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants. It also confirms that $\{y_1, y_2\}$ is a fundamental set of solutions for the differential equation on $\mathcal{I}$. This finishes verifying the third claim.

Given that we now know every linearly independent pair of solutions is a fundamental solution set, all that remains in verifying the second claim is to show that any set of solutions containing either less than or more than two solutions cannot be a fundamental set. First assume we have a set of just one solution $\{y_1\}$. To be linearly independent, y_1 cannot be the zero solution. So there is a point x_0 in $\mathcal{I}$ at which $y_1(x_0) \neq 0$. Now let y_2 be a solution to the initial-value problem

$$ay'' + by' + cy = 0 \qquad \text{with} \quad y(x_0) = 0 \quad \text{and} \quad y'(x_0) = 1 \quad .$$

Computing the Wronskian $W = W[y_1, y_2]$ at x_0 we get

$$W(x_0) = y_1(x_0){y_2}'(x_0) - {y_1}'(x_0)y_2(x_0) = y_1(x_0) \cdot 1 - {y_1}'(x_0) \cdot 0 = y_1(x_0) \quad ,$$

which is nonzero. Hence, by theorem 15.7 on page 320, we know $\{y_1, y_2\}$ is linearly independent on $\mathcal{I}$. Thus, y_2 is a solution to the differential equation that cannot be written as a linear combination of y_1 alone, and thus, $\{y_1\}$ is not a fundamental solution set for the differential equation.

On the other hand, suppose we have a set of three or more solutions $\{y_1, y_2, y_3, \ldots\}$. Clearly, if $\{y_1, y_2\}$ is not linearly independent, then neither is $\{y_1, y_2, y_3, \ldots\}$, while if the pair is linearly independent, then, as just shown above, y_3 is a linear combination of y_1 and y_2. Either way, $\{y_1, y_2, y_3, \ldots\}$ is not linearly independent, and, hence, is not a fundamental solution set.

Finally, consider the first claim — that fundamental solution sets exist. Because we've already verified the third claim, it will suffice to confirm the existence of a linearly independent pair of solutions. To do this, pick any point x_0 in $\mathcal{I}$. Let y_1 be the solution to

$$ay'' + by' + cy = 0 \qquad \text{with} \quad y(x_0) = 1 \quad \text{and} \quad y'(x_0) = 0 \quad ,$$

and let y_2 be the solution to

$$ay'' + by' + cy = 0 \qquad \text{with} \quad y(x_0) = 0 \quad \text{and} \quad y'(x_0) = 1 \quad .$$

We know these solutions exist because of theorem 15.2 on page 314. Moreover, letting $W = W[y_1, y_2]$,

$$W(x_0) \; = \; y_1(x_0){y_2}'(x_0) \; - \; {y_1}'(x_0)y_2(x_0) \; = \; 1 \cdot 1 \; - \; 0 \cdot 0 \; = \; 1 \; \neq \; 0 \quad .$$

Thus, from theorem 15.7, we know $\{y_1, \; y_2\}$ is a linearly independent pair of solutions. ∎

15.3 Arbitrary Homogeneous Linear Equations

In this section, we will verify the claims made in section 14.3 concerning the solutions and the fundamental sets of solutions for homogeneous linear differential equations of any order (i.e., theorem 14.2 on page 304 and in theorem 14.3 on page 305). As already noted, our discussion will make use of the theory normally developed in an elementary course on linear algebra. In particular, you should be acquainted with “matrix/vector equations”, determinants, and the basic theory of vector spaces.

The Differential Equation and Basic Assumptions

Throughout this section, we are dealing with a fairly arbitrary N^{th}-order homogeneous linear differential equation

$$a_0 y^{(N)} \; + \; a_1 y^{(N-1)} \; + \; \cdots \; + \; a_{N-2} y'' \; + \; a_{N-1} y' \; + \; a_N y \; = \; 0 \quad . \tag{15.6a}$$

Whether or not it is explicitly stated, we will always assume the a_k’s are continuous functions on some open interval $\mathcal{I}$, and that a_0 is never zero in this open interval. In addition, we may have an N^{th}-order set of initial conditions at some point x_0 in $\mathcal{I}$,

$$y(x_0) \; = \; A_1 \quad , \quad y'(x_0) \; = \; A_2 \quad , \quad y''(x_0) \; = \; A_2 \quad , \quad \ldots \quad \text{and} \quad y^{(N-1)}(x_0) \; = \; A_N \quad . \tag{15.6b}$$

For brevity, we may refer to the above differential equation as “our differential equation” and to the initial-value problem consisting of this differential equation and the above initial conditions as either “our initial-value problem” or initial-value problem (15.6). So try to remember them. To further simplify our verbage, let us also agree that, for the rest of this section, whenever “a function” is referred to, then this is “a function defined on the interval $\mathcal{I}$ which is differentiable enough to compute however many derivatives we need”. And if this function is further specified as ‘’a solution”, then it is “a solution to the above differential equation over the interval $\mathcal{I}$ ”.

Basic Existence and Uniqueness

Our first lemma simply states that our initial-value problems have unique solutions.

Lemma 15.9 (existence and uniqueness of solutions)
For each choice of constants A_1 , A_2 , $\ldots$ and A_N , initial-value problem (15.6) has exactly one solution, and this solution is valid over the interval $\mathcal{I}$.

PROOF: This lemma follows from theorem 11.4 on page 255. To see this, first algebraically solve differential equation (15.6a) for $y^{(N)}$. The result is

$$y^{(N)} = -\frac{a_N}{a_0}y - \frac{a_{N-1}}{a_0}y' - \cdots - \frac{a_1}{a_0}y^{(N-1)} \quad .$$

Letting p_1, p_2, ... and p_N be the functions

$$p_1(x) = -\frac{a_N(x)}{a_0(x)} \quad , \quad p_2(x) = -\frac{a_{N-1}(x)}{a_0(x)} \quad , \quad \cdots \quad \text{and} \quad p_N(x) = -\frac{a_1(x)}{a_0(x)} \quad ,$$

and letting

$$F(x, \tau_1, \tau_2, \tau_3, \ldots, \tau_N) = p_1(x)\tau_1 + p_2(x)\tau_2 + \cdots + p_N(x)\tau_N \quad ,$$

we can further rewrite our differential equation as

$$y^{(N)} = F\left(x, y, y', \ldots, y^{(n-1)}\right) \quad .$$

Observe that, because the a_k's are all continuous functions on $\mathcal{I}$ and a_0 is never zero on $\mathcal{I}$, each p_k is a continuous function on $\mathcal{I}$. Clearly then, the above $F(x, \tau_1, \tau_2, \tau_3, \ldots, \tau_N)$ and the partial derivatives

$$\frac{\partial F}{\partial \tau_1} = p_1(x) \quad , \quad \frac{\partial F}{\partial \tau_2} = p_2(x) \quad , \quad \ldots \quad \text{and} \quad \frac{\partial F}{\partial \tau_N} = p_N(x)$$

are all continuous functions on the region of all $(x, \tau_1, \tau_2, \tau_3, \ldots, \tau_N)$ with x in the open interval $\mathcal{I}$. Moreover, these partial derivatives depend only on x.

Checking back, we see that theorem 11.4 immediately applies and assures us that our initial-value problem has exactly one solution, and that this solution is valid over the interval $\mathcal{I}$. ∎

A "Matrix/Vector" Formula for Linear Combinations

In much of the following, we will be dealing with linear combinations of some set of functions

$$\{\, y_1(x) \,,\, y_2(x) \,,\, \ldots \,,\, y_M(x) \,\} \quad ,$$

along with corresponding linear combinations of the derivatives of these functions. Observe that

$$\begin{bmatrix} c_1y_1 + c_2y_2 + \cdots + c_My_M \\ c_1{y_1}' + c_2{y_2}' + \cdots + c_M{y_M}' \\ c_1{y_1}'' + c_2{y_2}'' + \cdots + c_M{y_M}'' \\ \vdots \\ c_1{y_1}^{(N-1)} + c_2{y_2}^{(N-1)} + \cdots + c_M{y_M}^{(N-1)} \end{bmatrix}$$

$$= \begin{bmatrix} y_1 & y_2 & \cdots & y_M \\ {y_1}' & {y_2}' & \cdots & {y_M}' \\ {y_1}'' & {y_2}'' & \cdots & {y_M}'' \\ \vdots & \vdots & \ddots & \vdots \\ {y_1}^{(N-1)} & {y_2}^{(N-1)} & \cdots & {y_M}^{(N-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} \quad .$$

That is,

$$\begin{bmatrix} c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x) \\ c_1 {y_1}'(x) + c_2 {y_2}'(x) + \cdots + c_M {y_M}'(x) \\ c_1 {y_1}''(x) + c_2 {y_2}''(x) + \cdots + c_M {y_M}''(x) \\ \vdots \\ c_1 {y_1}^{(N-1)}(x) + c_2 {y_2}^{(N-1)}(x) + \cdots + c_M {y_M}^{(N-1)}(x) \end{bmatrix} = [\mathbf{Y}(x)]\mathbf{c}$$

where

$$\mathbf{Y}(x) = \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_M(x) \\ {y_1}'(x) & {y_2}'(x) & \cdots & {y_M}'(x) \\ {y_1}''(x) & {y_2}''(x) & \cdots & {y_M}''(x) \\ \vdots & \vdots & \ddots & \vdots \\ {y_1}^{(N-1)}(x) & {y_2}^{(N-1)}(x) & \cdots & {y_M}^{(N-1)}(x) \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix} \quad . \tag{15.7}$$

The $N \times M$ matrix $\mathbf{Y}(x)$ defined in line (15.7) will be useful in many of our derivations. For lack of imagination, we will either call it *the* $N \times M$ *matrix formed from set* $\{y_1(x), y_2(x), \ldots, y_M(x)\}$, or, more simply *the matrix* $\mathbf{Y}(x)$ *described in line (15.7).*

Initial-Value Problems

Let

$$\{ y_1(x) , y_2(x) , \ldots , y_M(x) \}$$

be a set of M solutions to our N^{th}-order differential equation, equation (15.6a), and suppose we wish to find a linear combination

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x)$$

satisfying some N^{th}-order set of initial conditions

$$y(x_0) = A_1 \quad , \quad y'(x_0) = A_2 \quad , \quad y''(x_0) = A_3 \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(x_0) = A_N \quad .$$

Replacing $y(x_0)$ with its formula in terms of the y_k's, this set of initial conditions becomes

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) + \cdots + c_M y_M(x_0) &= A_1 \quad , \\ c_1 {y_1}'(x_0) + c_2 {y_2}'(x_0) + \cdots + c_M {y_M}'(x_0) &= A_2 \quad , \\ c_1 {y_1}''(x_0) + c_2 {y_2}''(x_0) + \cdots + c_M {y_M}''(x_0) &= A_3 \quad , \\ &\vdots \end{aligned}$$

and

$$c_1 {y_1}^{(N-1)}(x_0) + c_2 {y_2}^{(N-1)}(x_0) + \cdots + c_M {y_M}^{(N-1)}(x_0) = A_N \quad .$$

This is an algebraic system of N equations with the M unknowns c_1, c_2, ... and c_M (keep in mind that x_0 and the A_k's are 'given'). Our problem is to find those c_k's for any given choice of A_k's.

Letting $\mathbf{Y}(x)$ and $\mathbf{c}$ be the $N \times M$ and $M \times 1$ matrices from line (15.7), we can rephrase our problem as finding the solution $\mathbf{c}$ to

$$[\mathbf{Y}(x_0)]\mathbf{c} = \mathbf{A} \qquad \text{for any given} \quad \mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} \quad .$$

But this is a classic problem from linear algebra, and from linear algebra we know there is one and only solution $\mathbf{c}$ for each $\mathbf{A}$ if and only if both of the following hold:

1. *$M = N$ (i.e., $\mathbf{Y}(x_0)$ is actually a square $N \times N$ matrix).*
2. *The $N \times N$ matrix $\mathbf{Y}(t_0)$ is invertible.*

If these conditions are satisfied, then $\mathbf{c}$ can be determined from each $\mathbf{A}$ by

$$\mathbf{c} = [\mathbf{Y}(x_0)]^{-1}\mathbf{A} \quad .$$

(In practice, a "row reduction" method may be a more efficient way to solve for $\mathbf{c}$.)

This tells us that we will probably be most interested in sets of exactly N solutions whose corresponding matrix $\mathbf{Y}(x)$ is invertible when $x = x_0$. Fortunately, as you probably recall, there is a relatively simple test for determining if any square matrix $\mathbf{M}$ is invertible based on the determinant $\det(\mathbf{M})$ of that matrix; namely,

$$\mathbf{M} \text{ is invertible} \quad \iff \quad \det(\mathbf{M}) \neq 0 \quad .$$

To simplify discussion, we will give a name and notation for the determinant of the matrix formed from any set of N functions $\{y_1, y_2, \ldots, y_N\}$. We will call this determinant the *Wronskian* and denote it by $W(x)$,

$$W(x) = \det(\mathbf{Y}(x)) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_N(x) \\ {y_1}'(x) & {y_2}'(x) & \cdots & {y_N}'(x) \\ {y_1}''(x) & {y_2}''(x) & \cdots & {y_N}''(x) \\ \vdots & \vdots & \ddots & \vdots \\ {y_1}^{(N-1)}(x) & {y_2}^{(N-1)}(x) & \cdots & {y_N}^{(N-1)}(x) \end{vmatrix} \quad .$$

Unsurprisingly, this is the same definition as given for a Wronskian in section 14.4.

While we are at it, let us recall a few other related facts from linear algebra regarding the solving of

$$[\mathbf{Y}(x_0)]\mathbf{c} = \mathbf{A}$$

when $\mathbf{Y}(x_0)$ is an $N \times M$ matrix which is *not* an invertible:

1. If $M < N$, then there is an $\mathbf{A}$ for which there is no solution $\mathbf{c}$.

2. If $M > N$ and $\mathbf{0}$ is the zero $N \times 1$ matrix (i.e., the $N \times 1$ matrix whose entries are all zeros), then there is a nonzero $M \times 1$ matrix $\mathbf{c}$ such that $[\mathbf{Y}(x_0)]\mathbf{c} = \mathbf{0}$.

3. If $M = N$ but $\mathbf{Y}(x_0)$ is not invertible, then both of the following are true:

   (a) There is an $\mathbf{A}$ for which there is no solution $\mathbf{c}$.

(b) There is a $\mathbf{c}$ other than $\mathbf{c} = \mathbf{0}$ such that $[\mathbf{Y}(x_0)]\mathbf{c} = \mathbf{0}$.

Applying all the above to our initial-value problem immediately yields the next lemma.

Lemma 15.10 (solving initial-value problems with linear combinations)
Consider initial-value problem

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

with

$$y(x_0) = A_1 \quad , \quad y'(x_0) = A_2 \quad , \quad y''(x_0) = A_3 \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(x_0) = A_N \quad .$$

Let $\{y_1, y_2, \ldots, y_M\}$ be any set of M solutions to the above differential equation, and, if $M = N$, let $W(x)$ be the Wronskian for this set. Then all of the following hold:

1. *If $M < N$, then there are constants A_1, A_2, ... and A_N such that no linear combination of the given y_k's satisfies the above initial-value problem.*

2. *If $M > N$, then there are constants c_1, c_2, ... and c_M, not all zero, such that*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x)$$

satisfies the N^{th}-order set of zero initial conditions,

$$y(x_0) = 0 \quad , \quad y'(x_0) = 0 \quad , \quad y''(x_0) = 0 \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(x_0) = 0 \quad .$$

In fact, any multiple of this y satisfies this set of zero initial conditions.

3. *If $M = N$ and $W(x_0) = 0$, then both of the following hold:*
   (a) *There are constants A_1, A_2, ... and A_N such that no linear combination of the given y_k's satisfies the above initial-value problem.*
   (b) *There are constants c_1, c_2, ... and c_M, not all zero, such that*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x)$$

satisfies the N^{th}-order set of zero initial conditions,

$$y(x_0) = 0 \quad , \quad y'(x_0) = 0 \quad , \quad y''(x_0) = 0 \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(x_0) = 0 \quad .$$

4. *If $M = N$ and $W(x_0) \neq 0$, then, for each choice of constants A_1, A_2, ... and A_N, there is exactly one solution y to the above initial-value problem of the form of a linear combination of the y_k's,*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x) \quad .$$

Expressing All Solutions as Linear Combinations

Now assume we have a set of N solutions

$$\{ y_1(x) , y_2(x) , \ldots , y_N(x) \}$$

to our N^{th}-order differential equation (equation (15.6a) on page 323). Let's see what lemma 15.10 tells us about expressing any given solution y to our differential equation as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x)$$

for some choice of constants c_1 , c_2 , ... and c_N .

First, pick any x_0 in $\mathcal{I}$, and suppose

$$W(x_0) \neq 0 \quad .$$

Let y_0 be any particular solution to our differential equation; set

$$A_1 = y_0(x_0) \quad , \quad A_2 = {y_0}'(x_0) \quad , \quad \ldots \quad \text{and} \quad A_N = {y_0}^{(N-1)}(x_0) \quad ,$$

and consider our initial-value problem (problem (15.6) on page 323) with these choices for the A_k's . Clearly, $y(x) = y_0(x)$ is one solution. In addition, our last lemma tells us that there are constants c_1 , c_2 , ... and c_N such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x) \quad ,$$

is also a solution to this initial-value problem. So we seem to have two solutions to the above initial-value problem: $y_0(x)$ and the above linear combination of y_k's . But from lemma 15.9 (on the existence and uniqueness of solutions), we know there is only one solution to this initial-value problem. Hence, our two solutions must be the same,

$$y_0(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x) \quad .$$

This confirms that

$$W(x_0) \neq 0 \quad \Longrightarrow \quad \begin{array}{l}\text{Every solution to our differential equation}\\ \text{is a linear combination of the } y_k\text{'s .}\end{array} \tag{15.8}$$

On the other hand, if

$$W(x_0) = 0 \quad ,$$

then lemma 15.10 says that there are A_k's such that no linear combination of these y_k's is a solution to our initial-value problem. Still, lemma 15.9 assures us that there is a solution — all lemma 15.10 adds is that this solution is not a linear combination of the y_k's . Hence,

$$W(x_0) = 0 \quad \Longrightarrow \quad \begin{array}{l}\text{Not all solutions to our differential equation}\\ \text{are linear combination of the } y_k\text{'s .}\end{array} \tag{15.9}$$

With a little thought, you will realize that, together, implications (15.8) and (15.9), along with lemma 15.9, give us:

Lemma 15.11
Assume

$$\{ y_1(x) , y_2(x) , \ldots , y_N(x) \}$$

is a set of solutions to our N^{th}-order differential equation (equation (15.6a) on page 323), and let $W(x)$ be the corresponding Wronskian. Pick any point x_0 in $\mathcal{I}$. Then every solution to our differential equation is given by a linear combination of the above $y_k(x)$'s if and only if

$$W(x_0) \neq 0 \quad .$$

Moreover, if $W(x_0) \neq 0$ and $y(x)$ is any solution to our differential equation, then there is exactly one choice for c_1, c_2, ... and c_N such that

$$y(x) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_N y_N(t) \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

There is a useful corrollary to our last lemms. To derive it, let x_0 and x_1 be any two points in $\mathcal{I}$, and observe that, by the last lemma, we know

$$W(x_0) \neq 0 \quad \Longleftrightarrow \quad \begin{array}{l}\text{Every solution is a linear}\\ \text{combination of the } y_k\text{'s}.\end{array} \quad \Longleftrightarrow \quad W(x_1) \neq 0 \quad .$$

In other words, the Wronskian cannot be zero at one point in $\mathcal{I}$ and nonzero at another.

Corollary 15.12
Let

$$\{ y_1(x) , y_2(x) , \ldots , y_N(x) \}$$

be a set of solutions to our N^{th}-order differential equation, and let $W(x)$ be the corresponding Wronskian. Then

$$W(x) \neq 0 \text{ for one point in } \mathcal{I} \quad \Longleftrightarrow \quad W(x) \neq 0 \text{ for every point in } \mathcal{I} \quad .$$

Equivalently,

$$W(x) = 0 \text{ for one point in } \mathcal{I} \quad \Longleftrightarrow \quad W(x) = 0 \text{ for every point in } \mathcal{I} \quad .$$

Existence of Fundamental Sets

At this point, it should be clear that we will be able to show that any set of N solutions to our differential equation is a fundamental set of solutions if and only if its Wronskian is nonzero. Let's now construct such a set by picking any x_0 in $\mathcal{I}$, and considering a sequence of initial-value problems involving our N^{th}-order homogeneous linear differential equation.

For the first, the initial values are

$$y(x_0) = A_{1,1} \quad , \quad y'(x_0) = A_{1,2} \quad ,$$
$$y''(x_0) = A_{1,3} \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(x_0) = A_{1,N} \quad ,$$

where $A_{1,1} = 1$ and the other $A_{1,k}$'s are all zero. Lemma 15.9 on page 323 assures us that there is a solution — call it y_1.

For the second, the initial values are

$$y(x_0) = A_{2,1} \quad , \quad y'(x_0) = A_{2,2} \quad ,$$
$$y''(x_0) = A_{2,3} \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(x_0) = A_{2,N} \quad ,$$

where $A_{2,2} = 1$ and the other $A_{2,k}$'s are all zero. Let y_2 be the single solution that lemma 15.9 tells us exists.

And so on ....

In general, for $j = 1, 2, 3, \ldots, N$, we let y_j be the single solution to the given differential equation satisfying

$$y(x_0) = A_{j,1} \quad , \quad y'(x_0) = A_{j,2} \quad ,$$
$$y''(x_0) = A_{j,3} \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(x_0) = A_{j,N} \quad ,$$

where $A_{j,j} = 1$ and the other $A_{j,k}$'s are all zero.

This gives us a set of N solutions

$$\{ y_1(x), \ y_2(x), \ \ldots, \ y_N(x) \}$$

to our N^{th}-order differential equation. By the initial conditions which they satisfy,

$$\mathbf{Y}(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_N(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) & \cdots & y_N{}'(x_0) \\ y_1{}''(x_0) & y_2{}''(x_0) & \cdots & y_N{}''(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1{}^{(N-1)}(x_0) & y_2{}^{(N-1)}(x_0) & \cdots & y_N{}^{(N-1)}(x_0) \end{bmatrix}$$

$$= \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} & \cdots & A_{N,1} \\ A_{1,2} & A_{2,2} & A_{3,2} & \cdots & A_{N,2} \\ A_{1,3} & A_{2,3} & A_{3,3} & \cdots & A_{N,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{1,N} & A_{2,N} & A_{3,N} & \cdots & A_{N,N} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad .$$

Hence,

$$W(x_0) = \det(\mathbf{Y}(x_0) = = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0 \quad ,$$

and lemma 15.11 tells us that every other solution to our differential equation can be written as a linear combination of these y_k's. All that remains to showing $\{y_1, y_2, \ldots, y_N\}$ is a fundamental set of solutions for our differential equation is to verify that this set is linearly independent on $\mathcal{I}$, and all that we need to show is that none of the y_k's is a linear combination of the others.

Well, what if y_1 is a linear combination of the others,

$$y_1(x) = c_2 y_2(x) + c_3 y_3(x) + \cdots + c_N y_N(x) \quad \text{for all } x \text{ in } \mathcal{I} \quad ?$$

Then, plugging in $x = x_0$ and using the above initial values, we would get

$$\begin{aligned} 1 = y_1(x_0) &= c_2 y_2(x_0) + c_3 y_3(x_0) + \cdots + c_N y_N(x_0) \\ &= c_2 \cdot 0 + c_3 \cdot 0 + \cdots + c_N \cdot 0 = 0 \quad , \end{aligned}$$

which is impossible. So y_1 cannot be not a linear combination of the other y_k's.

Could y_2 be a linear combination of the others,

$$y_2(x) = c_1 y_1(x) + c_3 y_3(x) + \cdots + c_N y_N(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad ?$$

Taking the derivative of this, we get

$$y_2{}'(x) = c_1 y_1{}'(x) + c_3 y_3{}'(x) + \cdots + c_N y_N{}'(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

Plugging in $x = x_0$ and then using the above initial values yields

$$\begin{aligned} 1 = y_2{}'(x_0) &= c_1 y_1{}'(x_0) + c_3 y_3{}'(x_0) + \cdots + c_N y_N{}'(x_0) \\ &= c_1 \cdot 0 + c_3 \cdot 0 + \cdots + c_N \cdot 0 = 0 \quad , \end{aligned}$$

which, again, is impossible. So y_2 is not a linear combination of the other y_k's.

Clearly we can continue in this manner and verify that each y_k is not a linear combination of the others. So $\{y_1, y_2, \ldots, y_N\}$ is a linearly independent set of solutions.

In summary:

Lemma 15.13

Fundamental sets of N solutions exist. In fact, for each x_0 in $\mathcal{I}$, there is a fundamental set of solutions $\{y_1, y_2, \ldots, y_N\}$ satisfying

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_N(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) & \cdots & y_N{}'(x_0) \\ y_1{}''(x_0) & y_2{}''(x_0) & \cdots & y_N{}''(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1{}^{(N-1)}(x_0) & y_2{}^{(N-1)}(x_0) & \cdots & y_N{}^{(N-1)}(x_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad .$$

A Little More Linear Algebra

For convenience, let $\mathcal{S}$ denote the set of all solutions to our N^{th}-order homogeneous linear differential equation, equation (15.6a) on page 323. Using the principle of superposition, it is trivial to verify that $\mathcal{S}$ is a vector space of functions on $\mathcal{I}$. Now recall that a *basis* for vector space $\mathcal{S}$ is any linearly independent subset $\{y_1, y_2, \ldots, y_M\}$ of $\mathcal{S}$ such that any other y in $\mathcal{S}$ can be written as a linear combination of these y_k's. But that's completely equivalent to our definition of $\{y_1, y_2, \ldots, y_M\}$ being a fundamental set of solutions for our differential equation. So, for us, the phrases

"$\{y_1, y_2, \ldots, y_M\}$ is a fundamental set of solutions for our differential equation"

and

"$\{y_1, y_2, \ldots, y_M\}$ is a basis for $\mathcal{S}$"

are equivalent and interchangable.

Now, take another look at our last lemma, lemma 15.13. It tells us that $\mathcal{S}$ has a basis containing exactly N solutions. From this and the basic theory of vector spaces developed in most elementary linear algebra courses, it follows that:

1. $\mathcal{S}$ is an N-dimensional vector space.

2. Every basis for $\mathcal{S}$ is a set of exactly N solutions to our differential equation.

3. A set $\{y_1, y_2, \ldots, y_N\}$ of exactly N solutions to our differential equation is a basis for $\mathcal{S}$ if and only if the set is linearly independent on $\mathcal{I}$.

4. A set $\{y_1, y_2, \ldots, y_N\}$ of exactly N solutions to our differential equation is a basis for $\mathcal{S}$ if and only if every other y in $\mathcal{S}$ can be written as a linear combination of these y_k's.

This brings us to our last lemma.

Lemma 15.14
Let $\{y_1, y_2, \ldots, y_N\}$ be a set of N solutions to our N^{th}-order homogeneous linear differential equation (equation (15.6a) on page 323). Then, if any one of the following three statements is true, they all are true:

1. $\{y_1, y_2, \ldots, y_N\}$ *is linearly independent on* $\mathcal{I}$.

2. $\{y_1, y_2, \ldots, y_N\}$ *is a fundamental set of solutions for our differential equation.*

3. *Every solution to our differential equation can be written as a linear combination of these* y_k*'s.*

4. *The Wronskian,* $W(x)$, *of* $\{y_1, y_2, \ldots, y_N\}$ *is nonzero at some point in* $\mathcal{I}$

PROOF: From the discussion just before this lemma, and from lemma 15.11 on page 328,

$$\begin{aligned}
&\{y_1, y_2, \ldots, y_N\} \text{ is linearly independent on } \mathcal{I} \\
&\qquad \iff \quad \{y_1, y_2, \ldots, y_N\} \text{ is a basis for } \mathcal{S} \\
&\qquad \iff \quad \text{every other } y \text{ in } \mathcal{S} \text{ can be written as a linear combination of these } y_k\text{'s} \\
&\qquad \iff \quad W(x_0) \neq 0 \qquad \text{for some } x_0 \text{ in } \mathcal{I} \quad .
\end{aligned}$$

Summary and Final Results

Remember, our goal in this section is to verify the claims made in section 14.3 regarding the solutions and the fundamental sets of solutions for homogeneous linear differential equations of any order. We are almost there. All that remains is to restate the theorems in that section (theorems 14.2 and 14.3) and to show how they follow from the lemmas just developed.

Theorem 15.15 (same as theorem 14.2)
Let $\mathcal{I}$ be some open interval, and suppose we have an N^{th}-order homogeneous linear differential equation

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

where, on $\mathcal{I}$, the a_k's are all continuous functions with a_0 never being zero. Then the following statements all hold:

1. *Fundamental sets of solutions for this differential equation (over $\mathcal{I}$) exist.*
2. *Every fundamental solution set consists of exactly N solutions.*
3. *If $\{y_1, y_2, \ldots, y_N\}$ is any linearly independent set of N particular solutions over $\mathcal{I}$, then:*
   (a) $\{y_1, y_2, \ldots, y_N\}$ *is a fundamental set of solutions.*
   (b) *A general solution to the differential equation is given by*
   $$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x)$$
   where c_1, c_2, ... and c_N are arbitrary constants.
   (c) *Given any point x_0 in $\mathcal{I}$ and any N fixed values A_1, A_2, ... and A_N, there is exactly one ordered set of constants $\{c_1, c_2, \ldots, c_N\}$ such that*
   $$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x)$$
   also satisfies the initial conditions
   $$y(x_0) = A_1 \quad , \quad y'(x_0) = A_2 \quad ,$$
   $$y''(x_0) = A_2 \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(x_0) = A_N \quad .$$

PROOF: Lemma 15.13 on page 331 assures us of the existence of fundamental sets of solutions. That lemma also, as already noted, tells us that the set of all solutions to our differential equation is an N-dimensional vector space, and that, as we just saw, means that every fundamental set of solutions contains exactly N solutions. This proves parts 1 and 2 of the theorem.

The first two claims in part 3 follow directly from lemma 15.14. And the last claim? Since we now know the Wronskian of the set is nonzero, the last claim follows immediately from lemma 15.10 on page 327. ∎

Theorem 15.16 (same as theorem 14.3)
Let W be the Wronskian of any set $\{y_1, y_2, \ldots, y_N\}$ of N particular solutions to an N^{th}-order homogeneous linear differential equation

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

on some interval open $\mathcal{I}$. Assume further that the a_k's are continuous functions with a_0 never being zero on $\mathcal{I}$. Then:

1. *If $W(x_0) = 0$ for any single point x_0 in $\mathcal{I}$, then $W(x) = 0$ for every point x in $\mathcal{I}$, and the set $\{y_1, y_2, \ldots, y_N\}$ is not linearly independent (and, hence, is not a fundamental solution set) on $\mathcal{I}$.*
2. *If $W(x_0) \neq 0$ for any single point x_0 in $\mathcal{I}$, then $W(x) \neq 0$ for every point x in $\mathcal{I}$, and $\{y_1, y_2, \ldots, y_N\}$ is a fundamental solution set solutions for the given differential equation on $\mathcal{I}$.*

PROOF: See lemma 15.14 and corollary 15.12. ∎

Additional Exercises

15.1. Let y_1 and y_2 be the following functions on the entire real line:

$$y_1(x) = \begin{cases} -x^2 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \leq x \end{cases} \qquad \text{and} \qquad y_2(x) = \begin{cases} x^2 & \text{if} \quad x < 0 \\ 3x^2 & \text{if} \quad 0 \leq x \end{cases} \quad .$$

a. Verify that

i. $\{y_1, y_2\}$ is not linearly dependent on the entire real line, but

ii. the Wronskian for $\{y_1, y_2\}$ is zero over the entire real line (even at $x = 0$).

b. Why do the results in the previous part not violate either lemma 15.6 on page 318 or theorem 15.7 on page 320?

c. Is there an interval $\mathcal{I}$ on which $\{y_1, y_2\}$ is linearly dependent?

15.2. Let $\{y_1, y_2\}$ be a linearly independent pair of solutions over an interval $\mathcal{I}$ to some second-order homogeneous linear differential equation

$$ay'' + by' + cy = 0 \quad .$$

As usual, assume a , b and c are continuous functions on $\mathcal{I}$ with a never being zero that interval. Also, as usual, let

$$W = W[y_1, y_2] = y_1 {y_2}' - {y_1}' y_2 \quad .$$

Do the following, using the fact that W is never zero on $\mathcal{I}$.

a. Show that, if $y_1(x_0) = 0$ for some x_0 in $\mathcal{I}$, then ${y_1}'(x_0) \neq 0$ and $y_2(x_0) \neq 0$.

b. Show that, if ${y_1}'(x_0) = 0$ for some x_0 in $\mathcal{I}$, then $y_1(x_0) \neq 0$ and ${y_2}'(x_0) \neq 0$.

c. Why can we not have $W(x) > 0$ for some x in $\mathcal{I}$ and $W(x) < 0$ for other x in $\mathcal{I}$? That is, explain (briefly) why we must have either

$$W(x) > 0 \qquad \text{for all } x \text{ in } \mathcal{I}$$

or

$$W(x) < 0 \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

d. For the following, assume $W(x) > 0$ for all x in $\mathcal{I}$.[1] Let $[\alpha, \beta]$ be a subinterval of $\mathcal{I}$ such that

$$y_1(\alpha) = 0 \qquad , \qquad y_1(\beta) = 0$$

and

$$y_1(x) > 0 \qquad \text{whenever} \quad \alpha < x < \beta$$

[1] Similar results can be derived assuming $W(x) < 0$ for all x in $\mathcal{I}$.

i. *How do we know that neither* $y_1{}'(\alpha)$ *nor* $y_1{}'(\beta)$ *are zero? Which one is positive? Which one is negative? (It may help to draw a rough sketch of the graph of* y_1 *based on the above information.)*

ii. *Using the Wronskian, determine if* $y_2(\alpha)$ *is positive or negative. Then determine if* $y_2(\beta)$ *is positive or negative.*

iii. *Now show that there must be a point* x_0 *in the open interval* (α, β) *at which* y_2 *is zero.*

(What you've just shown is that there must be a zero of y_2 *between any two zeroes* α *and* β *of* y_1 *. You can easily expand this to the following statement:*

> *Between any two zeroes of* y_1 *is a zero of* y_2 *, and, likewise, between any two zeroes of* y_2 *is a zero of* y_1 *.*

This tells us something about the graphs of linearly independent pairs of solutions to second-order homogeneous differential equations. It turns out to be an important property of these solution pairs when considering a type of differential equation problem involving the values of solutions at pairs of points, instead of at single points.)