11

Higher-Order Equations: Extending First-Order Concepts

Let us switch our attention from first-order differential equations to differential equations of order two or higher. Our main interest will be with second-order differential equations, both because it is natural to look at second-order equations after studying first-order equations, and because second-order equations arise in applications much more often than do third-, or fourth- or eighty-third-order equations. Some examples of second-order differential equations are[1]

$$y'' + y = 0 \quad ,$$

$$y'' + 2xy' - 5\sin(x)\,y = 30e^{3x} \quad ,$$

and

$$(y+1)y'' = (y')^2 \quad .$$

Still, even higher order differential equations, such as

$$8y''' + 4y'' + 3y' - 83y = 2e^{4x} \quad ,$$

$$x^3 y^{(iv)} + 6x^2 y'' + 3xy' - 83\sin(x)y = 2e^{4x} \quad ,$$

and

$$y^{(83)} + 2y^3\,y^{(53)} - x^2 y'' = 18 \quad ,$$

can arise in applications, at least on occasion. Fortunately, many of the ideas used in solving these are straightforward extensions of those used to solve second-order equations. We will make use of this fact extensively in the following chapters.

Unfortunately, though, the methods we developed to solve first-order differential equations are of limited direct use in solving higher-order equations. Remember, most of those methods were based on integrating the differential equation after rearranging it into a form that could be legitimately integrated. This rarely is possible with higher-order equations, and that makes solving higher-order equations more of a challenge. This does not mean that those ideas developed in previous chapters are useless in solving higher-order equations, only that their use will tend to be subtle rather than obvious.

Still, there are higher-order differential equations that, after the application of a simple substitution, can be treated and solved as first-order equations. While our knowledge of first-order equations is still fresh, let us consider some of the more important situations in which this is

[1] For notational brevity, we will start using the 'prime' notation for derivatives a bit more. It is still recommended, however, that you use the '$^d/_{dx}$' notation when finding solutions just to help keep track of the variables involved.

possible. We will also take a quick look at how the basic ideas regarding first-order initial-value problems extend to higher-order initial-value problems. And finally, to cap off this chapter, we will briefly discuss the higher-order extensions of the existence and uniqueness theorems from section 3.3.

11.1 Treating Some Second-Order Equations as First-Order

Suppose we have a second-order differential equation (with y being the yet unknown function and x being the variable). With luck, it is possible to convert the given equation to a first-order differential equation for another function v via the substitution $v = y'$. With a little more luck, that first-order equation can then be solved for v using methods discussed in previous chapters, and y can then be obtained from v by solving the first-order differential equation given by the original substitution, $y' = v$.

This approach requires some luck because, typically, setting $v = y'$ does not lead to a differential equation for just the one unknown function v. Instead, it usually results in a differential equation with *two* unknown functions, y and v, along with the variable x. This does not simplify our equation at all! So, being lucky here means that the conversion does yield a differential equation just involving v and one variable.

It turns out that we get lucky with two types of second-order differential equations: those that do not explicitly contain a y, and those that do not explicitly contain an x. The first type will be especially important to us since solving this type of equation is part of an important method for solving more general differential equations (the "reduction of order" method in chapter 13). It is also, typically, the easier type of equation to solve. So let's now spend a few moments discussing how to solve these equations. (We'll say more about the second type in a few pages.)

Solving Second-Order Differential Equations Not Explicitly Containing y

If the equation explicitly involves x, ${}^{dy}/_{dx}$, and ${}^{d^2y}/_{dx^2}$ — but not y — then we can naturally view the differential equation as a "first-order equation for ${}^{dy}/_{dx}$". For convenience, we usually set

$$\frac{dy}{dx} = v \quad .$$

Consequently,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}[v] = \frac{dv}{dx} \quad .$$

Under these substitutions, the equation becomes a first-order differential equation for v. And since the original equation did not have a y, neither does the differential equation for v. This means we have a reasonable chance of solving this equation for $v = v(x)$ using methods developed in previous chapters. Then, assuming $v(x)$ can be so obtained,

$$y(x) = \int \frac{dy}{dx}\,dx = \int v(x)\,dx \quad .$$

When solving these equations, you normally end up with a formula involving two distinct arbitrary constants: one from the general solution to the first-order differential equation for v, and the other arising from the integration of v to get y. Don't forget them, and be sure to label them as *different* arbitrary constants.

!►Example 11.1: *Consider the second-order differential equation*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 30e^{3x} \quad .$$

Setting

$$\frac{dy}{dx} = v \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{dv}{dx} \quad ,$$

as suggested above, the differential equation becomes

$$\frac{dv}{dx} + 2v = 30e^{3x} \quad .$$

This is a linear first-order differential equation with integrating factor

$$\mu = e^{\int 2\,dx} = e^{2x} \quad .$$

Proceeding as normal with linear first-order equations,

$$e^{2x}\left[\frac{dv}{dx} + 2v = 30e^{3x}\right]$$

$$\hookrightarrow \qquad e^{2x}\frac{dv}{dx} + 2e^{2x}v = 30e^{3x}e^{2x}$$

$$\hookrightarrow \qquad \frac{d}{dx}\left[e^{2x}v\right] = 30e^{5x}$$

$$\hookrightarrow \qquad \int \frac{d}{dx}\left[e^{2x}v\right]dx = \int 30e^{5x}\,dx$$

$$\hookrightarrow \qquad e^{2x}v = 6e^{5x} + c_0 \quad .$$

Hence,

$$v = e^{-2x}\left[6e^{5x} + c_0\right] = 6e^{3x} + c_0e^{-2x} \quad .$$

But $v = {}^{dy}/_{dx}$ *, so the last equation can be rewritten as*

$$\frac{dy}{dx} = 6e^{3x} + c_0e^{-2x} \quad ,$$

which is easily integrated,

$$y = \int \left[6e^{3x} + c_0e^{-2x}\right]dx = 2e^{3x} - \frac{c_0}{2}e^{-2x} + c_2 \quad .$$

Thus (letting $c_1 = -{}^{c_0}/_2$ *), the solution to our original differential equation is*

$$y(x) = 2e^{3x} - c_1e^{-2x} + c_2 \quad .$$

If your differential equation for v is separable and you are solving as such, don't forget to check for the constant solutions to this differential equation, and to take these "constant-v" solutions into account when integrating $y' = v$.

!►Example 11.2: *Consider the second-order differential equation*

$$\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx} - 3\right)^2 \quad . \tag{11.1}$$

Letting

$$\frac{dy}{dx} = v \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{dv}{dx} \quad ,$$

the differential equation becomes

$$\frac{dv}{dx} = (v-3)^2 \quad . \tag{11.2}$$

This equation has a constant solution,

$$v = 3 \quad ,$$

which we can rewrite as

$$\frac{dy}{dx} = 3 \quad .$$

Integrating then gives us

$$y(x) = 3x + c_0 \quad .$$

This describes all the "constant-v" solutions to our original differential equation.

To find the nonconstant solutions to equation (11.2), divide through by $(v-3)^2$ and integrate:

$$\begin{aligned}
& & \frac{dv}{dx} &= (v-3)^2 \\
&\hookrightarrow & (v-3)^{-2}\frac{dv}{dx} &= -1 \\
&\hookrightarrow & \int (v-3)^{-2}\frac{dv}{dx}\,dx &= -\int 1\,dx \\
&\hookrightarrow & -(v-3)^{-1} &= -x + c_1 \\
&\hookrightarrow & v &= 3 + \frac{1}{x - c_1} \quad .
\end{aligned}$$

But, since $v = y'$, this last equation is the same as

$$\frac{dy}{dx} = 3 + \frac{1}{x - c_1} \quad ,$$

which is easily integrated, yielding

$$y(x) = 3x + \ln|x - c_1| + c_2 \quad .$$

Gathering all the solutions we've found gives us the set consisting of

$$y = 3x + \ln|x - c_1| + c_2 \qquad \text{and} \qquad y(x) = 3x + c_0 \tag{11.3}$$

describing all possible solutions to our original differential equation.

Equations of Even Higher Orders

With just a little imagination, the basic ideas discussed above can be applied to a few differential equations of even higher order. Here is an example:

!►Example 11.3: *Consider the third-order equation*

$$3\frac{d^3y}{dx^3} = \left(\frac{d^2y}{dx^2}\right)^{-2} \quad .$$

Set

$$v = \frac{d^2y}{dx^2} \quad .$$

Then

$$\frac{dv}{dx} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d^3y}{dx^3} \quad ,$$

and the original differential equation reduces to a simple separable first-order equation,

$$3\frac{dv}{dx} = v^{-2} \quad .$$

Multiplying both sides by v^2 *and proceeding as usual with such equations:*

$$3v^2\frac{dv}{dx} = 1$$

$$\hookrightarrow \qquad \int 3v^2\frac{dv}{dx}\,dx = \int 1\,dx$$

$$\hookrightarrow \qquad v^3 = x + c_1$$

$$\hookrightarrow \qquad v = (x+c_1)^{1/3} \quad .$$

So

$$\frac{d^2y}{dx^2} = v = (x+c_1)^{1/3} \quad .$$

Integrating once:

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx = \int (x+c_1)^{1/3}\,dx = \frac{3}{4}(x+c_1)^{4/3} + c_2$$

And once again:

$$y = \int \frac{dy}{dx}\,dx = \int \left[\frac{3}{4}(x+c_1)^{4/3} + c_2\right]dx = \frac{9}{28}(x+c_1)^{7/3} + c_2x + c_3 \quad .$$

Converting a Differential Equations to a System*

Consider what we actually have after taking a second-order differential equation (with y being the yet unknown function and x being the variable) and converting it to a first-order equation for v through the substitution $v = y'$. We actually have a pair of first-order differential equations involving the two unknown functions y and v. The first equation in the pair is simply the equation for the substitution, $y' = v$, and the second is what we obtain after using the substitution with the original second-order equation. If we are lucky, we can directly solve the second equation of this pair. But, as the next example illustrates, we have this pair whether or not either of the above cases apply. Together, this pair forms a "system" of first-order differential equations" whose solution is the pair $y(x)$ and $v(x)$.

!►*Example 11.4:* *Suppose our original differential equation is*

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 5\sin(x)\,y \quad .$$

Setting

$$v = \frac{dy}{dx} \qquad \text{and} \qquad \frac{dv}{dx} = \frac{d^2y}{dx^2} \quad ,$$

the original differential equation reduces to

$$\frac{dv}{dx} + 2xv = 5\sin(x)\,y \quad .$$

Thus, $v = v(x)$ and $y = y(x)$, together, must satisfy both

$$v = \frac{dy}{dx} \qquad \text{and} \qquad \frac{dv}{dx} + 2xv = 5\sin(x)\,y \quad .$$

This is a system of two first-order equations. Traditionally, each equation is rewritten in derivative formula form, and the system is then written as

$$\begin{aligned} \frac{dy}{dx} &= v \\ \frac{dv}{dx} &= 5\sin(x)\,y - 2xv \end{aligned} \quad .$$

As just illustrated, almost any second-order differential equation encountered in practice can be converted to a system of two first-order equations involving two unknown functions. In fact, almost any N^{th}-order differential equation can be converted using similar ideas to a system of N first-order differential equations involving N unknown functions. This is significant because methods exist for dealing with such systems. In many cases, these methods are analogous to methods we used with first-order differential equations. We will discuss some of these methods in the future (the distant future).

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* These comments relate the material in this chapter to more advanced concepts and methods that will be developed much later in this text. This discussion won't help you solve any differential equations now, but will give a little hint of an approach to dealing with higher-order equations we could take (and will explore in the distant future). You might find them interesting. Then, again, you may just want to ignore this discussion for now.

11.2 The Other Class of Second-Order Equations "Easily Reduced" to First-Order†

As noted a few pages ago, the substitution $v = y'$ can also be useful when no formulas of x explicitly appear in the given second-order differential equation. Such second-order differential equations are said to be *autonomous* (this extends the definition of "autonomous" given for first-order differential equations in chapter 3).

Solving Second-Order Autonomous Equations

Unfortunately, if you have an autonomous differential equation, then simply letting

$$v = \frac{dy}{dx}$$

and proceeding as in the previous section leads to a differential equation involving three entities — v, y and x. Unless that equation is *very* simple (say, $dv/dx = 0$), there won't be much you can do with it.

To avoid that impasse, take a second route: Eliminate all references to x by viewing the v as a function of y instead of x. This works because our equation contains no explicit formulas of x. This also means that the substitution for d^2y/dx^2 must be converted, via the above substitution and the chain rule, to a formula involving v and its derivative with respect to y as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}[v] = \frac{dv}{dx} \underset{\substack{\text{chain}\\ \text{rule}}}{=} \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v \quad .$$

The resulting equation will be a first-order differential equation for $v(y)$. Solving that equation, and plugging the resulting formula for $v(y)$ into our original substitution,

$$\frac{dy}{dx} = v(y) \quad ,$$

gives us another first-order differential equation, this time for y. Solving this yields $y(x)$.

Since this use of $v = y'$ is a little less natural than that in the previous section, let us outline the steps more explicitly while doing an example. For our example, we will solve the apparently simple equation

$$\frac{d^2y}{dx^2} + y = 0 \quad .$$

(By the way, this equation will turn out to be rather important to us. We will return to it several times in the next few chapters.)

1. Set

$$\frac{dy}{dx} = v$$

and remember that, by this and the chain rule,

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v \quad .$$

† The material in this section is interesting and occasionally useful, but not essential to the rest of this text. At least give this section a quick skim before going to the discussion of initial-value problems in the next section (and promise to return to this section when convenient or necessary).

Using these substitutions with our example,

$$\frac{d^2y}{dx^2} + y = 0$$

becomes

$$v\frac{dv}{dy} + y = 0 \quad .$$

2. Solve the resulting differential equation for v as a function of y.

For our example:

$$v\frac{dv}{dy} + y = 0$$

$$\hookrightarrow \qquad v\frac{dv}{dy} = -y$$

$$\hookrightarrow \qquad \int v\frac{dv}{dy}\,dy = -\int y\,dy$$

$$\hookrightarrow \qquad \frac{1}{2}v^2 = -\frac{1}{2}y^2 + c_0$$

$$\hookrightarrow \qquad v = \pm\sqrt{2c_0 - y^2} \quad .$$

3. Rewrite the original substitution,

$$\frac{dy}{dx} = v \quad ,$$

replacing the v with the formula just found for v. Then observe that this is another first-order differential equation. In fact, you should notice that it is a separable first-order equation.

Replacing the v in

$$\frac{dy}{dx} = v$$

with the formula just obtained in our example for v, we get

$$\frac{dy}{dx} = \pm\sqrt{2c_0 - y^2} \quad .$$

Why, yes, this is a separable first-order differential equation for y!

4. Solve the first-order differential equation just obtained. This gives the general solution to the original second-order differential equation.

For our example,

$$\frac{dy}{dx} = \pm\sqrt{2c_0 - y^2}$$

$$\hookrightarrow \qquad \frac{1}{\sqrt{2c_0 - y^2}}\frac{dy}{dx} = \pm 1$$

$$\hookrightarrow \qquad \int \frac{1}{\sqrt{2c_0 - y^2}}\frac{dy}{dx}\,dx = \pm\int 1\,dx \quad .$$

Evaluating these integrals (after, perhaps, consulting our old calculus text or a handy table of integrals) yields

$$\arcsin\left(\frac{y}{a}\right) = \pm x + b \quad .$$

where a *and* b *are arbitrary constants with* $a^2 = 2c_0$ *. (Note that* c_0 *had to be positive for square root to make sense.)*

Taking the sine of both sides and recalling that sine is an odd function, we see that

$$\frac{y}{a} = \sin(\pm x + b) = \pm\sin(x \pm b) \quad .$$

Thus, letting $c_1 = \pm a$ *and* $c_2 = \pm b$ *, we have*

$$y(x) = c_1 \sin(x + c_2)$$

as a general solution to our original differential equation,

$$\frac{d^2y}{dx^2} + y = 0 \quad .$$

(Later, after developing more theory, we will find easier ways to solve this and certain similar 'linear' equations.)

A Few Comments

Most of these should be pretty obvious:

1. Again, remember to check for the "constant-v" solutions of any separable differential equation for v .

2. It may be that the original differential equation does not explicitly contain either x or y . If so, then the approach just described and the approach described in the previous section may both be appropriate. Which one you choose is up to you.

3. To be honest, we won't be using the method just outlined in later sections or chapters. Many of the second-order autonomous equations arising in applications are also "linear", and we will develop better methods for dealing with these equations over the next few chapters (where we will also learn just what it means to say that a differential equation is "linear"). I should also mention that, much later, we will develop clever ways to analyze the possible solutions to fairly arbitrary autonomous differential equations after rewriting these equations as systems.

   Still, the method described here is invaluable for completely solving certain autonomous differential equations that are not "linear".

11.3 Initial-Value Problems
Initial Values with Higher-Order Equations

Remember, an N^{th}-order initial-value problem consists of an N^{th}-order differential equation along with the set of assignments

$$y(x_0) = y_0 \quad , \quad y'(x_0) = y_1 \quad , \quad y''(x_0) = y_2 \quad , \quad \ldots \quad \text{and} \quad y^{(N-1)}(x_0) = y_{N-1}$$

where x_0 is some single fixed number and the y_k's are the desired values of the function and its first few derivatives at position x_0.

In particular, a first-order initial-value problem consists of a first-order differential equation with a $y(x_0) = y_0$ initial condition. For example,

$$x\frac{dy}{dx} + 4y = x^3 \qquad \text{with} \quad y(1) = 3 \quad .$$

A second-order initial-value problem consists of a second-order differential equation along with $y(x_0) = y_0$ and $y'(x_0) = y_1$ initial conditions. For example,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 30e^{3x} \qquad \text{with} \quad y(0) = 5 \quad \text{and} \quad y'(0) = 14 \quad .$$

A third-order initial-value problem consists of a third-order differential equation along with $y(x_0) = y_0$, $y'(x_0) = y_1$ and $y''(x_0) = y_2$ initial conditions. For example,

$$3\frac{d^3y}{dx^3} = \left(\frac{d^2y}{dx^2}\right)^{-2} \qquad \text{with} \quad y(0) = 4 \quad , \quad y'(0) = 6 \quad \text{and} \quad y''(0) = 8 \quad .$$

And so on.

Solving Higher-Order Initial-Value Problems
The Basic Approach

The basic procedure for solving a typical higher-order initial-value problem is just about the same as the procedure for solving a first-order initial-value problem. You just need to account for the additional initial values.

First find the general solution to the differential equation. (Though we haven't proven it, you should expect the formula for the general solution to have as many arbitrary/undetermined constants as you have initial conditions.) Use the formula found for the general solution with each initial condition. This creates a system of algebraic equations for the yet-undetermined constants which can be solved for those constants. Solve the system by whatever means you can, and use those values for the constants in the formula for the differential equation's general solution. The resulting formula is the solution to the initial-value problem.

One example should suffice.

!►Example 11.5: *Consider the second-order initial-value problem*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 30e^{3x} \qquad \textit{with} \quad y(0) = 5 \quad \textit{and} \quad y'(0) = 14 \quad .$$

From example 11.1 we already know that

$$y(x) = 2e^{3x} - c_1e^{-2x} + c_2$$

is the general solution to the differential equation. Since the initial conditions include the value of $y'(x)$ *at* $x = 0$*, we will also need the formula for* y'*,*

$$y'(x) = \frac{dy}{dx} = 6e^{3x} + 2c_1e^{-2x} \quad ,$$

which we can obtain by either differentiating the above formula for y *or copying the formula for* y' *from the work done in the example. Combining these formulas with the given initial conditions yields*

$$5 = y(0) = 2e^{3\cdot 0} - c_1e^{-2\cdot 0} + c_2 = 2 - c_1 + c_2$$

and

$$14 = y'(0) = 6e^{3\cdot 0} + 2c_1e^{-2\cdot 0} = 6 + 2c_1 \quad .$$

That is,

$$5 = 2 - c_1 + c_2$$

and

$$14 = 6 + 2c_1 \quad .$$

Doing the obvious arithmetic, we get the system

$$\begin{aligned} -c_1 + c_2 &= 3 \\ 2c_1 &= 8 \end{aligned}$$

of two equations and two unknowns. This is an easy system to solve. From the second equation we immediately see that

$$c_1 = \frac{8}{2} = 4 \quad .$$

Then, solving the first equation for c_2 *and using the value just found for* c_1*, we see that*

$$c_2 = c_1 + 3 = 4 + 3 = 7 \quad .$$

Thus, for $y(x)$ *to satisfy the given differential equation and the two given initial conditions, we must have*

$$y(x) = 2e^{3x} - c_1e^{-2x} + c_2 \qquad \text{with} \quad c_1 = 4 \quad \text{and} \quad c_2 = 7 \quad .$$

That is,

$$y(x) = 2e^{3x} - 4e^{-2x} + 7$$

is the solution to our initial-value problem.

An Alternative Approach

At times, it may be a little easier to determine the values of the arbitrary/undetermined constants "as they arise" in solving the differential equation. This is especially true when using the methods discussed in sections 11.1 and 11.2, where we used the substitution $v = y'$ to convert the differential equation to a first-order differential equation for v. For the sort of equation considered in section 11.1, this substitution immediately gives a first-order initial-value problem with

$$v(x_0) = y'(x_0) = y_1 \quad .$$

For the type of equation considered in section 11.2 (the autonomous differential equations), the initial condition for $v = v(y)$ comes from combining $v = y'$ and the original initial conditions

$$y(x_0) = y_0 \qquad \text{and} \qquad y'(x_0) = y_1$$

into

$$v(y_0) = y'(x_0) = y_1 \quad .$$

Let's do one simple example.

!►Example 11.6: *Consider the initial-value problem*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 30e^{3x} \qquad \textit{with} \quad y(0) = 9 \quad \textit{and} \quad y'(0) = 2 \quad .$$

The differential equation is the same as in example 11.1 on page 243. Letting $v(x) = y'(x)$ yields the first-order initial-value problem

$$\frac{dv}{dx} + 2v = 30e^{3x} \qquad \textit{with} \quad v(0) = y'(0) = 2 \quad .$$

As shown in example 11.1, the general solution to the differential equation for v is

$$v = 6e^{3x} + c_0e^{-2x} \quad .$$

Combining this with the initial value yields

$$2 = v(0) = 6e^{3 \cdot 0} + c_0e^{-\cdot 0} = 6 + c_0 \quad .$$

So, $c_0 = 2 - 6 = -4$, and

$$\frac{dy}{dx} = v = 6e^{3x} - 4e^{-2x} \quad .$$

Integrating this,

$$y(x) = \int \left[6e^{3x} - 4e^{-2x}\right] dx = 2e^{3x} + 2e^{-2x} + c_1 \quad ,$$

and using this formula for y with the initial condition $y(0) = 9$ gives us

$$9 = y(0) = 2e^{3 \cdot 0} + 2e^{-2 \cdot 0} + c_1 = 2 + 2 + c_1 = 4 + c_1 \quad .$$

Thus, $c_1 = 9 - 4 = 5$, and

$$y(x) = 2e^{3x} + 2e^{-2x} + 5$$

is the solution to our initial-value problem.

As the example illustrates, one advantage of this approach is that you only deal with one unknown constant at a time. This approach also by-passes obtaining the general solution to the original differential equation. Consequently, if the general solution is also desired, then the slight advantages of this method are considerably reduced.

11.4 On the Existence and Uniqueness of Solutions

Second-Order Problems

When we were dealing with first-order differential equations, we often found it useful to rewrite a given first-order equation in the derivative formula form,

$$\frac{dy}{dx} = F(x, y) \quad .$$

Extending this form to higher-order differential equations is straightforward. In particular, if we algebraically solve a second-order differential equation for the second derivative, y'', in terms of x, y and the first derivative, y', then we will have rewritten our differential equation in the form

$$y'' = F\left(x, y, y'\right)$$

for some function F of three variables. We will call this the *second-derivative formula form* for the second-order differential equation.

!►*Example 11.7:* *Solving the equation*

$$y'' + 2xy' = 5\sin(x)\,y$$

for the second derivative yields

$$y'' = \underbrace{5\sin(x)\,y - 2xy'}_{F(x,y,y')} \quad .$$

Replacing the derivative on the right with the symbol z, we see that the formula for F is

$$F(x, y, z) = 5\sin(x)\,y - 2xz \quad .$$

To be honest, we won't find the second-derivative formula form particularly useful in solving second-order differential equations until we seriously start dealing with systems of differential equations. It is mentioned here because it is the form used in the following basic theorems on the existence and uniqueness of solutions to second-order initial-value problems:

Theorem 11.1 (existence and uniqueness for second-order initial-value problems)
Let x_0, y_0 and z_0 be three values, and let $F = F(x, y, z)$ be some function of three variables. Assume F and the partial derivatives $\partial F/\partial y$ and $\partial F/\partial z$ are all continuous in some open region containing the point $(x, y, z) = (x_0, y_0, z_0)$. Then the initial-value problem

$$y'' = F\left(x, y, y'\right) \qquad \text{with} \quad y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = z_0$$

has at least one solution $y = y(x)$. Moreover, there is an open interval (α, β) containing x_0 on which this y is the only solution to this initial-value problem.

Theorem 11.2 (existence and uniqueness for second-order initial-value problems)
Let x_0, y_0 and z_0 be three values, and let $F = F(x, y, z)$ be a function of three variables on the infinite slab

$$\mathcal{R} = \{\, (x, y, z) : \alpha < x < \beta \;,\; -\infty < y < \infty \;\text{ and }\; -\infty < z < \infty \,\}$$

where (α, β) *is some open interval containing the point* x_0 *. Assume that, on* $\mathcal{R}$ *, the functions* F *,* ${}^{\partial F}/_{\partial y}$ *and* ${}^{\partial F}/_{\partial z}$ *are all continuous with both* ${}^{\partial F}/_{\partial y}$ *and* ${}^{\partial F}/_{\partial z}$ *being functions of* x *only. Then the initial-value problem*

$$y'' = F\left(x, y, y'\right) \qquad \text{with} \quad y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = z_0 \quad .$$

has exactly one solution $y = y(x)$ *, and this solution is valid over the entire interval* (α, β) *.*

The above theorems are the second-order analogs of the theorems on existence and uniqueness for first-order differential equations given in section 3.3. They assure us that most of the second-order initial-value problems encountered in practice are, in theory, solvable. We will find the second theorem particularly important in rigorously establishing some useful results concerning the general solutions to an important class of second-order differential equations.

Theorems 11.1 and 11.2 can be proven using second-order analogs of the Picard iteration method developed in sections 3.4 and 3.5 for proving the theorems in section 3.3. We won't go into the details here, but will comment that, if you convert the second-order problems in the above theorems to systems of first-order differential equations with initial conditions (as indicated in the short discussion of "systems" beginning on page 246) and make use of the vector notation commonly used in multidimensional calculus, then the proofs given in sections 3.4 and 3.5 can be "easily" converted to proofs for the above theorems.

Problems of Any Order

The biggest difficulty in extending the above existence and uniqueness results for second-order problems to problems of arbitrary order N is that we quickly run out of letters to denote the variables and constants. So we will use subscripts.

Extending the idea of the "derivative formula form" for a first-order differential equation remains trivial: If, given a N^{th}-order differential equation, we can algebraically solve for the N^{th}-order order derivative $y^{(N)}$ in terms of x , y , and the other derivatives of y , then we will say that we've gotten the differential equation into the *highest-order derivative form* for the differential equation,

$$y^{(N)} = F(x, y, y', \ldots, y^{(N-1)}) \quad .$$

Note that F will be a function of $N+1$ variables, which we will denote by x , s_1 , s_2 ,..., s_N .

!►*Example 11.8:* *Solving the equation*

$$y^{(4)} - x^2 y''' - yy'' + 2xy^3 y' = 0$$

for the fourth derivative yields

$$y^{(4)} = \underbrace{x^2 y''' + yy'' - 2xy^3 y'}_{F(x,y,y',y'',y''')} \quad .$$

The formula for F *is then*

$$F(x, s_1, s_2, s_3, s_4) = x^2 s_4 + s_1 s_3 - 2x(s_1)^3 s_2 \quad .$$

Again, the main reason to mention this form is that it is used in the N^{th}-order analogs of the existence and uniqueness theorems given in section 3.3. These analogous theorems are

Theorem 11.3 (existence and uniqueness for N^{th}-order initial-value problems)
Let x_0 , σ_1 , σ_2 , ... and σ_N be fixed values, and let $F = F(x, s_1, s_2, \ldots, s_N)$ be some function of $N + 1$ variables. Assume F and the partial derivatives

$$\frac{\partial F}{\partial s_1} \quad , \quad \frac{\partial F}{\partial s_2} \quad , \quad \ldots \quad \text{and} \quad \frac{\partial F}{\partial s_N}$$

are all continuous functions in some open region containing the point $(x_0, \sigma_1, \sigma_2, \ldots, \sigma_N)$. Then the initial-value problem

$$y^{(N)} = F\left(x, y, y', \ldots, y^{(N-1)}\right)$$

with

$$y(x_0) = \sigma_1 \quad , \quad y'(x_0) = \sigma_2 \quad , \quad \ldots \quad \text{and} \quad y^{(N-1)}(x_0) = \sigma_N$$

has at least one solution $y = y(x)$. Moreover, there is an open interval (α, β) containing x_0 on which this y is the only solution to this initial-value problem.

Theorem 11.4 (existence and uniqueness for N^{th}-order initial-value problems)
Let x_0 , σ_1 , σ_2 , ... and σ_N be fixed values, and let $F = F(x, s_1, s_2, \ldots, s_N)$ be some function of $N + 1$ variables on

$$\mathcal{R} = \{ (x, s_1, s_2, \ldots, s_N) : \alpha < x < \beta \text{ and } -\infty < s_k < \infty \text{ for } k = 1, 2, \ldots, N \}$$

where (α, β) is some open interval containing the point x_0 . Assume F and the partial derivatives

$$\frac{\partial F}{\partial s_1} \quad , \quad \frac{\partial F}{\partial s_2} \quad , \quad \ldots \quad \text{and} \quad \frac{\partial F}{\partial s_N}$$

are all continuous functions on $\mathcal{R}$, and that each of the above partial derivatives depends only on x . Then the initial-value problem

$$y^{(N)} = F\left(x, y, y', \ldots, y^{(N-1)}\right)$$

with

$$y(x_0) = \sigma_1 \quad , \quad y'(x_0) = \sigma_2 \quad , \quad \ldots \quad \text{and} \quad y^{(N-1)}(x_0) = \sigma_N$$

has at exactly one solution $y = y(x)$, and this solution is valid over the entire interval (α, β) .

As with theorems 11.1 and 11.2, the above theorems can be proven using higher-order analogs of the Picard iteration method developed in sections 3.4 and 3.5. Again, we will not go through the details here.

Additional Exercises

11.1. *None of the following second-order equations explicitly contains y . Solve each using the substitution $v = y'$ as described in section 11.1.*

a. $xy'' + 4y' = 18x^2$ **b.** $xy'' = 2y'$

c. $y'' = y'$

d. $y'' + 2y' = 8e^{2x}$

e. $xy'' = y' - 2x^2y'$

f. $(x^2+1)y'' + 2xy' = 0$

11.2. *For each of the following, determine if the given differential equation explicitly contains* y *. If it does not, solve it.*

a. $y'' = 4x\sqrt{y'}$

b. $yy'' = -(y')^2$

c. $y'y'' = 1$

d. $xy'' = (y')^2 - y'$

e. $xy'' - y' = 6x^5$

f. $yy'' - (y')^2 = y'$

g. $y'' = 2y' - 6$

h. $(y-3)y'' = (y')^2$

i. $y'' + 4y' = 9e^{-3x}$

j. $y'' = y'\left(y' - 2\right)$

11.3. *Solve the following higher-order differential equations using the basic ideas from section 11.1 (as done in example 11.3 on page 245):*

a. $y''' = y''$

b. $xy''' + 2y'' = 6x$

c. $y''' = 2\sqrt{y''}$

d. $y^{(4)} = -2y'''$

11.4. *The following second-order equations are all autonomous. Solve each using the substitution* $v = y'$ *as described in section 11.2.*

a. $yy'' = (y')^2$

b. $3yy'' = 2(y')^2$

c. $\sin(y)y'' + \cos(y)(y')^2 = 0$

d. $y'' = y'$

e. $(y')^2 + yy'' = 2yy'$

f. $y^2y'' + y'' + 2y(y')^2 = 0$

11.5. *For each of the following, determine if the given differential equation is autonomous. If it is, then solve it.*

a. $y'' = 4x\sqrt{y'}$

b. $yy'' = -(y')^2$

c. $y'y'' = 1$

d. $xy'' = (y')^2 - y'$

e. $xy'' - y' = 6x^5$

f. $yy'' - (y')^2 = y'$

g. $y'' = 2y' - 6$

h. $(y-3)y'' = (y')^2$

i. $y'' + 4y' = 9e^{-3x}$

j. $y'' = y'\left(y' - 2\right)$

11.6 a. *Solve the following initial-value problems using the general solutions already found for the corresponding differential equations in exercise sets 11.1 and 11.3:*

i. $xy'' + 4y' = 18x^2$ *with* $y(1) = 8$ *and* $y'(1) = -3$

ii. $xy'' = 2y'$ *with* $y(-1) = 4$ *and* $y'(-1) = 12$

iii. $y'' = y'$ *with* $y(0) = 8$ *and* $y'(0) = 5$

iv. $y'' + 2y' = 8e^{2x}$ *with* $y(0) = 0$ *and* $y'(0) = 0$

v. $y''' = y''$ *with* $y(0) = 10$, $y'(0) = 5$ *and* $y''(0) = 2$

vi. $xy''' + 2y'' = 6x$ *with* $y(1) = 2$, $y'(1) = 1$ *and* $y''(1) = 4$

b. *Solve the following initial-value problems:*

i. $xy'' + 2y' = 6$ *with* $y(1) = 4$ *and* $y'(1) = 5$

ii. $2xy'y'' = (y')^2 - 1$ *with* $y(1) = 0$ *and* $y'(1) = \sqrt{3}$

11.7 a. *Solve the following initial-value problems using the general solutions already found for the corresponding differential equations in exercise set 11.4:*

i. $yy'' = (y')^2$ *with* $y(0) = 5$ *and* $y'(0) = 15$

ii. $3yy'' = 2(y')^2$ *with* $y(0) = 8$ *and* $y'(0) = 6$

iii. $3yy'' = 2(y')^2$ *with* $y(1) = 1$ *and* $y'(1) = 9$

b. *Solve the following initial-value problems:*

i. $yy'' + 2(y')^2 = 3yy'$ *with* $y(0) = 2$ *and* $y'(0) = \frac{3}{4}$

ii. $y'' = -y'e^{-y}$ *with* $y(0) = 0$ *and* $y'(0) = 2$

11.8. *In solving a second-order differential equation using the methods described in this chapter, we first solved a first-order differential equation for* $v = y'$*, obtaining a formula for* $v = y'$ *involving an arbitrary constant. Sometimes the value of the first 'arbitrary' constant affects how we solve* $v = y'$ *for* y*. You will illustrate this using*

$$y'' = -2x(y')^2 \tag{11.4}$$

in the following set of problems.

a. *Using the "alternative approach" to solving initial-value problems (as illustrated in example 11.6 on page 252), find the solution to differential equation (11.4) satisfying each of the following sets of initial values:*

i. $y(0) = 3$ *and* $y'(0) = 4$

ii. $y(0) = 3$ *and* $y'(0) = 0$

iii. $y(1) = 0$ *and* $y'(1) = 1$

iv. $y(0) = -\frac{1}{4}$ *and* $y'(1) = 5$

(Observe how different the solutions to these different initial-value problems are, even though they all involve the same differential equation.)

b. *Find the set of all possible solutions to differential equation (11.4).*

11.9. *We will again illustrate the issue raised at the beginning of exercise 11.8, but using differential equation*

$$y'' = 2yy' \quad . \tag{11.5}$$

a. *Using the "alternative approach" to solving initial-value problems (as illustrated in example 11.6 on page 252), find the solution to differential equation (11.5) satisfying each of the following sets of initial values:*

i. $y(0) = 0$ *and* $y'(0) = 1$

ii. $y(0) = 1$ *and* $y'(0) = 1$

iii. $y(0) = 1$ *and* $y'(0) = 0$

iv. $y(0) = 0$ *and* $y'(0) = -1$

(Again, observe how different the solutions to these different initial-value problems are, even though they all involve the same differential equation.)

b. *Find the set of all possible solutions to differential equation (11.5).*