18

Arbitrary Homogeneous Linear Equations with Constant Coefficients

In chapter 16, we saw how to solve any equation of the form

$$ay'' + by' + cy = 0$$

when a, b and c are real constants. Unsurprisingly, the same basic ideas apply when dealing with any equation of the form

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

when N is some positive integer and the a_k's are all real constants. Assuming $y = e^{rx}$ still leads to the corresponding "characteristic equation"

$$a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-2} r^2 + a_{N-1} r + a_N = 0 \quad ,$$

and a general solution to the differential equation can then be obtained using the solutions to the characteristic equation, much as we did in chapter 16. Computationally, the only significant difficulty is in the algebra needed to find the roots of the characteristic polynomial.

So let us look at that algebra, first.

18.1 Some Algebra

A basic fact of algebra is that any second-degree polynomial

$$p(r) = ar^2 + br + c$$

can be factored to

$$p(r) = a(r - r_1)(r - r_2)$$

where r_1 and r_2 are the roots of the polynomial (i.e., the solutions to $p(r) = 0$). These roots may be complex, in which case r_1 and r_2 are complex conjugates of each other (assuming a, b and c are real numbers). It is also possible that $r_1 = r_2$, in which case the factored form of the polynomial is more concisely written as

$$p(r) = a(r - r_1)^2 \quad .$$

The idea of "factoring", of course, extends to polynomials of higher degree. And to use this idea with these polynomials, it will help to introduce the "completely factored form" for an arbitrary N^{th}-degree polynomial

$$p(r) = a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-2} r^2 + a_{N-1} r + a_N \quad .$$

We will say that we've (re)written this polynomial into its *completely factored form* if and only if we've factored it to an expression of the form

$$p(r) = a_0 (r - r_1)^{m_1} (r - r_2)^{m_2} \cdots (r - r_K)^{m_K} \tag{18.1}$$

where

$$\{ r_1, r_2, \ldots, r_K \}$$

is the set of all different (possibly complex) roots of the polynomial (i.e., values of r satisfying $p(r) = 0$), and

$$\{ m_1, m_2, \ldots, m_K \}$$

is some corresponding set of positive integers.

Let's make a few simple observations regarding the above, and then look at a few examples:

1. It will be important for our discussion that
$$\{ r_1, r_2, \ldots, r_K \}$$
is the set of all *different* roots of the polynomial. If $j \neq k$, then $r_j \neq r_k$.

2. Each m_k is the largest integer such that $(r - r_k)^{m_k}$ is a factor of the original polynomial. Consequently, for each r_k, there is only one possible value for m_k. We call m_k the *multiplicity* of r_k.

3. As shorthand, we often say that r_k is a *simple root* if its multiplicity is 1, a *double root* if its multiplicity is 2, a *triple root* if its multiplicity is 3, and so on.

4. If you multiply out all the factors in the completely factored form in line (18.1), you get a polynomial of degree
$$m_1 + m_2 + \cdots + m_K \quad .$$
Since this polynomial is supposed to be $p(r)$, an N^{th}-degree polynomial, we must have
$$m_1 + m_2 + \cdots + m_K = N \quad .$$

!►Example 18.1: *By straightforward multiplication, you can verify that*

$$2(r-4)^3(r+5) = 2r^4 - 14r^3 - 24r^2 + 352r - 640 \quad .$$

This means

$$p(r) = 2r^4 - 14r^3 - 24r^2 + 352r - 640$$

can be written in completely factored form

$$p(r) = 2(r-4)^3(r-[-5]) \quad .$$

This polynomial has two distinct real roots, 4 *and* -5*. The root* 4 *has multiplicity* 3*, and* -5 *is a simple root.*

!► ***Example 18.2:*** *Straightforward multiplication also verifies that*

$$(r-3)^5 = r^5 - 15r^4 + 90r^3 - 270r^2 + 405r - 243 \quad .$$

Thus,

$$r^5 - 15r^4 + 90r^3 - 270r^2 + 405r - 243$$

has completely factored form

$$(r-3)^5 \quad .$$

Here, 3 *is the only distinct root, and this root has multiplicity* 5 *.*

!► ***Example 18.3:*** *As the last example, for now, you can show that*

$$\big(r-[3+4i]\big)^2\big(r-[3-4i]\big)^2 = r^4 - 12r^3 + 86r^2 - 300r + 625 \quad .$$

Hence,

$$\big(r-[3+4i]\big)^2\big(r-[3-4i]\big)^2$$

is the completely factored form for

$$r^4 - 12r^3 + 86r^2 - 300r + 625 \quad .$$

This time we have two complex roots, $3+4i$ *and* $3-4i$ *, with each being a double root.*

Can every polynomial be written in completely factored form? The next theorem says "yes".

Theorem 18.1 (Complete factorization theorem)
Every polynomial can be written in completely factored form.

Note that we are not requiring the coefficients of the polynomial be real. This theorem is valid for every polynomial with real or complex coefficients. Unfortunately, you will have to accept this theorem on faith. Its proof requires developing more theory than is appropriate in this text.[1]

Unfortunately, also, this theorem does not tell us how to find the completely factored form. Of course, if the polynomial is of degree 2,

$$ar^2 + br + c \quad ,$$

then we can find the roots via the quadratic formula,

$$r = r_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

Analogs of this formula do exist for polynomials of degrees 3 and 4, but these analogs are rather complicated, and not often used unless the user is driven by great need. For polynomials of degrees greater than 4, it has been shown that no such analogs exist.

This means that finding the completely factored form may require some of those "tricks for factoring" you learned long ago in your old algebra classes. We'll review a few of those tricks later in examples involving differential equations.

[1] Those who are interested may want to look up the "Fundamental Theorem of Algebra" in a text on complex variables. The complete factorization theorem given here is a corollary of that theorem.

18.2 Solving the Differential Equation

The Characteristic Equation

Suppose we have some N^{th}-order differential equation of the form

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0 \tag{18.2}$$

where the a_k's are all constants (and $a_0 \neq 0$). Since

$$\begin{aligned}
\left(e^{rx}\right)' &= re^{rx} \\
\left(e^{rx}\right)'' &= \left(re^{rx}\right)' = r \cdot re^{rx} = r^2 e^{rx} \\
\left(e^{rx}\right)''' &= \left(r^2 e^{rx}\right)' = r^2 \cdot re^{rx} = r^3 e^{rx} \\
&\ \ \vdots
\end{aligned}$$

for any constant r, it is easy to see that plugging $y = e^{rx}$ into the differential equation yields

$$a_0 r^N e^{rx} + a_1 r^{N-1} e^{rx} + \cdots + a_{N-2} r^2 e^{rx} + a_{N-1} r e^{rx} + a_N e^{rx} = 0 \quad ,$$

which, after dividing out e^{rx}, gives us the corresponding *characteristic equation*

$$a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-2} r^2 + a_{N-1} r + a_N = 0 \quad . \tag{18.3}$$

As before, we refer to the polynomial on the left,

$$p(r) = a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-2} r^2 + a_{N-1} r + a_N \quad ,$$

as the *characteristic polynomial* for the differential equation. Also, as in a previous chapter, it should be observed that the characteristic equation can be obtained from the original differential equation by simply replacing the derivatives of y with the corresponding powers of r.

According to the complete factorization theorem, the above characteristic equation can be rewritten in completely factored form,

$$a_0 (r - r_1)^{m_1} (r - r_2)^{m_2} \cdots (r - r_K)^{m_K} = 0 \tag{18.4}$$

where the r_k's are all the different roots of the characteristic polynomial, and the m_k's are the multiplicities of the corresponding roots. It turns out that, for each root r_k with multiplicity m_k, we can identify a corresponding linearly independent set of m_k particular solutions to the original differential equation. It will be obvious (once you see them) that no solution generated from one root can be written as a linear combination of solutions generated from the other roots. Hence, the set of all these particular solutions generated from all the r_k's will be a linearly independent set containing (according to our complete factorization theorem)

$$m_1 + m_2 + \cdots + m_K = N$$

solutions. From the big theorem on solutions to homogeneous equations (theorem 14.2 on page 304), we then know that this big set is a fundamental set of solutions for the differential equation, and that the general solution is given by an arbitrary linear combination of these particular solutions.

Exactly which particular solutions are generated from each individual root depends on the multiplicity and whether the root is real valued or not.

Particular Solutions Corresponding to One Root

In the following, we will assume r_k is a root of multiplicity m_k to our characteristic polynomial. That is,

$$(r - r_k)^{m_k}$$

is one factor in equation (18.4). However, since the choice of k will be irrelevant in this discussion, we will, for simplicity, drop the subscripts.

The Basic Result

Assume r is a root of multiplicity m to our characteristic polynomial. Then, as before,

$$e^{rx}$$

is one particular solution to the differential equation, and if $m = 1$, it is the only solution corresponding to this root we need to find.

So now assume $m > 1$. In the previous chapter, we found that

$$x\,e^{rx}$$

is a second solution to the differential equation when r is a repeated root and $N = 2$. This was obtained via reduction of order. For the more general case being considered here, it can be shown that $x\,e^{rx}$ is still a solution. In fact, it can be shown that the m particular solutions to the differential equation corresponding to root r can be generated one after the other by simply multiplying the previously found solution by x. That is, we have the following theorem:

Theorem 18.2
Let r be a root of multiplicity m to the characteristic polynomial for

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

where the a_k's are all constants. Then

$$\left\{ e^{rx}\,,\ x\,e^{rx}\,,\ x^2 e^{rx}\,,\ \ldots\,,\ x^{m-1} e^{rx} \right\}$$

is a linearly independent set of m solutions to the differential equation.

The proof of this theorem will be discussed later, in section 18.4. (And it probably should be noted that $x^m e^{rx}$ ends up *not* being a solution to the differential equation.)

Particular Solutions Corresponding to a Real Root

If r is a real root of multiplicity m to our characteristic polynomial, then theorem 18.2, above, tells us that

$$\left\{ e^{rx}\,,\ x\,e^{rx}\,,\ x^2 e^{rx}\,,\ \ldots\,,\ x^{m-1} e^{rx} \right\}$$

is the linearly independent set of m solutions to the differential equation corresponding to that root. No more need be said.

!►*Example 18.4:* *Consider the homogeneous differential equation*

$$y^{(5)} - 15y^{(4)} + 90y''' - 270y'' + 405y' - 243y = 0 \quad .$$

Its characteristic equation is

$$r^5 - 15r^4 + 90r^3 - 270r^2 + 405r - 243 = 0 \quad .$$

The left side of this equation is the polynomial from example 18.2. Checking back at that example, we discover that this characteristic equation can be factored to

$$(r-3)^5 = 0 \quad .$$

So 3 *is the only root, and it has multiplicity* 5*. Theorem 18.2 then tells us that the linearly independent set of* 5 *corresponding particular solutions to the differential equation is*

$$\left\{ e^{3x} , \; xe^{3x} , \; x^2e^{3x} , \; x^3e^{3x} , \; x^4e^{3x} \right\}$$

Since 5 *is also the order of the differential equation, we know (via theorem 14.2 on page 304) that the above set is a fundamental set of solutions to our homogeneous differential equation, and, thus,*

$$y(x) = c_1e^{3x} + c_2xe^{3x} + c_3x^2e^{3x} + c_4x^3e^{3x} + c_5x^4e^{3x}$$

is a general solution for our differential equation.

Particular Solutions Corresponding to a Complex Root

In chapter 16 we observed that complex roots to a second-degree polynomial always occur as a conjugate pair when the coefficients of the polynomial are real. With a little bit of work (see section 18.5), we can extend that observation to:

Theorem 18.3
Consider a polynomial

$$p(r) = a_0r^N + a_1r^{N-1} + \cdots + a_{N-1}r + a_N$$

in which a_0 *,* a_1 *, ... , and* a_N *are all real numbers. Let* λ *and* ω *be two real numbers, and let* m *be some positive integer. Then*

$$r_0 = \lambda + i\omega \quad \text{is a root of multiplicity } m \text{ for polynomial } p(r)$$

if and only if

$$r_0{}^* = \lambda - i\omega \quad \text{is a root of multiplicity } m \text{ for polynomial } p(r) \quad .$$

Now assume $\lambda + i\omega$ is a complex root of multiplicity m to our characteristic polynomial. Theorems 18.2 and 18.3, together, tell us that

$$\left\{ e^{(\lambda+i\omega)x} , \; x\,e^{(\lambda+i\omega)x} , \; x^2e^{(\lambda+i\omega)x} , \; \ldots , \; x^{m-1}e^{(\lambda+i\omega)x} \right\}$$

and

$$\left\{ e^{(\lambda-i\omega)x} , \; x\,e^{(\lambda-i\omega)x} , \; x^2e^{(\lambda-i\omega)x} , \; \ldots , \; x^{m-1}e^{(\lambda-i\omega)x} \right\}$$

are linearly independent sets of m solutions to the differential equation corresponding to roots $\lambda + i\omega$ and $\lambda - i\omega$.

For many problems, though, these are not particularly desirable sets of solutions because they introduce complex values into computations we expect to yield real values. But recall how we dealt with complex-exponential solutions for second-order equations, constructing alternative pairs of solutions via linear combinations. Let us try the same idea here, constructing an alternative pair of solutions $\{y_{k,1}, y_{k,2}\}$ from each pair

$$\left\{ x^k e^{(\lambda+i\omega)x} \,,\, x^k e^{(\lambda-i\omega)x} \right\}$$

by the linear combinations

$$y_{k,1}(x) \;=\; \frac{1}{2} x^k e^{(\lambda+i\omega)x} \;+\; \frac{1}{2} x^k e^{(\lambda-i\omega)x}$$

and

$$y_{k,2}(x) \;=\; \frac{1}{2i} x^k e^{(\lambda+i\omega)x} \;-\; \frac{1}{2i} x^k e^{(\lambda-i\omega)x} \quad .$$

Since

$$e^{(\lambda\pm i\omega)x} \;=\; e^{\lambda x}\left[\cos(\omega x) \mp i\sin(\omega x)\right] \quad ,$$

you can easily verify that

$$y_{k,1} \;=\; x^k e^{\lambda x}\cos(\omega x) \qquad \text{and} \qquad y_{k,2} \;=\; x^k e^{\lambda x}\sin(\omega x) \quad .$$

It is also "easily" verified that the set of these functions, with $k = 0, 1, 2, \ldots, m-1$, is linearly independent.

Thus, instead of using

$$\left\{ e^{(\lambda+i\omega)x} \,,\, x\,e^{(\lambda+i\omega)x} \,,\, x^2 e^{(\lambda+i\omega)x} \,,\, \ldots \,,\, x^{m-1} e^{(\lambda+i\omega)x} \right\}$$

and

$$\left\{ e^{(\lambda-i\omega)x} \,,\, x\,e^{(\lambda-i\omega)x} \,,\, x^2 e^{(\lambda-i\omega)x} \,,\, \ldots \,,\, x^{m-1} e^{(\lambda-i\omega)x} \right\}$$

as the two linearly independent sets corresponding to roots $\lambda + i\omega$ and $\lambda - i\omega$, we can use the sets of real-valued functions

$$\left\{ e^{\lambda x}\cos(\omega x) \,,\, x\,e^{\lambda x}\cos(\omega x) \,,\, x^2 e^{\lambda x}\cos(\omega x) \,,\, \ldots \,,\, x^{m-1} e^{\lambda x}\cos(\omega x) \right\}$$

and

$$\left\{ e^{\lambda x}\sin(\omega x) \,,\, x\,e^{\lambda x}\sin(\omega x) \,,\, x^2 e^{\lambda x}\sin(\omega x) \,,\, \ldots \,,\, x^{m-1} e^{\lambda x}\sin(\omega x) \right\} \quad .$$

!►*Example 18.5:* *Consider the differential equation*

$$y^{(4)} \;-\; 12y^{(3)} \;+\; 86y'' \;-\; 300y' \;+\; 625y \;=\; 0 \quad .$$

Its characteristic equation is

$$r^4 \;-\; 12r^3 \;+\; 86r^2 \;-\; 300r \;+\; 625 \;=\; 0 \quad ,$$

which, as we saw in example 18.3, can be factored to

$$\left(r - [3+4i]\right)^2 \left(r - [3-4i]\right)^2 \;=\; 0 \quad .$$

Here, we have a conjugate pair of roots, $3 + 4i$ *and* $3 - 4i$ *, each with multiplicity* 2 *. So the corresponding particular real-valued solutions to the differential equation are*

$$e^{3x}\cos(4x) \quad , \quad xe^{3x}\cos(4x) \quad , \quad e^{3x}\sin(4x) \quad \text{and} \quad xe^{3x}\sin(4x) \quad .$$

And since our homogeneous, linear differential equation is of order 4, its general solution is given by an arbitrary linear combination of these four solutions,

$$y(x) = c_1 e^{3x}\cos(4x) + c_2 x e^{3x}\cos(4x) + c_3 e^{3x}\sin(4x) + c_4 x e^{3x}\sin(4x) \quad ,$$

which, to save space, might also be written as

$$y(x) = [c_1 + c_2 x]e^{3x}\cos(4x) + [c_3 + c_4 x]e^{3x}\sin(4x) \quad .$$

18.3 Some More Examples

The most difficult part of solving a high-order, homogeneous linear differential equation with constant coefficients is the factoring of its characteristic polynomial. Unfortunately, the methods commonly used to factor second-degree polynomials do not nicely generalize to methods for factoring polynomials of higher degree. So we have to use whatever algebraic tricks we can think of. And if all else fails, we can run to the computer and let our favorite math package attempt the factoring.

Here are a few examples to help you recall some of the useful tricks for factoring polynomials of order three or above.

!► *Example 18.6:* *Consider the seventh-order, homogeneous differential equation*

$$y^{(7)} - 625y^{(3)} = 0 \quad .$$

The characteristic equation is

$$r^7 - 625r^3 = 0 \quad .$$

An obvious choice of action would be to first factor out r^3*,*

$$r^3\left(r^4 - 625\right) = 0 \quad .$$

Cleverly noting that $r^4 = [r^2]^2$ *and* $625 = 25^2$*, and then applying well-known algebraic formulas, we have*

$$r^3\left(r^4 - 625\right) = 0$$

$$\hookrightarrow \qquad r^3\left([r^2]^2 - [25]^2\right) = 0$$

$$\hookrightarrow \qquad r^3\left(r^2 - 25\right)\left(r^2 + 25\right) = 0$$

$$\hookrightarrow \qquad r^3(r-5)(r+5)\left(r^2 + 25\right) = 0 \quad .$$

Now

$$r^2 + 25 = 0 \quad \Longrightarrow \quad r^2 = -25 \quad \Longrightarrow \quad r^2 = \pm\sqrt{-25} = \pm 5i \quad .$$

So our characteristic equation can be written as

$$r^3(r-5)(r+5)(r-5i)(r+5i) \;=\; 0 \quad .$$

To be a little more explicit,

$$(r-0)^3(r-5)\big(r-[-5]\big)(r-5i)\big(r-[-5i]\big) \;=\; 0 \quad .$$

Thus, our characteristic polynomial has 5 different roots:

$$0 \quad , \quad 5 \quad , \quad -5 \quad , \quad 5i \quad \text{and} \quad -5i \quad .$$

The root 0 has multiplicity 3, and the differential equation has corresponding particular solutions

$$e^{0\cdot x} \qquad , \qquad xe^{0\cdot x} \qquad \text{and} \qquad x^2e^{0\cdot x} \quad ,$$

which most of us would rather write as

$$1 \qquad , \qquad x \qquad \text{and} \qquad x^2 \quad .$$

The roots 5 and -5 each have multiplicity 1. So the differential equation has corresponding particular solutions

$$e^{5x} \qquad \text{and} \qquad e^{-5x} \quad .$$

Finally, we have a pair of complex roots $5i$ and $-5i$, each with multiplicity 1. Since these are of the form $\lambda \pm i\omega$ with $\lambda = 0$ and $\omega = 5$, the corresponding real-valued particular solutions to our differential equation are

$$\cos(5x) \qquad \text{and} \qquad \sin(5x) \quad .$$

Taking an arbitrary linear combination of the above seven particular solutions, we get

$$y(x) \;=\; c_1\cdot 1 \;+\; c_2x \;+\; c_3x^2 \;+\; c_4e^{5x} \;+\; c_5e^{-5x} \;+\; c_6\cos(5x) \;+\; c_7\sin(5x)$$

as a general solution to our differential equation.

!► ***Example 18.7:*** Consider

$$y''' \;-\; 19y' \;+\; 30y \;=\; 0 \quad .$$

The characteristic equation is

$$r^3 \;-\; 19r \;+\; 30 \;=\; 0 \quad .$$

Few people can find a first factoring of this characteristic polynomial,

$$p(r) \;=\; r^3 \;-\; 19r \;+\; 30 \quad ,$$

by inspection. But remember,

$$(r-r_1) \text{ is a factor of } p(r) \quad \Longleftrightarrow \quad p(r_1) \;=\; 0 \quad .$$

This means we can test candidates for r_1 by just seeing if $p(r_1) = 0$. Good candidates here would be the integer factors of 30 (± 1, ± 2, ± 3, ± 5, ± 6, ± 10 and ± 15).

Trying $r_1 = 1$, *we get*

$$p(1) = 1^3 - 19 \cdot 1 + 30 = 12 \neq 0 \quad .$$

So $r_1 \neq 1$.

Trying $r_1 = -1$, *we get*

$$p(-1) = (-1)^3 - 19 \cdot (-1) + 30 = -1 + 19 + 30 \neq 0 \quad .$$

So $r_1 \neq -1$.

Trying $r_1 = 2$, *we get*

$$p(2) = (2)^3 - 19 \cdot (2) + 30 = 8 - 38 + 30 = 0 \quad .$$

Success! One root is $r_1 = 2$ *and one factor of our characteristic polynomial is* $(r-2)$. *To get our first factoring, we then divide* $r - 2$ *into the characteristic polynomial:*

$$\begin{array}{rl}
 & r^2 + 2r - 15 \\ \cline{2-2}
r - 2 \big) & r^3 \qquad\quad - 19r + 30 \\
 & -r^3 + 2r^2 \\ \cline{2-2}
 & 2r^2 - 19r \\
 & -2r^2 + 4r \\ \cline{2-2}
 & -15r + 30 \\
 & 15r - 30 \\ \cline{2-2}
 & 0
\end{array}$$

Thus,

$$r^3 - 19r + 30 = (r-2)(r^2 + 2r - 15) \quad .$$

By inspection, we see that

$$r^2 + 2r - 15 = (r+5)(r-3) \quad .$$

So, our characteristic equation

$$r^3 - 19r + 30 = 0$$

factors to

$$(r-2)(r-[-5])(r-3) = 0 \quad ,$$

and, thus,

$$y(x) = c_1 e^{2x} + c_2 e^{-5x} + c_3 e^{3x}$$

is a general solution to our differential equation.

18.4 On Verifying Theorem 18.2

Theorem 18.2 claims to give a linearly independent set of solutions to a linear homogeneous differential equation with constant coefficients corresponding to a repeated root for the equation's characteristic polynomial. Our task of verifying this claim will be greatly simplified if we slightly expand our discussion of "factoring" linear differential operators from section 12.4. (You may want to go back and quickly review that section.)

Linear Differential Operators with Constant Coefficients

First, we need to expand our terminology a little: When we refer to L as being an N^{th}*-order linear differential operator with constant coefficients*, we just mean that L is an N^{th}-order linear differential operator

$$L = a_0 \frac{d^N}{dx^N} + a_1 \frac{d^{N-1}}{dx^{N-1}} + \cdots + a_{N-2} \frac{d^2}{dx^2} + a_{N-1} \frac{d}{dx} + a_N$$

in which all the a_k's are constants. Its *characteristic polynomial* $p(r)$ is simply the polynomial

$$p(r) = a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-2} r^2 + a_{N-1} r + a_N \quad .$$

It turns out that factoring a linear differential operator with constant coefficients is remarkably easy if you already have the factorization for its characteristic polynomial.

!►*Example 18.8:* *Consider the linear differential operator*

$$L = \frac{d^2}{dx^2} - 5\frac{d}{dx} + 6 \quad .$$

It's characteristic polynomial is

$$r^2 - 5r + 6 \quad ,$$

which factors to

$$(r-2)(r-3) \quad .$$

Now, consider the analogous composition product

$$\left(\frac{d}{dx} - 2\right)\left(\frac{d}{dx} - 3\right) \quad .$$

Letting ϕ *be any suitably differentiable function, we see that*

$$\begin{aligned}
\left(\frac{d}{dx} - 2\right)\left(\frac{d}{dx} - 3\right)[\phi] &= \left(\frac{d}{dx} - 2\right)\left[\frac{d\phi}{dx} - 3\phi\right] \\
&= \frac{d}{dx}\left[\frac{d\phi}{dx} - 3\phi\right] - 2\left[\frac{d\phi}{dx} - 3\phi\right] \\
&= \frac{d^2\phi}{dx^2} - 3\frac{d\phi}{dx} - 2\frac{d\phi}{dx} + 6\phi \\
&= \frac{d^2\phi}{dx^2} - 5\frac{d\phi}{dx} + 6\phi \\
&= L[\phi] \quad .
\end{aligned}$$

Thus,

$$L = \left(\frac{d}{dx} - 2\right)\left(\frac{d}{dx} - 3\right) \quad .$$

Redoing this last example with the numbers 2 and 3 replaced by constants r_1 and r_2 leads to the first result of this section:

Lemma 18.4

Let L be a second-order linear differential operator with constant coefficients,

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad ,$$

and let

$$p(r) = a(r - r_1)(r - r_2)$$

be a factorization of the characteristic polynomial for L (the roots r_1 and r_2 need not be different, nor must they be real). Then the operator L has factorization

$$L = a\left(\frac{d}{dx} - r_1\right)\left(\frac{d}{dx} - r_2\right) \quad .$$

PROOF: First of all, by the definition of p and elementary algebra,

$$ar^2 + br + c = p(r) = a(r - r_1)(r - r_2) = ar^2 - a[r_1 + r_2]r + ar_1r_2 \quad .$$

So,

$$b = -a[r_1 + r_2] \qquad \text{and} \qquad c = ar_1r_2 \quad .$$

Now, let ϕ be any sufficiently differentiable function. By the above,

$$\begin{aligned}
a\left(\frac{d}{dx} - r_1\right)\left(\frac{d}{dx} - r_2\right)[\phi] &= a\left(\frac{d}{dx} - r_1\right)\left[\frac{d\phi}{dx} - r_2\phi\right] \\
&= a\left(\frac{d}{dx}\left[\frac{d\phi}{dx} - r_2\phi\right] - r_1\left[\frac{d\phi}{dx} - r_2\phi\right]\right) \\
&= a\left(\frac{d^2\phi}{dx^2} - r_2\frac{d\phi}{dx} - r_1\frac{d\phi}{dx} + r_1r_2\phi\right) \\
&= a\frac{d^2\phi}{dx^2} - a[r_1 + r_2]\frac{d\phi}{dx} + ar_1r_2\phi \\
&= a\frac{d^2\phi}{dx^2} + b\frac{d\phi}{dx} + c\phi \\
&= L[\phi] \quad .
\end{aligned}$$

∎

Clearly, straightforward extensions of these arguments will show that, for any factorization of the characteristic polynomial of any linear differential operator with constant coefficients, there is a corresponding factorization of that operator. To simplify writing the factors of the operator when r_k is a multiple root of the characteristic polynomial, let us agree that

$$\begin{aligned}
\left(\frac{d}{dx} - r_1\right)^0 &= 1 \quad , \\
\left(\frac{d}{dx} - r_1\right)^1 &= \left(\frac{d}{dx} - r_1\right) \quad , \\
\left(\frac{d}{dx} - r_1\right)^2 &= \left(\frac{d}{dx} - r_1\right)\left(\frac{d}{dx} - r_1\right) \quad , \\
\left(\frac{d}{dx} - r_1\right)^3 &= \left(\frac{d}{dx} - r_1\right)\left(\frac{d}{dx} - r_1\right)\left(\frac{d}{dx} - r_1\right) \quad , \\
&\vdots
\end{aligned}$$

Using this notation along with the obvious extension of the above proof yields the next theorem.

Theorem 18.5 (factorization of constant coefficient operators)
Let L be a linear differential operator with constant coefficients, and let

$$p(r) = a(r - r_1)^{m_1}(r - r_2)^{m_2} \cdots (r - r_K)^{m_K}$$

be the completely factored form for the characteristic polynomial for L. Then

$$L = a\left(\frac{d}{dx} - r_1\right)^{m_1}\left(\frac{d}{dx} - r_2\right)^{m_2} \cdots \left(\frac{d}{dx} - r_K\right)^{m_K} \quad .$$

Let us make two observations regarding the polynomial p, one of the roots r_j of this polynomial, and the operator L from the last theorem:

1. Because the order in which we write the factors of a polynomial is irrelevant, we have

$$p(r) = a(r - r_1)^{m_1}(r - r_2)^{m_2} \cdots (r - r_K)^{m_K} = ap_j(r)(r - r_j)^{m_j}$$

where $p_j(r)$ is the product of all the $(r - r_k)$'s with $r_k \neq r_j$. Hence, L can be factored by

$$L = aL_j\left(\frac{d}{dx} - r_j\right)^{m_j}$$

where L_j is the composition product of all the $\left(d/dx - r_k\right)$'s with $r_k \neq r_j$.

2. If y is a solution to

$$\left(\frac{d}{dx} - r_j\right)^{m_j}[y] = 0 \quad ,$$

then

$$L[y] = aL_j\left(\frac{d}{dx} - r_j\right)^{m_j}[y] = aL_j\left[\left(\frac{d}{dx} - r_j\right)^{m_j}[y]\right] = aL_j[0] = 0 \quad .$$

Together, these observations give us the actual result we will use.

Corollary 18.6

Let L be a linear differential operator with constant coefficients; let

$$p(r) = a(r - r_1)^{m_1}(r - r_2)^{m_2} \cdots (r - r_K)^{m_K}$$

be the completely factored form for the characteristic polynomial for L, and let r_j be any one of the roots of p. Suppose, further, that y is a solution to

$$\left(\frac{d}{dx} - r_j\right)^{m_j}[y] = 0 \quad .$$

Then y is also solution to

$$L[y] = 0 \quad .$$

Proof of Theorem 18.2

Theorem 18.2 claims that, if r is a root of multiplicity m for the characteristic polynomial of some linear homogeneous differential equation with constant coefficients, then

$$\left\{ e^{rx} , x e^{rx} , x^2 e^{rx} , \ldots , x^{m-1} e^{rx} \right\}$$

is a linearly independent set of solutions to that differential equation. First we will verify that each of these $x^k e^{rx}$'s is a solution to the differential equation. Then we will confirm the linear independence of this set.

Verifying the Solutions

If you look back at corollary 18.6, you will see that we need only show that

$$\left(\frac{d}{dx} - r\right)^{m}[x^k e^{rx}] = 0 \tag{18.5}$$

whenever k is a nonnegative integer less than m. To expedite our main computations, we'll do two preliminary computations. And, since at least one may be useful in a later chapter, we'll describe the results in an easily referenced lemma.

Lemma 18.7

Let r, α and β be constants with α being a positive integer and β being real valued. Then

$$\left(\frac{d}{dx} - r\right)^{\alpha}\left[e^{rx}\right] = 0 \qquad \text{and} \qquad \left(\frac{d}{dx} - r\right)^{\alpha}\left[x^{\beta} e^{rx}\right] = \beta\left(\frac{d}{dx} - r\right)^{\alpha-1}\left[x^{\beta-1} e^{rx}\right] \quad .$$

PROOF: For the first:

$$\left(\frac{d}{dx} - r\right)^{\alpha}\left[e^{rx}\right] = \left(\frac{d}{dx} - r\right)^{\alpha-1}\left(\frac{d}{dx} - r\right)\left[e^{rx}\right]$$

$$= \left(\frac{d}{dx} - r\right)^{\alpha-1} \left[re^{rx} - re^{rx}\right] \; = \; \left(\frac{d}{dx} - r\right)^{\alpha-1} [0] \; = \; 0 \quad .$$

For the second:

$$\begin{aligned}
\left(\frac{d}{dx} - r\right)^{\alpha} \left[x^{\beta} e^{rx}\right] \; &= \; \left(\frac{d}{dx} - r\right)^{\alpha-1} \left(\frac{d}{dx} - r\right) \left[x^{\beta} e^{rx}\right] \\
&= \; \left(\frac{d}{dx} - r\right)^{\alpha-1} \left[\frac{d}{dx}\left[x^{\beta} e^{rx}\right] - rx^{\beta} e^{rx}\right] \\
&= \; \left(\frac{d}{dx} - r\right)^{\alpha-1} \left[\beta x^{\beta-1} e^{rx} + x^{\beta} re^{rx} - rx^{\beta} e^{rx}\right] \\
&= \; \beta \left(\frac{d}{dx} - r\right)^{\alpha-1} \left[x^{\beta-1} e^{rx}\right] \quad .
\end{aligned}$$

$\blacksquare$

Now let k be an positive integer less than m. Using the above lemma, we see that

$$\begin{aligned}
\left(\frac{d}{dx} - r\right)^{m} \left[x^{k} e^{rx}\right] \; &= \; k \left(\frac{d}{dx} - r\right)^{m-1} \left[x^{k-1} e^{rx}\right] \\
&= \; k(k-1) \left(\frac{d}{dx} - r\right)^{m-2} \left[x^{k-2} e^{rx}\right] \\
&= \; k(k-1)(k-2) \left(\frac{d}{dx} - r\right)^{m-3} \left[x^{k-3} e^{rx}\right] \\
&\;\;\vdots \\
&= \; k(k-1)(k-2)\cdots(k-[k-1]) \left(\frac{d}{dx} - r\right)^{m-k} \left[x^{k-k} e^{rx}\right] \\
&= \; k! \left(\frac{d}{dx} - r\right)^{m-k} \left[e^{rx}\right] \\
&= \; 0 \quad ,
\end{aligned}$$

verifying equation (18.5).

Verifying Linear Independence

To finish verifying the claim of theorem 18.2, we need only confirm that

$$\left\{\, e^{rx} \,,\; x\, e^{rx} \,,\; x^2 e^{rx} \,,\; \ldots \,,\; x^{m-1} e^{rx} \,\right\}$$

is a linearly independent set of functions on the real line. Well, let's ask if this set could be, instead, a linearly *dependent* set of functions on the real line. Then one of these functions, say, $x^{\kappa} e^{rx}$, would be a linear combination of the others,

$$x^{\kappa} e^{rx} \; = \; \text{linear combination of the other } x^{k} e^{rx}\text{'s} \quad .$$

Subtract $x^{\kappa} e^{rx}$ from both sides, and you get

$$0 \; = \; \text{linear combination of the other } x^{k} e^{rx}\text{'s} \; - \; 1 \cdot x^{\kappa} e^{rx} \quad ,$$

which we can rewrite as

$$0 \;=\; c_0 e^{rx} \;+\; c_1 x\, e^{rx} \;+\; c_2 x^2 e^{rx} \;+\; \cdots \;+\; c_m x^{m-1} e^{rx}$$

where the c_k's are constants with

$$c_\kappa \;=\; -1 \quad .$$

Dividing out e^{rx} reduces the above to

$$0 \;=\; c_0 \;+\; c_1 x \;+\; c_2 x^2 \;+\; \cdots \;+\; c_{m-1} x^{m-1} \quad . \tag{18.6}$$

Since this is supposed to hold for all x, it should hold for $x = 0$, giving us

$$0 \;=\; c_0 \;+\; c_1 \cdot 0 \;+\; c_2 \cdot 0^2 \;+\; \cdots \;+\; c_{m-1} \cdot 0^{m-1} \;=\; c_0 \quad .$$

Now differentiate both sides of equation (18.6) and plug in $x = 0$:

$$\frac{d}{dx}[0] \;=\; \frac{d}{dx}\left[c_0 \;+\; c_1 x \;+\; c_2 x^2 \;+\; \cdots \;+\; c_{m-1} x^{m-1}\right]$$

$$\hookrightarrow \qquad 0 \;=\; 0 \;+\; c_1 \;+\; 2c_2 x \;+\; \cdots \;+\; (m-1) c_{m-1} x^{m-2}$$

$$\hookrightarrow \qquad 0 \;=\; 0 \;+\; c_1 \;+\; 2c_2 \cdot 0 \;+\; \cdots \;+\; (m-1) c_{m-1} \cdot 0^{m-2}$$

$$\hookrightarrow \qquad 0 \;=\; c_1 \quad .$$

Differentiating both sides of equation (18.6) twice and plugging in $x = 0$:

$$\frac{d^2}{dx^2}[0] \;=\; \frac{d^2}{dx^2}\left[c_0 \;+\; c_1 x \;+\; c_2 x^2 \;+\; c_3 x^3 \cdots \;+\; c_{m-1} x^{m-1}\right]$$

$$\hookrightarrow \qquad 0 \;=\; \frac{d}{dx}\left[0 \;+\; c_1 \;+\; 2c_2 x \;+\; 3c_3 x^2 \cdots \;+\; (m-1) c_{m-1} x^{m-2}\right]$$

$$\hookrightarrow \qquad 0 \;=\; 0 \;+\; 0 \;+\; 2c_2 \;+\; 6c_3 x \cdots \;+\; (m-1) c_{m-1} (m-2) x^{m-2}$$

$$\hookrightarrow \qquad 0 \;=\; 0 \;+\; 0 \;+\; 2c_2 \;+\; 6c_3 \cdot 0 \;+\; \cdots \;+\; (m-1)(m-2) c_{m-1} \cdot 0^{m-2}$$

$$\hookrightarrow \qquad 0 \;=\; c_2 \quad .$$

Clearly, we can differentiate equation (18.6) again and again, plug in $x = 0$, and, eventually, obtain

$$0 \;=\; c_k \qquad \text{for} \quad k \;=\; 0,\, 1,\, 2,\, \ldots,\, m-1 \quad .$$

But, one of these c_k's is c_κ which we know is -1 (assuming our set of $x^k e^{rx}$'s is linearly *dependent*). In other word, for our set of $x^k e^{rx}$'s to be linearly *dependent*, we must have

$$0 \;=\; c_\kappa \;=\; -1 \quad ,$$

which is impossible. So our set of $x^k e^{rx}$'s cannot be linearly dependent. It must be linearly independent, just as theorem 18.2 claimed.

18.5 On Verifying Theorem 18.3

Theorem 18.3 is a theorem about complex conjugation in the algebra of complex numbers. So let's start with a brief discussion of that topic.

Algebra with Complex Conjugates

Recall that a complex number z is something that can be written as

$$z = x + iy$$

where x and y are real numbers, which we generally refer to as, respectively, the *real* and the *imaginary parts* of z. Along these lines, we say z is *real* if and only if $z = x$ (i.e., $y = 0$), and we say z is *imaginary* if and only if $z = iy$ (i.e., $x = 0$).

The corresponding *complex conjugate* of z — denoted z^* — is z with the sign of its complex part switched,

$$z = x + iy \quad \Longrightarrow \quad z^* = x + i(-y) = x - iy \quad .$$

Note that

$$\left(z^*\right)^* = (x - iy)^* = x + iy = z \quad ,$$

and that

$$z^* = z \qquad \text{if } z \text{ is real} \quad .$$

We will use these facts in a moment. We will also use formulas involving the complex conjugates of sums and products. To derive them, let

$$z = x + iy \qquad \text{and} \qquad c = a + ib$$

where x, y, a and b are all real, and compute out

$$(c + z)^* \quad , \quad c^* + z^* \quad , \quad (cz)^* \quad \text{and} \quad c^* z^*$$

in terms of a, b, x and y. You'll quickly discover that

$$(c + z)^* = c^* + z^* \qquad \text{and} \qquad (cz)^* = c^* z^* \quad .$$

It then follows that

$$\begin{aligned}
\left(z^2\right)^* &= (z \cdot z)^* = z^* \cdot z^* = \left(z^*\right)^2 \quad , \\
\left(z^3\right)^* &= \left(z^2 \cdot z\right)^* = \left(z^2\right)^* \cdot z^* = \left(z^*\right)^2 \cdot z^* = \left(z^*\right)^3 \quad , \\
&\vdots
\end{aligned}$$

Continuing along these lines, it is a straightforward exercise to confirm that, given any polynomial

$$c_0 z^N + c_1 z^{N-1} + \cdots + c_{N-1} z + c_N \quad ,$$

then

$$\begin{aligned}&\left[c_0 z^N \;+\; c_1 z^{N-1} \;+\; \cdots \;+\; c_{N-1} z \;+\; c_N\right]^* \\ &\qquad = \; {c_0}^* \left(z^*\right)^N \;+\; {c_1}^* \left(z^*\right)^{N-1} \;+\; \cdots \;+\; {c_{N-1}}^* z^* \;+\; {c_N}^* \quad .\end{aligned}$$

If, in addition, each c_k is a real number, then ${c_k}^* = c_k$ and the above reduces even more.

Lemma 18.8
Let

$$c_0 z^N \;+\; c_1 z^{N-1} \;+\; \cdots \;+\; c_{N-1} z \;+\; c_N$$

be a polynomial in which each c_k is a real number. Then, for any complex number z,

$$\begin{aligned}&\left[c_0 z^N \;+\; c_1 z^{N-1} \;+\; \cdots \;+\; c_{N-1} z \;+\; c_N\right]^* \\ &\qquad = \; c_0 \left(z^*\right)^N \;+\; c_1 \left(z^*\right)^{N-1} \;+\; \cdots \;+\; c_{N-1} z^* \;+\; c_N \quad .\end{aligned}$$

The Proof of Theorem 18.3

Let me remind you of the statement of theorem 18.3:

> *Consider a polynomial*
>
> $$p(r) \;=\; a_0 r^N \;+\; a_1 r^{N-1} \;+\; \cdots \;+\; a_{N-2} r^2 \;+\; a_{N-1} r \;+\; a_N$$
>
> *in which a_0, a_1, ..., and a_N are all real numbers. Let λ and ω be two real numbers, and let m be some positive integer. Then*
>
> $$r_0 \;=\; \lambda + i\omega \quad \text{is a root of multiplicity } m \text{ for polynomial } p(r)$$
>
> *if and only if*
>
> $$r_0{}^* \;=\; \lambda - i\omega \quad \text{is a root of multiplicity } m \text{ for polynomial } p(r) \quad .$$

To start our proof of this theorem, assume r_0 is a root of multiplicity m of p. Then

$$0 \;=\; a_0 (r_0)^N \;+\; a_1 (r_0)^{N-1} \;+\; \cdots \;+\; a_{N-2} (r_0)^2 \;+\; a_{N-1} r_0 \;+\; a_N \quad .$$

But

$$\begin{aligned}0 \;=\; 0^* \;&=\; \left[a_0 (r_0)^N \;+\; a_1 (r_0)^{N-1} \;+\; \cdots \;+\; a_{N-2} (r_0)^2 \;+\; a_{N-1} r_0 \;+\; a_N\right]^* \\ &=\; a_0 \left(r_0{}^*\right)^N \;+\; a_1 \left(r_0{}^*\right)^{N-1} \;+\; \cdots \;+\; a_{N-2} \left(r_0{}^*\right)^2 \;+\; a_{N-1} r_0{}^* \;+\; a_N \quad ,\end{aligned}$$

showing that $r_0{}^*$ is also a root of p. This also means that $r - r_0$ and $r - r_0{}^*$ are both factors of $p(r)$, and, hence,

$$p(r) \;=\; p_1(r)\,(r - r_0)\,(r - r_0{}^*)$$

where p_1 is the polynomial of degree $N-2$ that can be obtained by dividing these two factors out of p,

$$p_1(r) \;=\; \frac{p(r)}{(r - r_0)\,(r - r_0{}^*)} \quad .$$

Now

$$(r - r_0)\left(r - r_0{}^*\right) = (r - [\lambda + i\omega])(r - [\lambda - i\omega]) = \cdots = r^2 - 2\lambda r + \omega^2 \quad .$$

So the coefficients of both the denominator and the numerator in the fraction defining $p_1(r)$ are real-valued constants. If you think about how one actually computes this fraction (via, say, long division), you will realize that all the coefficients of $p_1(r)$ must also be real.

If $m > 1$ then $(r - r_0)^{m-1}$ — but not $(r - r_0)^m$ — will be a factor of $p_1(r)$. Thus, r_0 will be a root of multiplicity $m - 1$ for p_1. Repeating the above arguments with p_1 replacing p leads to the conclusions that

1. $r_0{}^*$ is also a root of p_1

and

2. there is an $(N - 4)^{\text{th}}$ degree polynomial p_2 with real coefficients such that

$$p(r) = p_1(r)(r - r_0)\left(r - r_0{}^*\right) = p_2(r)(r - r_0)^2 (r - r_0{}^*)^2 \quad .$$

Clearly, we can continue repeating these arguments, ultimately obtaining the formula

$$p(r) = p_m(r)(r - r_0)^m (r - r_0{}^*)^m$$

where p_m is a polynomial of degree $N - 2m$ with just real coefficients and for which r_0 is not a root.

Could $r_0{}^*$ be a root of p_m? If so, then the argument given at the start of this proof would show that $(r_0{}^*)^*$ is also a root of p_m. But $(r_0{}^*)^* = r_0$ and we know r_0 is not a root of p_m. So it is not possible for $r_0{}^*$ to be a root of p_m.

All this shows that

$$r_0 \text{ is a root of multiplicity } m \text{ for } p(r) \quad \Longrightarrow \quad r_0{}^* \text{ is a root of multiplicity } m \text{ for } p(r) \quad .$$

Replacing r_0 with $r_0{}^*$ then gives us

$$r_0{}^* \text{ is a root of multiplicity } m \text{ for } p(r) \quad \Longrightarrow \quad \left(r_0{}^*\right)^* \text{ is a root of multiplicity } m \text{ for } p(r) \quad .$$

Together with the fact that $(r_0{}^*)^* = r_0$, these two implications give us

$$r_0 \text{ is a root of multiplicity } m \text{ for } p(r) \quad \Longleftrightarrow \quad r_0{}^* \text{ is a root of multiplicity } m \text{ for } p(r) \quad ,$$

completing our proof of theorem 18.3.

Additional Exercises

18.1. *Using clever factoring of the characteristic polynomials (such as done in example 18.6 on page 382), find the general solution to each of the following:*

a. $y^{(4)} - 4y^{(3)} = 0$

b. $y^{(4)} + 4y'' = 0$

c. $y^{(4)} - 34y'' + 225y = 0$

d. $y^{(4)} - 81y = 0$

e. $y^{(4)} - 18y'' + 81y = 0$

f. $y^{(5)} + 18y^{(3)} + 81y' = 0$

18.2. *For each of the following differential equations, one or more roots to the corresponding characteristic polynomial can be found by "testing candidates" (as illustrated in example 18.7 on page 383). Using this fact, find the general solution to each.*

a. $y''' - y'' + y' - y = 0$

b. $y''' - 6y'' + 11y' - 6y = 0$

c. $y''' - 8y'' + 37y' - 50y = 0$

d. $y^{(4)} + 2y^{(3)} + 10y'' + 18y' + 9y = 0$

18.3. *Find the solution to each of the following initial-value problems:*

a. $y''' + 4y' = 0$ *with* $y(0) = 4$, $y'(0) = 6$ *and* $y''(0) = 8$

b. $y''' - 6y'' + 12y' - 8y = 0$
with $y(0) = 5$, $y'(0) = 13$ *and* $y''(0) = 86$

c. $y^{(4)} + 26y'' + 25y = 0$
with $y(0) = 6$, $y'(0) = -28$, $y''(0) = -102$ *and* $y^{(3)}(0) = 628$

18.4. *Find the general solution to each of the following:*

a. $y''' - 8y = 0$

b. $y^{(4)} + 13y'' + 36y = 0$

c. $y^{(6)} - 3y^{(4)} + 3y'' - y = 0$

d. $y^{(6)} - 2y^{(3)} + y = 0$