8

Slope Fields: Graphing Solutions Without the Solutions

Up to now, our efforts have been directed mainly towards finding formulas or equations describing solutions to given differential equations. Then, sometimes, we sketched the graphs of these solutions using those formulas or equations. In this chapter, we will do something quite different. Instead of solving the differential equations, we will use the differential equations, directly, to sketch the graphs of their solutions. No other formulas or equations describing the solutions will be needed.

The graphic techniques and underlying ideas that will be developed here are, naturally, especially useful when dealing with differential equations that can not be readily solved using the methods already discussed. But these methods can be valuable even when we can solve a given differential equation since they yield "pictures" describing the general behavior of the possible solutions. Sometimes, these pictures can be even more enlightening than formulas for the solutions.

8.1 Motivation and Basic Concepts

Suppose we have a first-order differential equation that, for motivational purposes, "just cannot be solved" using the methods already discussed. For illustrative purposes, let's pretend

$$16\frac{dy}{dx} + xy^2 = 9x$$

is that differential equation. (True, this is really a simple separable differential equation. But it is also a good differential equation for illustrating the ideas being developed.)

For our purposes, we need to algebraically solve the differential equation to get it into the derivative formula form, $y' = F(x, y)$. Doing so with the above differential equation, we get

$$\frac{dy}{dx} = \frac{x}{16}\left(9 - y^2\right) \quad . \tag{8.1}$$

Remember, there are infinitely many particular solutions (with different particular solutions typically corresponding to different values for the general solution's 'arbitrary' constant). Let's now pick some point in the plane, say, $(x, y) = (1, 2)$, let $y = y(x)$ be the particular solution to the differential equation whose graph passes through that point, and consider sketching a short line tangent to this graph at this point. From elementary calculus, we know the slope of this tangent line is given by the derivative of $y = y(x)$ at that point. And fortunately, equation (8.1) gives us a formula for computing this very derivative without the bother of actually solving for $y(x)$! So, for the graph of this particular $y(x)$,

$$\begin{aligned}\text{Slope of the tangent line at } (1,2) &= \frac{dy}{dx} \quad \text{at} \quad (x,y) = (1,2) \\ &= \frac{x}{16}\left(9 - y^2\right) \quad \text{at} \quad (x,y) = (1,2) \\ &= \frac{1}{16}\left(9 - 2^2\right) \\ &= \frac{5}{16} \quad .\end{aligned}$$

Thus, if we draw a short line with slope $5/16$ through the point $(1, 2)$, that line will be tangent at that point to the graph of a solution to our differential equation.

So what? Well, consider further: At each point (x, y) in the plane, we can draw a short line whose slope is given by the right side of equation (8.1). For convenience, let's call each of these short lines the *slope line* for the differential equation at the given point. Now consider any curve drawn so that, at each point (x, y) on the curve, the slope line there is tangent to the curve. If this curve is the graph of some function $y = y(x)$, then, at each point (x, y) ,

$$\frac{dy}{dx} = \text{slope of the slope line at } (x,y) \quad .$$

But we constructed the slope lines so that

$$\text{slope of the slope line at } (x,y) = \text{right side of equation (8.1)} = \frac{x}{16}\left(9 - y^2\right) \quad .$$

So the curve drawn is the graph of a function $y(x)$ satisfying

$$\frac{dy}{dx} = \frac{x}{16}\left(9 - y^2\right) \quad .$$

That is, the curve drawn is the graph of a solution to our differential equation, and we've managed to draw this curve without actually solving the differential equation.

In practice, of course, we cannot draw the slope line at *every* point in the plane. But we can construct the slope lines at the points of any finite grid of points, and then sketch curves that "parallel" these slope lines — that is, sketch curves so that, at each point on each curve, the slope of the tangent line is closely approximated by the slopes of the nearby slope lines. Each of these curves would then approximate the graph of a solution to the differential equation. These curves may not be perfect, but, if we are careful, they should be close to the actual graphs, and, consequently, will give us a good picture of what the solutions to our differential equation look like. Moreover, we can construct these graphs without actually solving the differential equation.

By the way, the phrase "graph of a solution to the differential equation" is a bit long to constantly repeat. For brevity, we will misuse terminology slightly and call these graphs *solution curves* (for the given differential equation).[1]

[1] If you recall the discussion on graphing implicit solutions (section 4.7 starting on page 92), you may realize that, strictly speaking, the curves being sketched are "integral curves" containing "solution curves". However, we will initially make an assumption that makes the distinction between integral and solution curves irrelevant.

8.2 The Basic Procedure

What follows is a procedure for systematically constructing approximate graphs of solutions to a first-order differential equation using the ideas just developed. We assume that we have a first-order differential equation, possibly with some initial condition $y(x_0) = y_0$, and that we wish to sketch some of the solution curves for this differential equation in some "region of interest" in the XY–plane. To avoid a few complicating issues (which will be dealt with later), an additional requirement will be imposed (in the first step) on the sort of differential equations being considered. Later, we'll discuss what can be done when this requirement is not met. These steps will be illustrated using the initial-value problem

$$16\frac{dy}{dx} + xy^2 = 9x \qquad \text{with} \quad y(0) = 1 \quad . \tag{8.2}$$

The Procedure:

1. Algebraically solve the differential equation for the derivative to get it into the form

$$\frac{dy}{dx} = F(x, y)$$

where $F(x, y)$ is some formula involving x and/or y.

For now, let us limit our discussion to differential equations for which $F(x, y)$ is well defined and continuous throughout the region of interest. In particular, then, we are requiring $F(x, y)$ to be a finite number for each (x, y) in the region we are trying to graph solutions.[2] What may happen when this requirement is not satisfied will be discussed later (in section 8.4).

Solving equation (8.2) for the derivative, we get

$$\frac{dy}{dx} = \frac{x}{16}\left(9 - y^2\right) \quad .$$

So here,

$$F(x, y) = \frac{x}{16}\left(9 - y^2\right) \quad .$$

There is certainly no problem with computing this for any pair of values x and y; so our differential equation meets the requirement that "$F(x, y)$ be well defined" in whatever region we end up using.

2. Pick a grid of points

$$\begin{array}{l}
(x_1, y_1),\ (x_2, y_1),\ (x_3, y_1),\ \ldots,\ (x_J, y_1), \\
(x_1, y_2),\ (x_2, y_2),\ (x_3, y_2),\ \ldots,\ (x_J, y_2), \\
\vdots \\
(x_1, y_K),\ (x_2, y_K),\ (x_3, y_K),\ \ldots,\ (x_J, y_K).
\end{array}$$

on which to plot the slope lines. Just which points are chosen is largely a matter of judgement. If the problem involves an initial condition $y(x_0) = y_0$, then the corresponding point, (x_0, y_0), should be one point in the grid. In addition, the grid should:

[2] Since it prevents points with "infinite slope", this requirement ensures that the curves will be "solution curves" in the strict sense of the phrase.

i. 'cover' the region over which you plan to graph the solutions, and

ii. have enough points so that the slope lines at those points will give a good idea of the curves to be drawn.

(More points can always be added later.)

In our example, we have the initial condition $y(0) = 1$ *, so we want our grid to contain the point* $(0, 1)$ *. For our grid let us pick the set of all points in the region* $0 \leq x \leq 4$ *and* $0 \leq y \leq 4$ *with integral coordinates:*

$$\begin{array}{l} (0,4)\,,\ (1,4)\,,\ (2,4)\,,\ (3,4)\,,\ (4,4) \\ (0,3)\,,\ (1,3)\,,\ (2,3)\,,\ (3,3)\,,\ (4,3)\,, \\ (0,2)\,,\ (1,2)\,,\ (2,2)\,,\ (3,2)\,,\ (4,2)\,, \\ (0,1)\,,\ (1,1)\,,\ (2,1)\,,\ (3,1)\,,\ (4,1)\,, \\ (0,0)\,,\ (1,0)\,,\ (2,0)\,,\ (3,0)\,,\ (4,0)\,. \end{array}$$

Note that this does contain the point $(0, 1)$ *, as desired.*

3. For each grid point (x_j, y_k) :

(a) Compute $F(x_j, y_k)$, the right side of the differential equation from step 1.

(b) Using the value $F(x_j, y_k)$ just computed, carefully draw a short line at (x_j, y_k) with slope $F(x_j, y_k)$. (As already stated, this short line is called the *slope line* for the differential equation at (x_j, y_k) . Keep in mind that the slope line at each point is tangent to the solution curve passing through this point.)

More Terminology: The collection of all the slope lines at all points on the grid is called a *slope field* for the differential equation.[3]

Glancing back at our example from step 1, we see that

$$F(x, y) = \frac{x}{16}\left(9 - y^2\right) \quad .$$

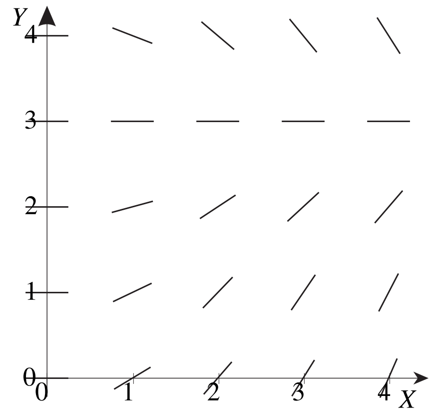
Systematically computing this at each grid point (and noting that these values give us the slopes of the slope lines at these points):

$$\begin{aligned} \textit{slope of slope line at } (0,0) &= F(0,0) = \frac{0}{16}\left(9 - 0^2\right) = 0 \\ \textit{slope of slope line at } (1,0) &= F(1,0) = \frac{1}{16}\left(9 - 0^2\right) = \frac{9}{16} \\ \textit{slope of slope line at } (2,0) &= F(2,0) = \frac{2}{16}\left(9 - 0^2\right) = \frac{9}{8} \\ &\vdots \\ \textit{slope of slope line at } (1,2) &= F(1,2) = \frac{1}{16}\left(9 - 2^2\right) = \frac{5}{16} \\ &\vdots \end{aligned}$$

The results of all these slope computations are contained in the table in figure 8.1a. Sketching the corresponding slope line at each grid point then gives us the slope field sketched in figure 8.1b.

4	0	$-7/16$	$-7/8$	$-21/16$	$-7/4$
3	0	0	0	0	0
2	0	$5/16$	$5/8$	$15/16$	$5/4$
1	0	$1/2$	1	$3/2$	2
0	0	$9/16$	$9/8$	$27/16$	$9/4$
y/x	0	1	2	3	4

(a)



(b)

Figure 8.1: The **(a)** table of slopes of the slope lines at the grid points and **(b)** the corresponding slope lines (and resulting slope field) for $y'(x) = \frac{1}{16}x\left(9 - y^2\right)$.

4. Using the slope field just constructed, sketch curves that "parallel" the slope field. To be precise:

   (a) Pick a convenient grid point as a starting point. Then, as well as can be done freehanded, sketch a curve through that point which "parallels" the slope field. This curve must go through the starting point and must be tangent to the slope line there. Beyond that, however, there is no reason to expect this curve to go through any other grid point — simply draw this curve so that, at each point on the curve, the curve's tangent there closely matches the nearby slope lines. In other words, do *not* attempt to "connect the dots"! Instead, "go with the flow" indicated by the slope field.

   (b) If desired, repeat the previous step and sketch another curve using a different starting point. Continue sketch curves with different starting points until you get as many curves as seems appropriate.

   If done carefully, the curves sketched will be reasonably good approximations of solution curves for the differential equation. If your original problem involves an initial value, be sure that one of your starting points corresponds to that initial value. The resulting curve will be (approximately) the graph of the solution to that initial-value problem.

   Figure 8.2 shows the slope field just constructed, along with four curves sketched according to the instructions just given. The starting points for these curves were chosen to be $(0,0)$, $(0,1)$, $(0,3)$ *, and* $(0,4)$ *. Each of these curves approximates the graph of one solution to our differential equation,*

   $$\frac{dy}{dx} = \frac{x}{16}\left(9 - y^2\right) \quad ,$$

   with the one passing through $(0,1)$ *being (approximately) the graph of the solution to the initial-value problem*

   $$\frac{dy}{dx} = \frac{x}{16}\left(9 - y^2\right) \qquad \text{with} \quad y(0) = 1 \quad .$$

[3] Some texts also refer to a slope field as a "direction field".

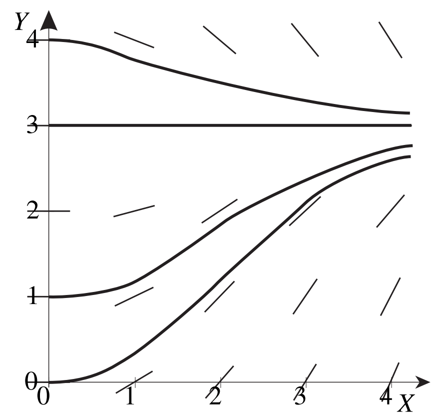


Figure 8.2: A slope field for $y'(x) = \frac{1}{16}x(9 - y^2)$, along with four curves sketched "parallel" to the field.

5. At this point (or possibly some point in the previous step) decide whether there are enough slope lines to accurately draw the desired curves. If not, add more points to the grid, repeat step 3 with the new grid points, and redraw the curves with the improved slope field (but, first, see some of the notes below on this procedure).

   It must be admitted that the graphs obtained in figure 8.2 are somewhat crude and limited. Clearly, we should have used a much bigger grid to cover more area, and should have used more grid points per unit area so we can get better detail (especially in the region where $y \approx 3$). So let us try (sometime in the near future) something like, say, a 19×16 grid covering the region where $0 \leq x \leq 6$ and $0 \leq y \leq 5$. This will give us a field of 504 slope lines instead of the measly 25 used in figure 8.2.

Though the above procedure took several pages to describe and illustrate, it is really quite simple, and, eventually, yields a picture that gives a fairly good idea of how the solutions of interest behave. Just how good a picture depends on the slope field generated and how carefully the curves are chosen and drawn. Here are a few observations that may help in generating this picture.

1. As indicated in the example, generating a good slope field can require a great deal of tedious computations and careful drawing — if done by hand. But why do it by hand? This is just the sort of tedious, mind-numbing work computers do so well. Program a computer to generate the slope field. Better yet, check your favorite computer math package. There is a good chance that it will already have commands to generate these fields. Use those commands (or find a math package that has those commands). That is how the direction field in figure 8.3 was generated.

2. As long as $F(x, y)$ is well defined at every point in the region being considered, solution curves cannot cross each other at nonzero angles. This is because any solution curve through any point (x, y) must be tangent to the one and only slope line there, whether or not that slope line is drawn in. Thus, at worst, two solution curves can become tangent to each other at a point. Even this, the merging of two or more solution curves with a common tangent, is not something you should often expect.

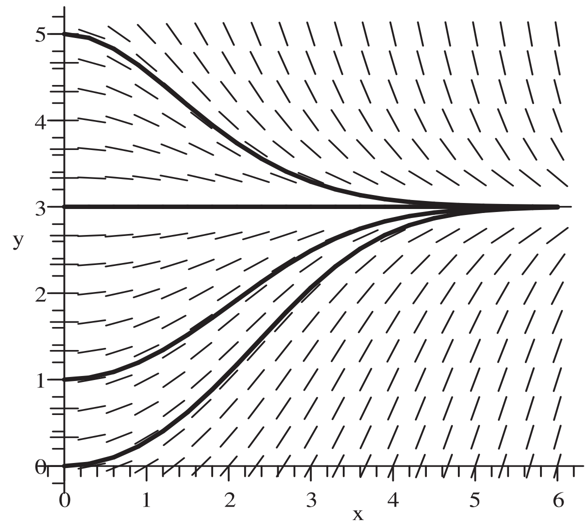


Figure 8.3: A better slope field (based on a 19×16 grid) for $y'(x) = \frac{1}{16}x\left(9 - y^2\right)$, along with four curves sketched "parallel" to the field.

3. Just which curves you choose to sketch depends on your goal. If your goal is to graph the solution to an initial-value problem, then it may suffice to just draw that one curve passing through the point corresponding to the initial condition. That curve approximates the graph of the desired solution $y(x)$ and, from it, you can find the approximate value of $y(x)$ for other values of x.

   On the other hand, by drawing a collection of well chosen curves following the slope field, you can get a fair idea of how all the solutions of interest generally behave and how they depend on the initial condition. Choosing those curves is a matter of judgement, but do try to identify any curves that are horizontal lines. These are the graphs of constant solutions and are likely to be particularly relevant. In fact, it's often worthwhile to identify and sketch the graphs of all constant solutions in the region, even if they do not pass through any of your grid points.

   Consider finding the values of $y(4)$ and $y(6)$ when $y(x)$ is the solution to the initial-value problem

$$\frac{dy}{dx} = \frac{x}{16}\left(9 - y^2\right) \qquad \textit{with} \quad y(0) = 1 \quad .$$

   Since this differential equation was the one used to generate the slope fields in figures 8.2 and 8.3, we can use the curve drawn through $(0, 1)$ in either of these figures as an approximate graph for $y(x)$. On this curve in the better slope field of figure 8.3, we see that $y \approx 2.6$ when $x = 4$, and that $y \approx 3$ when $x = 6$. Thus, according to our sketch, if $y(x)$ satisfies the above initial-value problem, then

$$y(4) \approx 2.6 \qquad \textit{and} \qquad y(6) \approx 3 \quad .$$

   More generally, after looking at figure 8.3, it should be apparent that any curve in the sketched region that "parallels" the slope field will approach $y = 3$

when x becomes large. This strongly suggests that, if $y(x)$ is any solution to our differential equation with $0 \leq y(0) \leq 5$, then

$$\lim_{x \to \infty} y(x) = 3 \quad .$$

Do observe that $y = 3$ is a constant solution to our differential equation.

8.3 Observing Long-Term Behavior in Slope Fields

Basic Notions

A slope field of a differential equation gives a picture of the general behavior of the possible solutions to that differential equation, at least in the region covered by that slope field. In many cases, this picture may even give you a good idea of the "long-term" behavior of the solutions.

!►*Example 8.1:* *Consider the differential equation*

$$\frac{dy}{dx} = \frac{x}{4}(3 - y) \quad .$$

A slope field (and some solution curves) for this equation is sketched in figure 8.4a. Now let $y = y(x)$ be any solution to this differential equation, and look at the slope field. It clearly suggests that

$$y(x) \rightarrow 3 \quad \text{as} \quad x \rightarrow \infty \quad .$$

On the other hand, the slope field sketched in figure 8.4b for

$$\frac{dy}{dx} = \frac{1}{3}(y - 3)^{1/3}$$

suggests a rather different sort of behavior for this equation's solutions as x gets large. Here, it looks as if almost no solutions approach any constant value as $x \rightarrow \infty$. Instead, we appear to have

$$\lim_{x \to \infty} y(x) = +\infty \qquad \text{if} \quad y(0) > 3$$

and

$$\lim_{x \to \infty} y(x) = -\infty \qquad \text{if} \quad y(0) < 3 \quad .$$

Of course, one should be cautious about using a slope field to predict the value of $y(x)$ when x is outside the range of x-values used in the slope field. In general, a slope field for a given differential equation sketched on one region of the XY–plane can be quite different from a slope field for that differential equation over a different region. So it is important to be sure that the general pattern of slope lines on which you are basing your prediction does not significantly change as you consider points outside the region of your slope field.

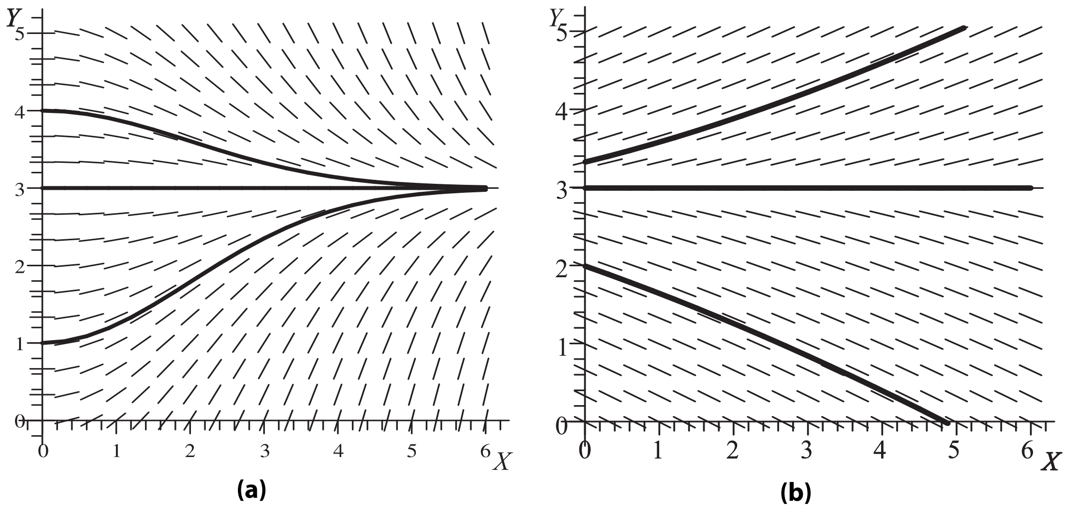


Figure 8.4: **(a)** A slope field and some solutions for $y'(x) = \frac{1}{4}x(3-y)$, and **(b)** a slope field and some solutions for $y'(x) = \frac{1}{3}(y-3)^{1/3}$

!►*Example 8.2:* *If you look at the differential equation for the slope field in figure 8.4a,*

$$\frac{dy}{dx} = \frac{x}{4}(3-y) \quad ,$$

you can see that the magnitude of the right side

$$\left|\frac{x}{4}(3-y)\right|$$

becomes larger as either $|x|$ *or* $|y-3|$ *becomes larger, but the sign of the right side remains*

$$\text{negative if } x > 0 \text{ and } y > 3 \qquad \text{and} \qquad \text{positive if } x > 0 \text{ and } y < 3 \quad .$$

Thus, the slope lines may become steeper as we increase x *or as we go up or down with* y*, but they continue to "direct" all sketched solution curves towards the line* $y = 3$ *as* $x \to \infty$*.*

There is one class of differential equations whose slope fields are especially suited for making long-term predictions: the *autonomous* first-order differential equations. Remember, such a differential equation can be written as

$$\frac{dy}{dx} = g(y)$$

where $g(y)$ is a known formula of y only. The fact that the right side of this equation does not depend on x means that the vertical column of slope lines at any one value of x is identically repeated at every other value of x. So if the slope field tells you that the solution curve through, say, the point $(x, y) = (1, 4)$ has slope $1/2$, then you are certain that the solution curve through any point (x, y) with $y = 4$ also has slope $1/2$. Moreover, if there is a horizontal slope line at a point (x_0, y_0), then there will be a horizontal slope line wherever $y = y_0$; that is, $y = y_0$ will be a constant solution to the differential equation.

!►*Example 8.3:* *The differential equation for the slope field sketched in figure 8.4b,*

$$\frac{dy}{dx} = \frac{1}{3}(y-3)^{1/3} \quad ,$$

is autonomous since this formula for the derivative does not explicitly involve x. So the pattern of slope lines in any vertical column in the given slope field will be repeated identically in every vertical column in any slope field covering a larger region (provided we use the same y-values). Moreover, from the right side of our differential equation, we can see that the slopes of the slope lines

1. *remain positive and steadily increase as y increases above $y = 3$,*

and

2. *remain negative and steadily decrease as y decreases below $y = 3$.*

Consequently, no matter how large a region we choose for our the slope field, we will see that

1. *the slope lines at points above $y = 3$ will be "directing" the solution curves more and more steeply upwards as y increases,*

and

2. *the slope lines at points above $y = 3$ will be "directing" the solution curves more and more steeply upwards as y increases.*

Thus, we can safely say that, if $y = y(x)$ is any solution to this differential equation, then

$$\lim_{x\to\infty} y(x) = \begin{cases} +\infty & \textit{if} \quad y(0) > 3 \\ -\infty & \textit{if} \quad y(0) < 3 \end{cases} .$$

Constant Solutions and Stability

The "long-term behavior" of a constant solution

$$y(x) = y_0 \qquad \text{for all} \quad x$$

is quite straightforward: the value of $y(x)$ remains y_0 as $x \to \infty$. What is more varied, and often quite important, is the long-term behavior of the other solutions that are initially "close" to this constant solution. The slope fields in figures 8.4a and 8.4b clearly illustrate how different this behavior may be.

In figure 8.4a, the graph of every solution $y = y(x)$ with $y(0) \approx 3$ remains close to the horizontal line $y = 3$ as x increases. Thus, if you know $y(x)$ satisfies the given differential equation, but only know that $y(0) \approx 3$, then it is still safe to expect that $y(x) \approx 3$ for all $x > 0$. In fact, it appears that $y(x) \to 3$ as $x \to \infty$.

In figure 8.4b, by contrast, the graph of every nonconstant solution $y = y(x)$ with $y(0) \approx 3$ diverges from the horizontal line $y = 3$ as x increases. Thus, if $y(x)$ is a solution to the differential equation for this slope field, but you only know that $y(0) \approx 3$, then you have very little idea what $y(x)$ is for large values of x. This could be a significant concern in real-world applications where, often, initial values are only known approximately.

This leads to the notion of the "stability" of a given constant solution for a first-order differential equation. This concerns the tendency of solutions having initial values close to that of that constant solution to continue having values close to that constant as the variable increases. Whether or not the initially nearby solutions remain nearby determines whether a constant solution is classified as being "stable", "asymptotically stable" or "unstable". Basically, we will say that a constant solution $y = y_0$ to some given first-order differential equation is:

- *stable* if (and only if) every other solution $y = y(x)$ having an initial value $y(0)$ "sufficiently close" to y_0 remains reasonably close to y_0 as x increases.[4]
- *asymptotically stable* if (and only if) any other solution $y = y(x)$ satisfies

$$\lim_{x \to \infty} y(x) = y_0$$

whenever the initial value $y(0)$ is "sufficiently close" to y_0.[5] (Typically, this means the horizontal line $y = y_0$ is the horizontal asymptote for these solutions — that's where the term "asymptotically stable" comes from.)

- *unstable* whenever it is not a stable constant solution.

Of course, the above definitions assume the differential equation is "reasonably well-defined in a region about the constant solution $y = y_0$".[6]

Often, the stability or instability of a constant solution is readily apparent from a given slope field, with rigorous confirmation easily done by fairly simple analysis. Asymptotically stable constant solutions are also often easily identified in slope fields, though rigourously verifying asymptotic stability may require a bit more analysis.

!►*Example 8.4:* *Recall that the slope field in figure 8.4a is for*

$$\frac{dy}{dx} = \frac{x}{4}(3 - y) \quad .$$

From our discussions in examples 8.1 and 8.2, we already know $y = 3$ is a constant solution to this differential equation, and that, if $y = y(x)$ is any other solution satisfying $y(0) \approx 3$, then $y(x) \approx 3$ for all $x > 0$. In fact, because the slope lines are all angled towards $y = 3$ as x increases, it should be clear that, for every $x > 0$, $y(x)$ will be closer to 3 than is $y(0)$. So $y = 3$ is a stable constant solution to the above differential equation.

Is $y = 3$ an asymptotically stable solution? That is, do we have

$$\lim_{x \to \infty} y(x) = 3$$

whenever $y = y(x)$ is a solution with $y(0)$ is sufficiently close to 3 ? The slope field certainly suggests so. Fortunately, this differential equation is a fairly simple separable equation which you can easily solve to get

$$y(x) = 3 + Ae^{-x^2/2}$$

as a general solution. Taking the limit, we see that

$$\lim_{x \to \infty} y(x) = \lim_{x \to \infty} 3 + Ae^{-x^2/2} = 3 + 0 \quad ,$$

no matter what $y(0)$ is. So, yes, $y = 3$ is not just a stable constant solution to the above differential equation, it is an asymptotically stable constant solution.

---

[4] To be more precise: $y = y_0$ is a *stable* constant solution if and only if, for every $\epsilon > 0$, there is a corresponding $\delta > 0$ such that, whenever $y = y(x)$ is a solution to the differential equation satisfying $|y(0) - y_0| < \delta$, then $|y(x) - y_0| < \epsilon$ for all $x > 0$.

[5] More precisely: $y = y_0$ is an *asymptotically stable* constant solution if and only if there is a corresponding $\delta > 0$ such that, whenever $y = y(x)$ is a solution to the differential equation satisfying $|y(0) - y_0| < \delta$, then $\lim_{x \to \infty} y(x) = y_0$.

[6] e.g., that the differential equation can be written as $y' = F(x, y)$ where F is continuous at every (x, y) with $x \geq 0$ and $|y - y_0| < \delta$ for some $\delta > 0$.

!►Example 8.5: *Now, again consider the slope field in figure 8.4b, which is for*

$$\frac{dy}{dx} = \frac{1}{3}(y-3)^{1/3} \quad .$$

Again, we know $y = 3$ is a constant solution for this differential equation. However, from our discussion in example 8.3, we also know that, if $y = y(x)$ is any other solution, then

$$\lim_{x\to\infty} y(x) = \pm\infty \quad ,$$

Clearly, then, even if $y(0)$ is very close (but not equal) to 3, $y(x)$ will not remain close to 3 as x increases. Thus, $y = 3$ is an unstable constant solution to this differential equation.

In the two examples given so far, all the solutions starting near a stable constant solution converged to that solution, while all nonconstant solutions starting near an unstable solution diverged to $\pm\infty$ as $x \to \infty$. The next two examples show that somewhat different behavior can occur.

!►Example 8.6: *The slope field and solution curves sketched in figure 8.5a are for*

$$\frac{dy}{dx} = \frac{y-2}{6e^{x/2}-2} \quad .$$

Here, $y = 2$ is the only constant solution. Following the slope lines in this figure, it appears that, although the graph of each nonconstant solution $y = y(x)$ starts at $x = 0$ by moving away from $y = 2$ as x increases, this graph quickly levels out so that $y(x)$ approaches some constant as $x \to \infty$. This behavior can be confirmed by solving the differential equation. With a little work, you can solve this differential equation and show that, if y is any solution to this differential equation, then

$$y(x) - 2 = \left[3 - e^{-x/2}\right][y(0) - 2] \quad .$$

You can also easily verify that

$$\left|3 - e^{-x/2}\right| < 3 \qquad \text{for} \quad x > 0 \quad .$$

So,

$$|y(x) - 2| = \left|3 - e^{-x/2}\right| |y(0) - 2| < 3\,|y(0) - 2| \quad .$$

In other words, the distance between $y(x)$ and $y = 2$ when $x > 0$ is never more than three times the distance between $y(x)$ and $y = 2$ when $x = 0$. So, if we wish $y(x)$ to stay within a certain distance of $y = 2$ for all positive values of x, we merely need to be sure that $y(0)$ is no more than a third of that distance from 2.

This confirms that $y = 2$ is a stable constant solution. However, it is not asymptotically stable because

$$\lim_{x\to\infty} y(x) = 2 + 3[y(0) - 2] \neq 2 \qquad \text{whenever} \quad y(0) \neq 2 \quad .$$

?►Exercise 8.1: *Let $y(x)$ be a solution to the differential equation discussed in the last example. Using the solution formula given above:*

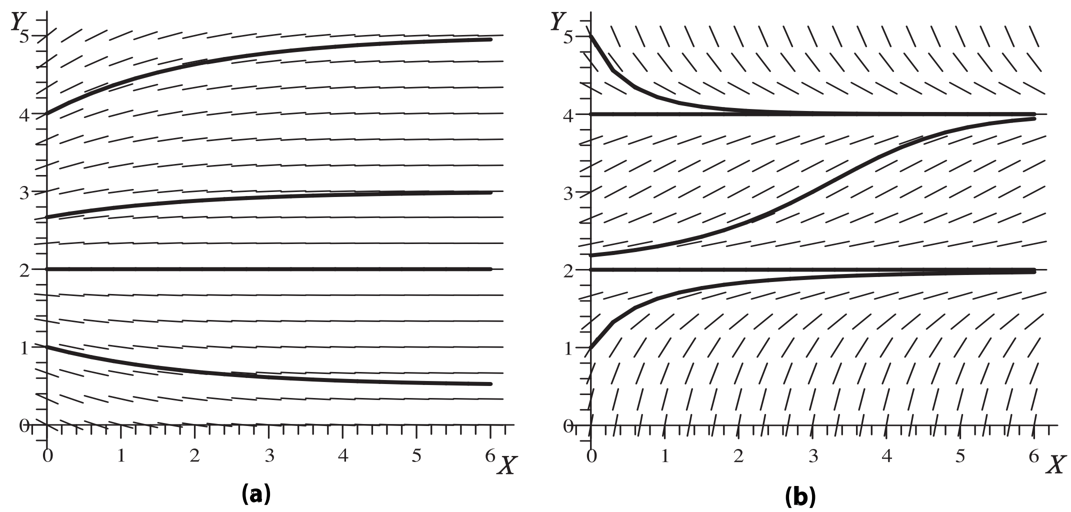


Figure 8.5: Slope fields **(a)** for example 8.6, and **(b)** for example 8.7.

a: *Show that*

$$|y(x) - 2| < 1 \qquad \textit{for all} \quad x > 0$$

whenever

$$|y(0) - 2| < \frac{1}{3} \quad .$$

b: *How close should $y(0)$ be to 2 so that*

$$|y(x) - 2| < \frac{1}{2} \qquad \textit{for all} \quad x > 0 \quad ?$$

In the next example, there are two constant solutions, and the analysis is done without the benefit of having a general solution to the given differential equation.

!►*Example 8.7:* *The slope field and solution curves sketched in figure 8.5b are for*

$$\frac{dy}{dx} = \frac{1}{2}(4 - y)(y - 2)^{4/3} \quad .$$

Technically, this separable equation can be solved for an implicit solution by the methods discussed for separable equations, but the resulting equation is too complicated to be of much value here. Fortunately, from a quick examination of the right side of this differential equation, we can see that:

1. *There are two constant solutions, $y = 2$ and $y = 4$.*
2. *The differential equation is autonomous. So the pattern of slope lines seen in figure 8.5b continues unchanged throughout the entire horizontal strip with $0 \le y \le 5$.*

Following the slope lines in figure 8.5b, it seems clear that $y = 4$ is a stable constant solution. In fact, it appears that

$$\lim_{x \to \infty} y(x) = 4$$

whenever y is a solution satisfying

$$2 < y(0) < 5 \quad .$$

This strongly suggests that $y = 4$ *is an asymptotically stable constant solution.*

On the other hand, if

$$\lim_{x\to\infty} y(x) = 4 \qquad \text{whenever} \quad 2 < y(0) < 5 \quad ,$$

then the constant solution $y = 2$ *cannot be stable. True, it appears that*

$$\lim_{x\to\infty} y(x) = 2 \qquad \text{whenever} \quad 0 < y(0) \leq 2 \quad ,$$

but, if $y(0)$ *is just a tiny bit larger than* 2 *, then* $y(x)$ *does not stay close to* 2 *as* x *increases — it gets close to* 4 *. So we must consider this constant solution as being unstable. (We will later see that this type of instability can cause serious problems when attempting to numerically solve a differential equation.)*

In the last example, we did not do the analysis to rigorously verify that $y = 4$ is an asymptotically stable constant solution, and that $y = 2$ is an unstable constant solution. Still, you are probably pretty confident that more rigorous analysis will confirm this. If so, good — you are correct. We'll verify this in section 8.5 using the more rigorous tests developed there.

Finally, a few comments that should be made regarding, not stability, but our discussion of "stability":

1. Strictly speaking, we've been discussing the stability of solutions to initial-value problems with the initial value of $y(x)$ is given at $x = 0$. To convert our discussion to a discussion of the stability of solutions to initial-value problems with the initial value of $y(x)$ given at some other point $x = x_0$, just repeat the above with $x = 0$ replaced by $x = x_0$. There will be no surprises.

2. Traditionally, discussions of "stability" only involve autonomous differential equations. We did not do so here because there seemed little reason to do so (provided we are careful about taking into account how the differential equation depends on x). Admittedly, limiting discussion to autonomous equations would have simplified things since the slope fields of autonomous differential equations do not depend on x. In addition, constant solutions to autonomous equations are traditionally called equilibrium solutions, and, to this author at least, "stable and unstable equilibriums" sounds more interesting than "stable and unstable constant solutions". Still, that did not justify limiting our discussion to just autonomous equations.

8.4 Problem Points in Slope Fields, and Issues of Existence and Uniqueness

In sketching and using a slope field for

$$\frac{dy}{dx} = F(x, y)$$

we have, up to this point, assumed $F(x, y)$ is well defined and continuous throughout the region of interest. This will not always be the case. So let us look at what can happen when F is not so well behaved at certain points. This, by the way, will naturally lead to a brief continuation of our discussion of "existence" and "uniqueness" that we began in the later part of chapter 3.

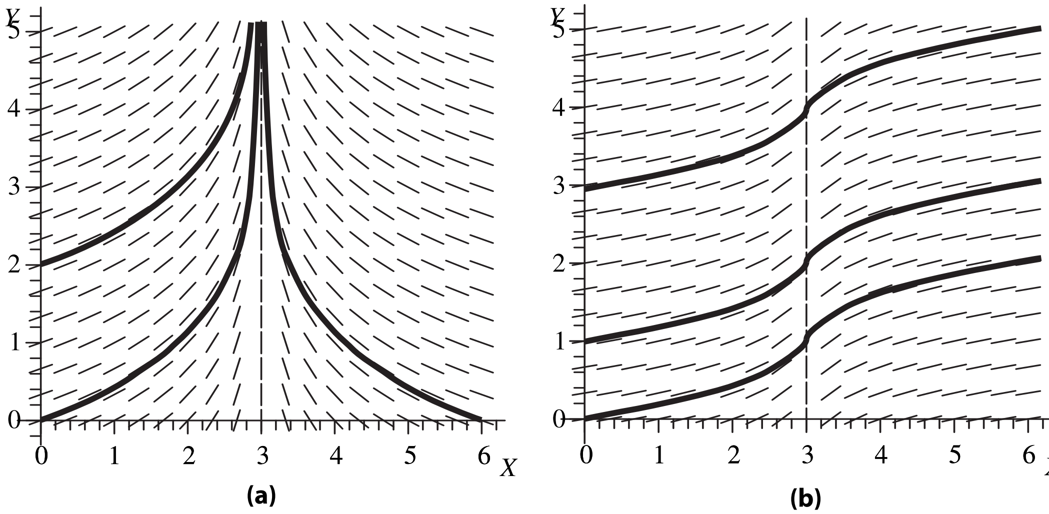


Figure 8.6: Slope fields **(a)** for $y'(x) = (3-x)^{-1}$ from example 8.8, and **(b)** for $y'(x) = \frac{1}{3}(x-3)^{-2/3}$ from example 8.8.

Infinite Slopes

Often, a given $F(x, y)$ becomes infinite at certain points in the XY–plane. This, in turn, means that the corresponding slope lines have "infinite slope", that is, they are vertical. One practical problem is that the software you are using to create your slope fields might object to 'division by zero' and not be able to deal with these points. On a more fundamental level, these infinite slopes may be warning you that something very significant is occurring in the solutions whose graphs include or are near these points.

In particular, these vertical slope lines may be telling you that solutions are, themselves, becoming infinite for finite values of x .

!►*Example 8.8:* *A slope field for*

$$\frac{dy}{dx} = \frac{1}{3-x}$$

is sketched in figure 8.6a. Since

$$\lim_{x \to 3} \frac{1}{3-x} = \pm\infty \quad ,$$

there are vertical slope lines at every point (x, y) *with* $x = 3$ *. This, along with the pattern of the other nearby slope lines, suggests that the solutions to this differential equation are "blowing up" as* x *approaches* 3 *. Fortunately, this differential equation is easily solved — just integrating it yields*

$$y = c - \ln|3-x| \quad ,$$

which does, indeed, "blow up" at $x = 3$ *for any choice of* c *.*

Consequently, the vertical slope lines in figure 8.6a form a vertical asymptote for the graphs of the solutions to the given differential equation. This further means that no solution to the differential equation passes through a point (x, y) *with* $x = 3$ *. In particular, if you are asked to solve the initial-value problem*

$$\frac{dy}{dx} = \frac{1}{3-x} \qquad \text{with} \quad y(3) = 2 \quad ,$$

you have every right to respond: "Nonsense, there is no solution to this initial-value problem."

On the other hand, the vertical slope lines might not be harbingers of particularly bad behavior in our solutions. Instead, the solutions may be fairly ordinary functions whose graphs just happen to have vertical tangent lines at a few points.

!► Example 8.9: *In figure 8.6b, we have a slope field for*

$$\frac{dy}{dx} = \frac{1}{3(x-3)^{2/3}} \quad .$$

Again, "division by zero" when $x = 3$ gives us vertical slope lines at every (x, y) with $x = 3$. This time, however, integrating the differential equation yields

$$y = (x-3)^{1/3} + c \quad .$$

For each value of c, this is a continuous function on the entire real line (including at $x = 3$) which just happens to have a vertical tangent when $x = 3$.

In particular, as you can easily verify,

$$y = (x-3)^{1/3} + 2$$

is the one and only solution on $(-\infty, \infty)$ to the initial-value problem

$$\frac{dy}{dx} = \frac{1}{3(x-3)^{2/3}} \qquad \text{with} \quad y(3) = 2 \quad .$$

Another possibility involving infinite slopes is illustrated in the next example.

!► Example 8.10: *The slope field in figure 8.7a is for*

$$\frac{dy}{dx} = \frac{x-2}{2-y} \quad .$$

This time, the vertical slope lines occur wherever $y = 2$ (excluding the point $(2, 2)$, which we will discuss later). It should be clear that these slope lines do not correspond to asymptotes of the graphs of solutions that "blow up", nor does it appear possible for a curve going from left to right to pass through these points and still parallel the slope lines. Instead, if we carefully sketch the curve that "follows the slope field" through, say, the point $(x, y) = (0, 2)$, then we end up with the circle sketched in the figure (which also has a vertical tangent at $(x, y) = (4, 2)$). But such a circle cannot be the graph of a function $y = y(x)$ since it corresponds to two different values for $y(x)$ for each x in the interval $(0, 4)$.

Fortunately, again, our differential equation is a simple separable equation. Solving it (as you can easily do), yields

$$y = 2 \pm \sqrt{A - (x-2)^2} \quad .$$

In particular, if we further require that $y(0) = 2$, then we obtain exactly two solutions,

$$y = 2 + \sqrt{4 - (x-2)^2} \qquad \text{and} \qquad y = 2 - \sqrt{4 - (x-2)^2} \quad ,$$

with each defined and continuous on the closed interval $[0, 4]$. The first satisfies the differential equation on the interval $(0, 4)$, and its graph is the upper half of the sketched circle. The second also satisfies the differential equation on the interval $(0, 4)$, but its graph is the lower half of the sketched circle.

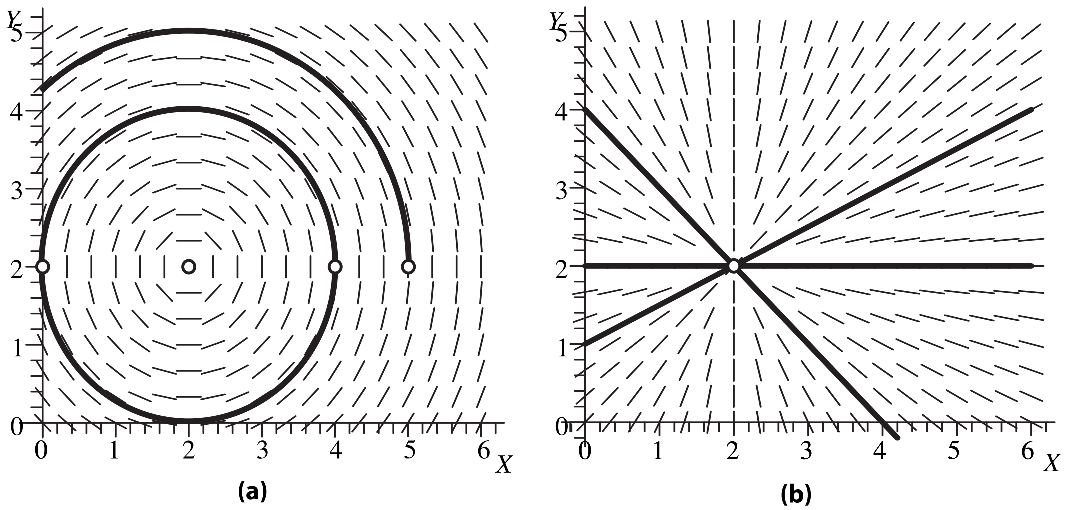


Figure 8.7: Slope fields **(a)** for $y'(x) = (x-2)(2-y)^{-1}$ from examples 8.10 and 8.11, and **(b)** for $y'(x) = (y-2)(x-2)^{-1}$ from example 8.12.

Undefined and Indeterminant Slopes

Let's now look at two examples involving points at which slope lines simply cannot be drawn because $F(x,y)$ is neither finite nor infinite at those points.

!►*Example 8.11:* *Again, consider the slope field in figure 8.7a for*

$$\frac{dy}{dx} = \frac{x-2}{2-y} \quad .$$

If $(x,y) = (2,2)$ *, this becomes the indeterminant expression*

$$\frac{dy}{dx} = \frac{0}{0} \quad .$$

Moreover, the slopes of the slope lines at points near $(x,y) = (2,2)$ *range from* 0 *to* $\pm\infty$ *. In fact, the point* $(x,y) = (2,2)$ *appears to be the center of the circles made up of the graphs of the solutions to this differential equation— a fact that can be confirmed using the solution formulas from example 8.10. Clearly, no real curve can pass through the point* $(x,y) = (2,2)$ *and remain parallel to the slope lines near this point. So if we really wanted a solution to*

$$\frac{dy}{dx} = \frac{x-2}{2-y} \quad \text{with} \quad y(2) = 2 \quad ,$$

which is valid on some interval (α,β) *, then we would be disappointed. There is no such solution.*

!►*Example 8.12:* *We also get*

$$\frac{dy}{dx} = \frac{0}{0}$$

when we let $(x,y) = (2,2)$ *in*

$$\frac{dy}{dx} = \frac{y-2}{x-2} \quad .$$

This time, however, the slope field (sketched in figure 8.7b) suggests that every solution curve passes through this point. And, indeed, solving this simple separable equation yields the formula

$$y = 2 + A(x-2)$$

where A *is an arbitrary constant. This formula gives* $y = 2$ *when* $x = 2$ *no matter what* A *is. Consequently, the initial-value problem*

$$\frac{dy}{dx} = \frac{y-2}{x-2} \qquad \text{with} \quad y(2) = 2$$

has infinitely many solutions.

In both of the above examples, the slope lines were all well defined (possibly with infinite slope) at all but one point in the XY–plane. They are fairly representative examples of what can happen when $F(x, y)$ is undefined at isolated points. Of course, we can easily give examples in which $F(x, y)$ is undefined on vast regions of the XY–plane. There isn't much to be said about these cases, but we'll provide one example for the sake of completeness.

!►*Example 8.13:* *Consider the differential equation*

$$\frac{dy}{dx} = \sqrt{1 - \left(x^2 + y^2\right)} \quad .$$

The right side only makes sense if $x^2 + y^2 \leq 1$ *. Obviously, there can be no "slope field" in any region outside the circle* $x^2 + y^2 = 1$ *(that's why we didn't attempt to sketch it), and it is just plain silly to ask for a solution to this differential equation satisfying, say,* $y(x_0) = y_0$ *whenever* (x_0, y_0) *is a point outside the circle* $x^2 + y^2 = 1$ *.*

Curves Diverging From or Converging To a Point

In example 8.12 (figure 8.7b), we have solution curves converging to and diverging from the point $(2, 2)$. In that case, $F(x, y)$ was indeterminant at that point. As the next example illustrates, we can have solution curves converging to and diverging from a point even though $F(x, y)$ is a nice well-defined, finite number at that point. Fortunately, for reasons to be explained, this is not very common.

!►*Example 8.14:* *Consider*

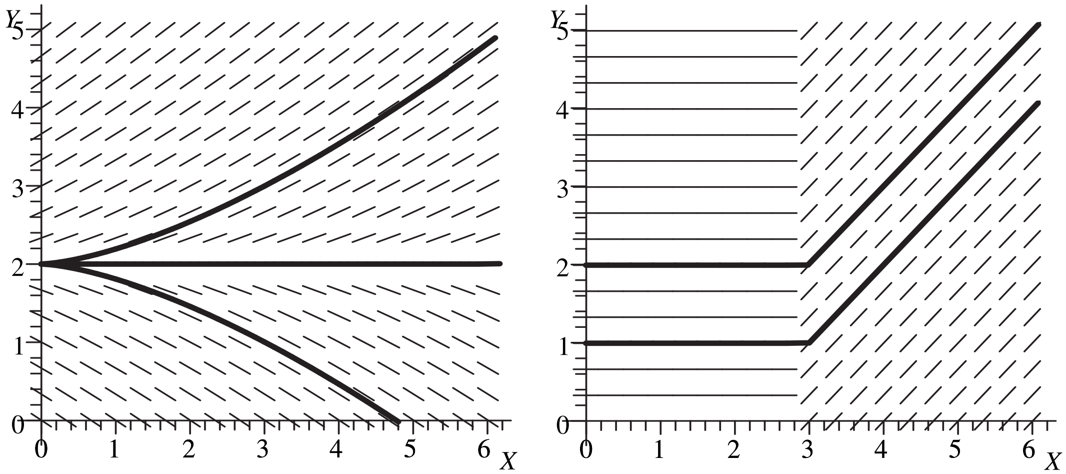
$$\frac{dy}{dx} = \frac{1}{2}(y-2)^{1/3} \quad .$$

A slope field and some solutions for this differential equation are sketched in figure 8.8a. Note that we've sketched three curves diverging from the point $(0, 2)$ *. These curves are the graphs of*

$$y = 2 \quad , \quad y = 2 + \left(\frac{x}{3}\right)^{3/2} \quad \text{and} \quad y = 2 - \left(\frac{x}{3}\right)^{3/2} \quad ,$$

all of which are solutions on $[0, \infty)$ *to the initial-value problem*

$$\frac{dy}{dx} = \frac{1}{2}(y-2)^{1/3} \qquad \text{with} \quad y(0) = 2 \quad .$$



(a) **(b)**

Figure 8.8: Slope fields **(a)** for $y'(x) = \frac{1}{2}(y-2)^{1/3}$ from example 8.14, and **(b)** for the differential equation in example 8.15.

What distinguishes this from example 8.12 (figure 8.7b) is that the right side of the above differential equation is not indeterminant at the point $(0, 2)$ *. Instead, at* $(x, y) = (0, 2)$ *we have*

$$\frac{dy}{dx} = \frac{1}{2}(2-2)^{1/3} = 0 \quad ,$$

which is a perfectly reasonable finite value.

On Existence and Uniqueness

Let us return to the issues of the "existence" and "uniqueness" of the solutions to a generic initial-value problem

$$\frac{dy}{dx} = F(x, y) \qquad \text{with} \quad y(x_0) = y_0 \quad . \tag{8.3}$$

We first discussed these issues in the later part of chapter 3. In particular, you may recall theorem 3.1 on page 48. That theorem assures us that:

> *If both* $F(x, y)$ *and* $\partial F/\partial y$ *are continuous functions on some open region of the* XY*–plane containing the point* (x_0, y_0) *, then:*
>
> 1. *(existence) The above initial-value problem has at least one solution* $y = y(x)$ *.*
>
> 2. *(uniqueness) There is an open interval* (a, b) *containing* x_0 *on which this* $y = y(x)$ *is the only solution to this initial-value problem.*

Now consider every slope field for this differential equation in some region around (x_0, y_0) on which F is continuous. The continuity of F ensures that the slope lines will be well defined with finite slope at every point, and that these slopes will vary continuously as you move throughout the region. Clearly, there is a curve through the point (x_0, y_0) that is "parallel" to every possible slope field, and this curve will have to be the graph of a function satisfying

$$\frac{dy}{dx} = F(x, y) \qquad \text{with} \quad y(x_0) = y_0 \quad .$$

This graphically verifies the "existence" part of theorem 3.1. In fact, a good mathematician can take the above argument, and construct a rigorous proof that

> *If F is a continuous functions on some open region of the XY–plane containing the point (x_0, y_0), then*
>
> $$\frac{dy}{dx} = F(x, y) \qquad \textit{with} \quad y(x_0) = y_0 \quad .$$
>
> *has at least one solution on some interval (a, b) with $a < x_0 < b$.*

So we can use our slope fields to visually convince ourselves that initial-value problem (8.3) has a solution whenever F is reasonably well behaved. But what about uniqueness? Will the curve drawn be the only possible curve matching the slope fields? Well, in example 8.14 (figure 8.8a) we had three different curves passing through the point $(0, 2)$, all of which matched the slope field. Thus, we have (at least) three different solutions to the initial-value problem given in that example. And this occurred even though the even though the $F(x, y)$ is a continuous function on all of the XY–plane.

This is where the second part of theorem 3.1 can help us in using slope fields. It assures us that there is only one solution (over some interval containing x_0) to

$$\frac{dy}{dx} = F(x, y) \qquad \text{with} \quad y(x_0) = y_0$$

provided both $F(x, y)$ and $^{\partial F}/_{\partial y}$ are continuous in a region around (x_0, y_0). In example 8.14,

$$F(x, y) = \frac{1}{2}(y - 2)^{1/3} \quad .$$

While this F is continuous throughout the XY–plane, the corresponding $^{\partial F}/_{\partial y}$,

$$\frac{\partial F}{\partial y} = \frac{1}{2 \cdot 3}(y - 2)^{-2/3} = \frac{1}{6(y - 2)^{2/3}} \quad ,$$

is not continuous at any (x, y) with $y = 2$. Consequently, theorem 3.1 does not assure us that the initial-value problem given in example 8.14 has only one solution. And, indeed, we discovered three solutions.

So, what can we say about using slope fields to sketch solutions to

$$\frac{dy}{dx} = F(x, y) \qquad \text{with} \quad y(x_0) = y_0 \quad ?$$

Based on the example we've seen and the discussion above, we can safely make the following three statements:

1. If $F(x, y)$ is reasonably well behaved in some region around the point (x_0, y_0) (i.e., $F(x, y)$ well defined, finite and continuous at each point (x, y) in this region), then we can use slope fields to sketch a curve that will be a reasonable approximation to a solution to the initial-value problem over some interval.

2. If $F(x, y)$ is not reasonably well behaved in some region around the point (x_0, y_0), in particular, if $F(x_0, y_0)$ is not a well-defined finite value, then we may or may not have a solution to the given initial-value problem. The slope field will probably give us an idea of the nature of solution curves passing through points near (x_0, y_0), but more analysis may be needed to be needed to determine if the given initial-value problem has a solution, and, if it exists, the nature of that solution.

3. Even if $F(x, y)$ is reasonably well behaved in some region around the point (x_0, y_0), it is worthwhile to see if ${}^{\partial F}/_{\partial y}$ is also well defined everywhere in that region. If so, then the curve drawn using a decent slope field will be a reasonably good approximation of the graph to the only solution to the initial-value problem. Otherwise, there is a possibility of multiple solutions.

Finally, let us observe that we can have unique, reasonably well-behaved solutions even though both F and ${}^{\partial F}/_{\partial y}$ have discontinuities. This was evident in example 8.9 on page 172 (figure 8.6b), and is evident in the following example.

!►*Example 8.15:* *The right side of*

$$\frac{dy}{dx} = \begin{cases} 0 & \text{if} \quad x < 3 \\ 1 & \text{if} \quad 3 \leq x \end{cases}$$

is discontinuous at every point (x, y) with $x = 3$. This differential equation yields the simple, yet striking, slope field in figure 8.8b. And from this slope field, it should be clear that there is exactly one solution to this differential equation satisfying, say, $y(3) = 2$. That is one of the curves sketched, and (as you can verify) that curve is the graph of

$$y(x) = \begin{cases} 2 & \text{if} \quad x < 3 \\ x - 1 & \text{if} \quad 3 \leq x \end{cases} .$$

8.5 Tests for Stability

In section 8.3, we discussed the stability of constant solutions, using slope fields to visually distinguish between constant solutions that were stable, asymptotically stable or unstable. That was good for developing a basic understanding of stability, but, as we saw in the examples, it is not always possible to determine the stability of a given constant solution from just a slope field. So let us take a closer look at the geometry of the solution curves to a first-order differential equation

$$\frac{dy}{dx} = F(x, y)$$

which start out near the graph of a constant solution $y = y_0$, and see if we can derive some relatively simple "computational" tests for verifying the stability or instability suggested by such slope fields as in figures 8.10 and 8.10.

Throughout this section, we'll assume we have three finite numbers y_0, y_l and y_h with

$$y_l < y_0 < y_h \quad .$$

The constant solution to our differential equation will be $y = y_0$, and the strips of interest will be those strips bounded by the horizontal lines

$$y = y_0 \quad , \quad y = y_l \quad \text{and} \quad y = y_h \quad .$$

We will also assume $F(x, y)$ is at least a continuous function of both x and y on these strips. This ensures that we need not worry about any truly "bad" problem points in the strips, and can safely assume that no solution curve "ends" at a point in one of our strips.

Autonomous Equations

Since the analysis is much easier with autonomous equations, we will start with those. Accordingly, we assume $y = y_0$ is a constant solution to a differential equation of the form

$$\frac{dy}{dx} = g(y)$$

where g is a continuous function on the closed interval $[y_l, y_h]$.

The Single Crossing Point Lemma

We start by observing that no solution curve can cross a horizontal line $y = y_c$ more than once if $g(y_c)$ is a finite, nonzero value. In particular, suppose $g(y_c) > 0$ (as we have for $y_c = y_h$ in figure 8.9), and suppose $y = y(x)$ is a solution to our autonomous differential equation whose graph crosses the horizontal line $y = y_c$ at the point $(x, y) = (x_c, y_c)$. At this point, the slope of the solution curve is positive, telling us that the solution curve goes from below to above this horizontal line as x goes from the left to the right of x_c. And since $g(y) > 0$ at every point on the horizontal line $y = y_c$, there is no point where the solution curve can come back below this horizontal line as x increases.

Likewise, if $g(y_c) < 0$ (see figure 8.10), then each solution curve crossing $y = y_c$ goes from above to below $y = y_c$, and can never "come back up" to cross $y = y_c$ a second time.

We'll use this observation several times in what follows, so let us dignify it as a lemma:

Lemma 8.1
Let $y = y(x)$ be a solution to

$$\frac{dy}{dx} = g(y)$$

on some interval $(0, x_{max})$ whose graph crosses a horizontal line $y = y_c$ when $x = x_c$. Suppose, further, that $g(y_c)$ is a finite, nonzero value. Then,

$$g(y_c) > 0 \quad \Longrightarrow \quad y(x) > y_c \quad \text{whenever} \quad x_c < x < x_{max} \quad ,$$

while

$$g(y_c) < 0 \quad \Longrightarrow \quad y(x) < y_c \quad \text{whenever} \quad x_c < x < x_{max} \quad .$$

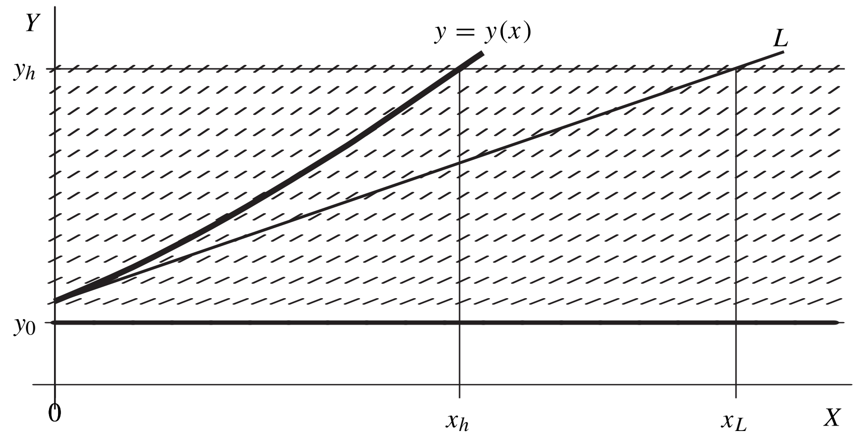


Figure 8.9: A slope field on the strip $y_0 \leq y \leq y_h$ for $y' = g(y)$ when $g(y_0) = 0$ and g is an increasing function on $[y_0, y_h]$.

Instability

Consider the case illustrated in figure 8.9. Here, $y = y_0$ is a constant solution to

$$\frac{dy}{dx} = g(y) \quad ,$$

and the slope of the slope line at (x, y) (i.e., the value of $g(y)$) increases as y increases from $y = y_0$ to $y = y_h$. So if

$$y_0 < y_1 < y_2 < y_h \quad ,$$

then

$$0 = g(y_0) < g(y_1) < g(y_2) < g(y_h) \quad . \tag{8.4}$$

Now take any solution $y = y(x)$ to

$$\frac{dy}{dx} = g(y) \qquad \text{with} \quad y_0 < y(0) < y_h \quad ,$$

and let L be the straight line tangent to the graph of this solution at the point where $x = 0$ (see figure 8.9). From inequality set (8.4) (and figure 8.9), we see that:

1. The slope of tangent line L is positive. Hence, L crosses the horizontal line $y = y_h$ at some point (x_L, y_h) with $0 < x_L < \infty$.

2. At each point in the strip, the slope of the tangent to the graph of $y = y(x)$ is at least as large as the slope of L. So, as x increases, the graph of $y = y(x)$ goes upwards faster than L. Consequently, this solution curve crosses the horizontal line $y = y_h$ at a point (x_h, y_h) with $0 < x_h < x_L$.

From this and lemma 8.1, it follows that, if x is a point in the domain of our solution $y = y(x)$, then

$$y(x) \geq y_h \qquad \text{whenever} \quad x > x_h \quad .$$

That is,

$$y(x) - y_0 > y_h - y_0$$

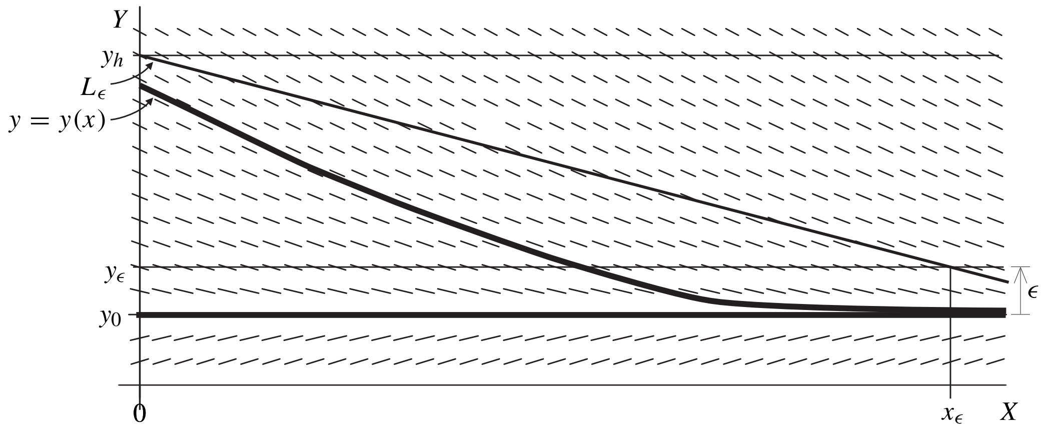


Figure 8.10: Slope field for $y' = g(y)$ when $g(y_0) = 0$ and g is a decreasing function on $[y_l, y_h]$ with $y_l < y_0 < y_h$.

whenever x is a point in the domain of $y = y(x)$ with $x_h < x$.

This tells us that, no matter how close we pick $y(0)$ to y_0 (at least with $y(0) > y_0$), the graph of our solution will, as x increases, diverge to a distance of at least $y_h - y_0$ from y_0 . This means we can *not* choose a distance ϵ with

$$\epsilon < y_h - y_0 \quad ,$$

and find a solution $y = y(x)$ to

$$\frac{dy}{dx} = g(y) \qquad \text{with} \quad y(0) > y_0$$

that remains within ϵ of y_0 for all values of x . In other words, $y = y_0$ is not a stable constant solution.

This, along with analogous arguments when $g(y)$ is an increasing function on $[y_l, y_0]$, gives us:

Theorem 8.2
Let $y = y_0$ be a constant solution to

$$\frac{dy}{dx} = g(y) \quad .$$

where g is a continuous function on some interval $[y_l, y_h]$ with $y_l < y < y_h$. Then $y = y_0$ is an unstable constant solution if either of the following holds:

1. *$g(y)$ is an increasing function on $[y_l, y_0]$ for some $y_l < y_0$.*

2. *$g(y)$ is an increasing function on $[y_0, y_h]$ for some $y_0 < y_h$.*

Stability

Now consider the case illustrated in figure 8.10. Here, $y = y_0$ is a constant solution to

$$\frac{dy}{dx} = g(y)$$

when $g(y)$ (the slope of the slope line at point (x, y)) is a decreasing function on an interval $[y_l, y_h]$. So, if

$$y_l < y_{-2} < y_{-1} < y_0 < y_1 < y_2 < y_h \quad ,$$

then

$$g(y_l) > g(y_{-2}) > g(y_{-1}) > 0 > g(y_1) > g(y_2) > g(y_h) \quad .$$

Thus, the slope lines just below the horizontal line $y = y_0$ have positive slope, those just above $y = y_0$ have negative slope, and the slopes become steeper as the distance from the horizontal line $y = y_0$ increases.

The fact that $y = y_0$ is a stable constant solution should be obvious from the figure. After all, the slope lines are all angled toward $y = y_0$ as x increases, "directing" the solutions curves toward $y = y_0$ as x increases.

Figure 8.10 also suggests that, if $y = y(x)$ is any solution to

$$\frac{dy}{dx} = g(y) \qquad \text{with} \quad y_l < y(0) < y_h \quad ,$$

then

$$\lim_{x \to \infty} y(x) = y_0 \quad ,$$

suggesting that $y = y_0$ is also asymptotically stable. To rigorously confirm this, it is convenient to separately consider the three cases

$$y(0) = y_0 \qquad , \qquad y_0 < y(0) < y_h \qquad \text{and} \qquad y_l < y(0) < y_0 \quad .$$

The first case is easily taken care of. If $y(0) = y_0$, then our solution $y = y(x)$ must be the constant solution $y = y_0$ (the already noted stability of this constant solution prevents any other possible solutions). Hence,

$$\lim_{x \to \infty} y(x) = \lim_{x \to \infty} y_0 = y_0 \quad .$$

Next, assume $y = y(x)$ is a solution to

$$\frac{dy}{dx} = g(y) \qquad \text{with} \quad y_0 < y(0) < y_h \quad .$$

To show

$$\lim_{x \to \infty} y(x) = y_0 \quad ,$$

it helps to remember that the above limit is equivalent to saying that we can make $y(x)$ as close to y_0 as desired (say, within some small, positive distance ϵ) by simply picking x large enough.

So let ϵ be any small, positive value, and let us show that there is a corresponding "large enough value" x_ϵ so that $y(x)$ is within a distance ϵ of y_0 whenever x is bigger than x_ϵ. And since we are only concerned with ϵ being "small", let's go ahead and assume

$$\epsilon < y_h - y_0 \quad .$$

Now, for notational convenience, let $y_\epsilon = y_0 + \epsilon$, and let L_ϵ be the straight line through the point $(x, y) = (0, y_h)$ with the same slope as the slope lines along the horizontal line $y = y_\epsilon$ (see figure 8.10). Because $y_h > y_\epsilon > y_0$, the slope lines along the line $y = y_\epsilon$ have negative slope. Hence, so does L_ϵ. Consequently, the line L_ϵ goes downwards from point $(0, y_h)$, intersecting the horizontal line $y = y_\epsilon$ at some point to the right of the Y–axis. Let x_ϵ be the X–coordinate of that point.

Next, consider the graph of our solution $y = y(x)$ when $0 \leq x \leq x_\epsilon$. Observe that:

1. This part of this solution curve starts at the point $(0, y(0))$, which is between the lines L_ϵ and $y = y_0$.
2. The slope at each point of this solution curve above $y = y_\epsilon$ is less than the slope of the line L_ϵ. Hence, this part of the solution curve must go downwards faster than L_ϵ as x increases.
3. If $y(x) < y_\epsilon$ for some value of x, then $y(x) < y_\epsilon$ for all larger values of x. (This is from lemma 8.1.)
4. The graph of $y = y(x)$ cannot go below the horizontal line $y = y_0$ because the slope lines at points just below $y = y_0$ all have positive slope.

These observations tell us that, at least when $0 \leq x \leq x_\epsilon$, our solution curve must remain between the lines L_ϵ and $y = y_0$. In particular, since L_ϵ crosses the horizontal line $y = y_\epsilon$ at $x = x_\epsilon$, we must have

$$y_0 \leq y(x_\epsilon) \leq y_\epsilon = y_0 + \epsilon \quad .$$

From this along with lemma 8.1, it follows that

$$y_0 \leq y(x) \leq y_0 + \epsilon \qquad \text{for all} \quad x > x_\epsilon \quad .$$

Equivalently,

$$0 \leq y(x) - y_0 \leq \epsilon \qquad \text{for all} \quad x > x_\epsilon \quad ,$$

which tells us that $y(x)$ is within ϵ of y_0 whenever $x > x_\epsilon$. Hence, we can make $y(x)$ as close as desired to y_0 by choosing x large enough. That is,

$$\lim_{x\to\infty} y(x) = y_0 \quad .$$

That leaves the verification of

$$\lim_{x\to\infty} y(x) = y_0$$

when $y = y(x)$ satisfies

$$\frac{dy}{dx} = g(y) \qquad \text{with} \quad y_l < y(0) < y_0 \quad .$$

This will be left to the interested reader (just use straightforward modifications of the arguments in the last few paragraphs — start by vertically flipping figure 8.10).

To summarize our results:

Theorem 8.3

Let $y = y_0$ be a constant solution to an autonomous differential equation

$$\frac{dy}{dx} = g(y) \quad .$$

This constant solution is both stable and asymptotically stable if there is an interval $[y_l, y_h]$, with $y_l < y_0 < y_h$, on which $g(y)$ is a decreasing continuous function.

Differential Tests for Stability

Recall from elementary calculus that you can determine whether a function g is an increasing or decreasing function by just checking to see if its derivative is positive or negative. To be precise,

$$g'(y) > 0 \quad \text{for} \quad a \leq y \leq b \quad \Longrightarrow \quad g \text{ is an increasing function on } [a, b]$$

and

$$g'(y) < 0 \quad \text{for} \quad a \leq y \leq b \quad \Longrightarrow \quad g \text{ is a decreasing function on } [a, b] \quad .$$

Consequently, we can replace the lines in theorems 8.3 and 8.2 about g being increasing or decreasing with corresponding conditions on g', obtaining the following:

Theorem 8.4
Let $y = y_0$ be a constant solution to an autonomous differential equation

$$\frac{dy}{dx} = g(y)$$

in which g is a differentiable function on some interval $[y_l, y_h]$ with $y_l < y_0 < y_h$. Then $y = y_0$ is both a stable and asymptotically stable constant solution if

$$g'(y) < 0 \qquad \text{for} \quad y_l \leq y_0 \leq y_h \quad .$$

Theorem 8.5
Let $y = y_0$ be a constant solution to an autonomous differential equation

$$\frac{dy}{dx} = g(y)$$

in which g is a differentiable function on some interval $[y_l, y_h]$ with $y_l < y_0 < y_h$. Then $y = y_0$ is an unstable constant solution if either

$$g'(y) > 0 \qquad \text{for} \quad y_l < y < y_0$$

or

$$g'(y) > 0 \qquad \text{for} \quad y_0 < y < y_h \quad .$$

But now recall that, if a function is sufficiently continuous and is positive (or negative) at some point, then that function remains positive (or negative) over some interval surrounding that point. With this we can reduce the above theorems to the following single theorem

Theorem 8.6
Let $y = y_0$ be a constant solution to an autonomous differential equation

$$\frac{dy}{dx} = g(y)$$

in which g is differentiable and g' is continuous on some interval $[y_l, y_h]$ with $y_l < y_0 < y_h$. Then:

1. *$y = y_0$ is a stable and asymptotically stable constant solution if $g'(y_0) < 0$.*

2. $y = y_0$ *is an unstable constant solution if* $g'(y_0) > 0$.

!►Example 8.16: *Let us again consider the autonomous differential equation considered earlier in example 8.7 on page 169,*

$$\frac{dy}{dx} = \frac{1}{2}(4-y)(y-2)^{4/3} \quad ,$$

and whose slope field was sketched in figure 8.5b on page 169.

Because the right side,

$$g(y) = \frac{1}{2}(4-y)(y-2)^{4/3}$$

is zero when y *is either* 2 *or* 4 *, this differential equation has constant solutions*

$$y = 2 \qquad \text{and} \qquad y = 4 \quad .$$

So as to apply any of the above theorems, we compute $g'(y)$ *:*

$$g'(y) = \frac{d}{dy}\left[\frac{1}{2}(4-y)(y-2)^{4/3}\right] = -\frac{1}{2}(y-2)^{4/3} + \frac{2}{3}(4-y)(y-2)^{1/3} \quad .$$

After a bit of algebra, this simplifies to

$$g'(y) = \frac{7}{6}\left(\frac{22}{7} - y\right)(y-2)^{1/3} \quad .$$

Plugging in $y = 4$ *, we get*

$$g'(4) = \frac{7}{6}\left(\frac{22}{7} - 4\right)(4-2)^{1/3} = \frac{7}{6}\left(\frac{22}{7} - \frac{28}{7}\right)\sqrt[3]{2} = -\sqrt[3]{2} < 0 \quad .$$

Theorem 8.6 then tells us that the constant solution $y = 4$ *is stable and asymptotically stable, just as we suspected from looking at the slope field in figure 8.5b.*

Unfortunately, we cannot apply theorem 8.6 to determine the stability of the other constant solution, $y = 2$ *, since*

$$g'(2) = \frac{7}{6}\left(\frac{22}{7} - 2\right)(2-2)^{1/3} = 0 \quad .$$

Instead, we must look a little more closely at the formula for $g'(y)$ *, and observe that, if*

$$2 < y < \frac{22}{7} \quad ,$$

then

$$g'(y) = \frac{7}{6}\underbrace{\left(\frac{22}{7} - y\right)}_{>0}\underbrace{(y-2)^{1/3}}_{>0} > 0 \quad .$$

The test given in theorem 8.5 (with $[y_0, y_h] = \left[2, {}^{22}/_{7}\right]$ *) applies and assures us that* $y = 2$ *is an unstable constant solution, just as we suspected from looking at figure 8.5b.*

Nonautonomous Equations

Take a quick look at figures 8.9 and 8.10, only imagine that the slope lines are also becoming steeper as x increases. With a little thought, you will realize that the arguments leading to theorems 8.2 and 8.3 remain valid even if these slope lines so depend on x. Combined with the relationship between the sign of the derivatives and the increasing/decreasing behavior of functions then leads to the following analogs of theorems 8.4 and 8.5:

Theorem 8.7
Let $y = y_0$ be a constant solution to an autonomous differential equation

$$\frac{dy}{dx} = F(x, y) \quad .$$

in which $F(x, y)$ is differentiable with respect to both x and y at every point in some strip

$$\{(x, y) : 0 \leq x \text{ and } y_l \leq y \leq y_h\}$$

with $y_l < y_0 < y_h$. Further suppose that, at each point in this strip above the line $y = y_0$,

$$\frac{\partial F}{\partial y} < 0 \qquad \text{and} \qquad \frac{\partial F}{\partial x} \leq 0 \quad ,$$

and that, at each point in this strip below the line $y = y_0$,

$$\frac{\partial F}{\partial y} < 0 \qquad \text{and} \qquad \frac{\partial F}{\partial x} \geq 0 \quad .$$

Then $y = y_0$ is both a stable and asymptotically stable constant solution.

Theorem 8.8
Let $y = y_0$ be a constant solution to

$$\frac{dy}{dx} = F(x, y)$$

in which $F(x, y)$ is differentiable with respect to both x and y at every point in some strip

$$\{(x, y) : 0 \leq x \text{ and } y_l \leq y \leq y_h\}$$

with $y_l < y_0 < y_h$. Further suppose that, at each point in this strip above the line $y = y_0$,

$$\frac{\partial F}{\partial y} > 0 \qquad \text{and} \qquad \frac{\partial F}{\partial x} \geq 0 \quad ,$$

or that, at each point in this strip below the line $y = y_0$,

$$\frac{\partial F}{\partial y} > 0 \qquad \text{and} \qquad \frac{\partial F}{\partial x} \leq 0 \quad .$$

Then $y = y_0$ is an unstable constant solution.

Additional Exercises

8.2. *For each of the following, construct the slope field for the given differential equation on the indicated* 2×2 *grid of listed points:*

a. $\frac{dy}{dx} = \frac{1}{2}\left(x^2 + y^2\right)$ *at* $(x, y) = (0, 0), (1, 0), (0, 1)$ *and* $(1, 1)$

b. $2\frac{dy}{dx} = x^2 - y^2$ *at* $(x, y) = (0, 0), (1, 0), (0, 1)$ *and* $(1, 1)$

c. $\frac{dy}{dx} = \frac{y}{x}$ *at* $(x, y) = (1, 1), (3/2, 1), (1, 3/2)$ *and* $(3/2, 3/2)$

d. $(2x + 1)\frac{dy}{dx} = x^2 - 2y^2$ *at* $(x, y) = (0, 1), (1, 1), (0, 2)$ *and* $(1, 2)$

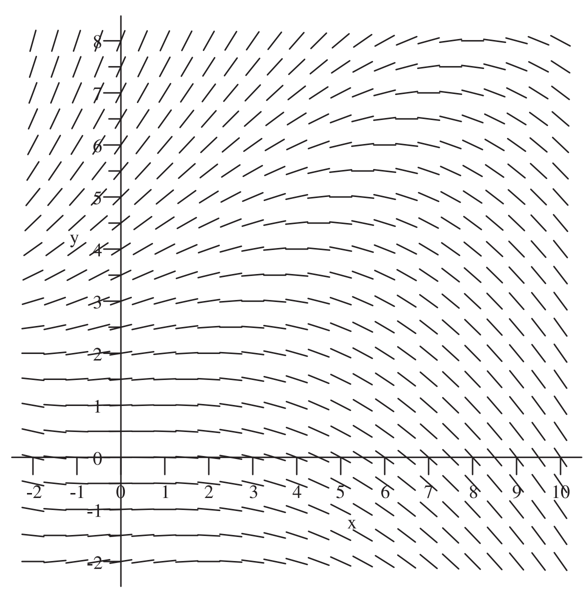
Several slope fields for unspecified first-order differential equations have been given below. For sketching purposes, you may want to use an enlarged photocopy of each given slope field.

8.3. *A slope field for an unspecified first-order differential equation is given to the right. Using this slope field:*

a i. *Sketch the graph of the solution to this differential equation that satisfies* $y(0) = 2$.

ii. *Using your sketch, find (approximately) the value of* $y(8)$, *where* $y(x)$ *is the solution just sketched.*

b. *Sketch the graphs of two other solutions to this unspecified differential equation.II*



8.4. *A slope field for an unspecified first-order differential equation is given to the right. Using this slope field:*

a. *Sketch the graphs of the solutions to this differential equation that satisfy*

i. $y(0) = 2$

ii. $y(0) = 4$

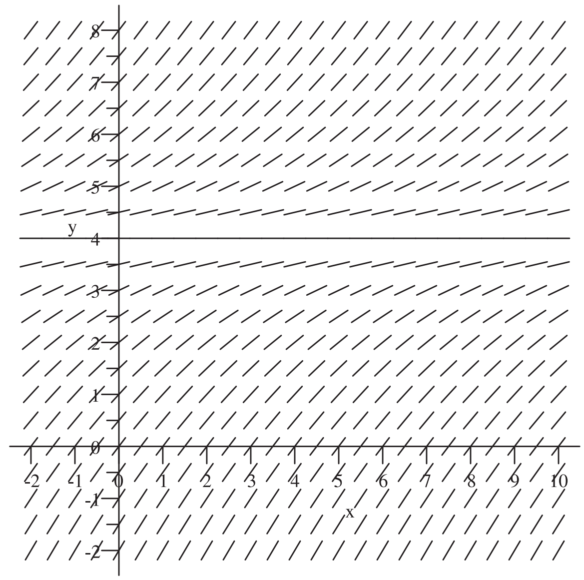
iii. $y(0) = 4.5$

b. *What, approximately, is* $y(4)$ *if* y *is the solution to this unspecified differential equation satisfying*

i. $y(0) = 2$?

ii. $y(0) = 4$?

iii. $y(0) = 4.5$?



8.5. *A slope field for an unspecified first-order differential equation is given to the right. Using this slope field:*

a. *Let* $y(x)$ *be the solution to the differential equation with* $y(0) = 5$.

i. *Sketch the graph of this solution.*

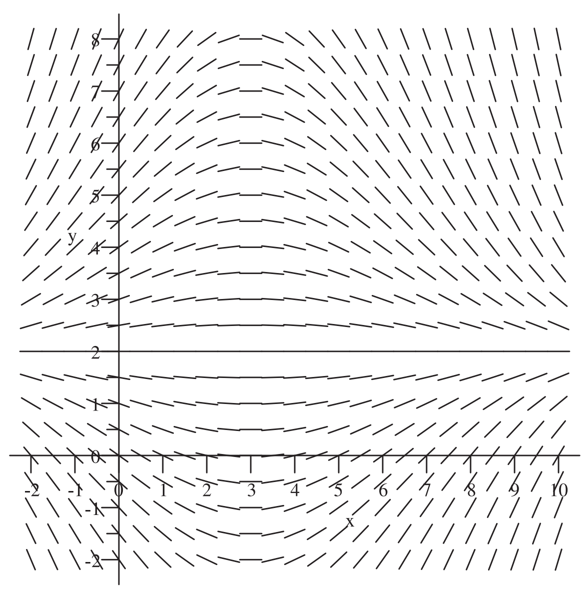
ii. *What (approximately) is the maximum value of* $y(x)$ *on the interval* $(-2, 10)$ *, and where does it occur?*

iii. *What (approximately) is* $y(10)$?

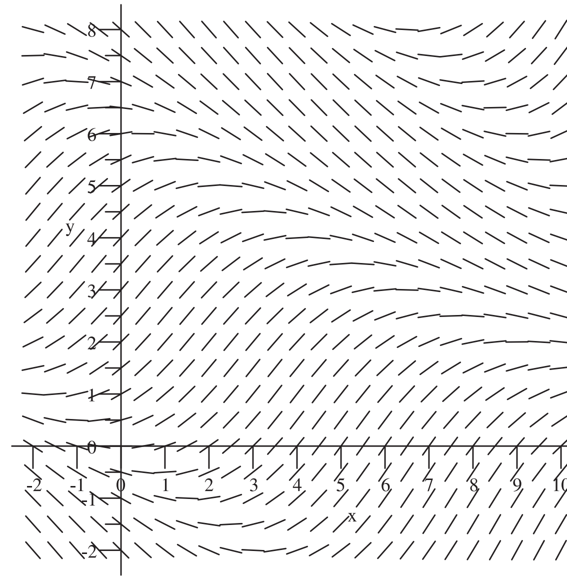
b. *Now let* $y(x)$ *be the solution to the differential equation with* $y(0) = 0$.

i. *Sketch the graph of this solution.*

ii. *What (approximately) is* $y(10)$?



8.6. *A slope field for an unspecified first-order differential equation is given to the right. Using this slope field:*



a. *Let* $y(x)$ *be the solution to the differential equation with* $y(0) = 4$ *.*

i. *Sketch the graph of this solution.*

ii. *What (approximately) is the maximum value of* $y(x)$ *on the interval* $(-2, 10)$ *, and where does it occur?*

iii. *What (approximately) is* $y(10)$ *?*

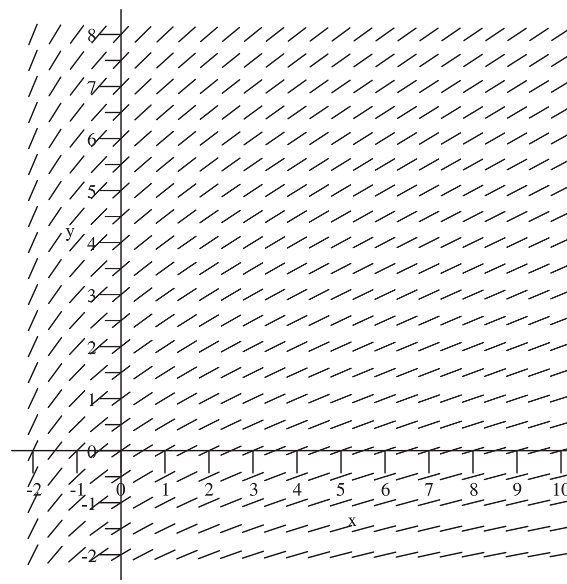
b. *Now let* $y(x)$ *be the solution to the differential equation with* $y(2) = 0$ *.*

i. *Sketch the graph of this solution.*

ii. *What (approximately) is the maximum value of* $y(x)$ *on the interval* $(-2, 10)$ *, and where does it occur?*

iii. *What (approximately) is* $y(10)$ *?*

8.7. *A slope field for some first-order differential equation is given to the right. Using this slope field:*



a. *Let* $y(x)$ *be the solution to the differential equation with* $y(0) = 2$ *.*

i. *Sketch the graph of this solution.*

ii. *What (approximately) is* $y(3)$ *?*

b. *Now let* $y(x)$ *be the solution to the differential equation with* $y(3) = 1$ *.*

i. *Sketch the graph of this solution.*

ii. *What (approximately) is* $y(0)$ *?*

8.8. *Look up the commands for generating slope fields for first-order differential equations in your favorite computer math package (they may be the same commands for generating "direction fields").*[7] *Then, use these commands to do the following for each problem below:*

i. Sketch the indicated slope field for the given differential equation.

[7] In Maple, these commands are dfieldplot and DEplot. The command DEplot can even be used to sketch approximations to solution curves.

ii. Use the resulting slope field to sketch (by hand) some of the solution curve for the given differential equation.

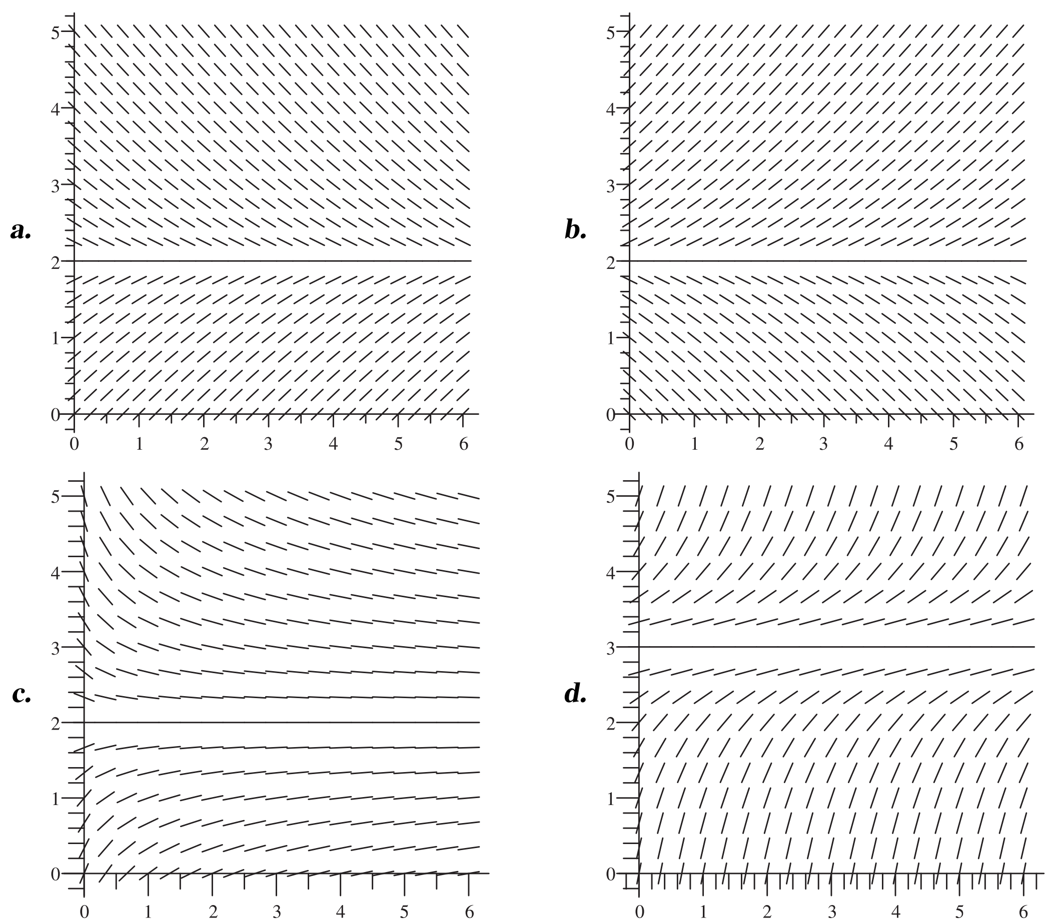
a. $\dfrac{dy}{dx} = \sin(x+y)$ *using a* $25{\times}25$ *grid on the region* $-2 \leq x \leq 10$ *and* $-2 \leq y \leq 10$

b. $10\dfrac{dy}{dx} = y(5-y)$ *using a* $25{\times}25$ *grid on the region* $-2 \leq x \leq 10$ *and* $-2 \leq y \leq 10$

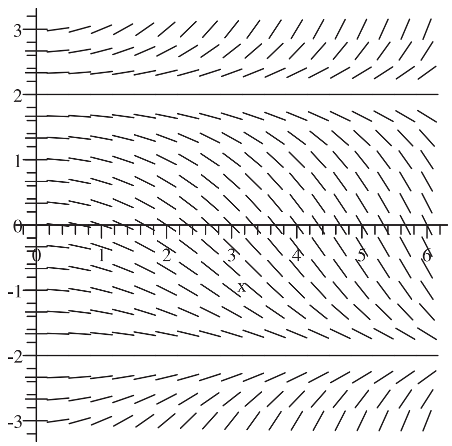
c. $10\dfrac{dy}{dx} = y(y-5)$ *using a* $25{\times}25$ *grid on the region* $-2 \leq x \leq 10$ *and* $-2 \leq y \leq 10$

d. $2\dfrac{dy}{dx} = y(y-2)^2$ *using a* $25{\times}17$ *grid on the region* $-2 \leq x \leq 10$ *and* $-1 \leq y \leq 3$

8.9. *Slope fields for several (unspecified) first-order differential equations have be sketched below. Assume that each horizontal line is the graph of a constant solution to the corresponding differential equation. Identify each of these constant solutions, and, for each constant solution, decide whether the slope field is indicating that it is a stable, asymptotically stable, or unstable constant solution.*



e.



f.