7

The Exact Form and General Integrating Factors

In the previous chapters, we've seen how separable and linear differential equations can be solved using methods for converting them to forms that can be easily integrated. In this chapter, we will develop a more general approach to converting a differential equation to a form (the "exact form") that can be integrated through a relatively straightforward procedure. We will see just what it means for a differential equation to be in exact form and how to solve differential equations in this form. Because it is not always obvious when a given equation is in exact form, a practical "test for exactness" will also be developed. Finally, we will generalize the notion of integrating factors to help us find exact forms for a variety of differential equations.

The theory and methods we will develop here are more general than those developed earlier for separable and linear equations. In fact, the procedures developed here can be used to solve any separable or linear differential equation (though you'll probably prefer using the methods developed earlier). More importantly, the methods developed in this chapter can, in theory at least, be used to solve a great number of other first-order differential equations. As we will see though, practical issues will reduce the applicability of these methods to a somewhat smaller (but still significant) number of differential equations.

By the way, the theory, the computational procedures, and even the notation that we will develop for equations in exact form are all very similar to that often developed in the later part of many calculus courses for two-dimensional conservative vector fields. If you've seen that theory, look for the parallels between it and what follows.

7.1 The Chain Rule

The exact form for a differential equation comes from one of the chain rules for differentiating a composite function of two variables. Because of this, it may be wise to briefly review these differentiation rules.

First, suppose ϕ is a differentiable function of a single variable y (so $\phi = \phi(y)$), and that y, itself, is a differentiable function of another variable t (so $y = y(t)$). Then the composite function $\phi(y(t))$ is a differentiable function of t whose derivative is given by the (elementary) chain rule

$$\frac{d}{dt}[\phi(y(t))] = \phi'(y(t))\ y'(t) \quad .$$

A less precise (but more suggestive) description of this chain rule is

$$\frac{d}{dt}[\phi(y(t))] = \frac{d\phi}{dy}\frac{dy}{dt} \quad .$$

!►*Example 7.1:* *Let*

$$y(t) = t^2 \qquad \textit{and} \qquad \phi(y) = \sin(y) \quad .$$

Then

$$\phi(y(t)) = \sin\left(t^2\right) \quad ,$$

and

$$\frac{d}{dt}\sin\left(t^2\right) = \frac{d}{dt}[\phi(y(t))] = \frac{d\phi}{dy}\frac{dy}{dt}$$
$$= \frac{d}{dy}[\sin(y)] \cdot \frac{d}{dt}\left[t^2\right] = \cos(y)\cdot 2t = \cos\left(t^2\right)2t \quad .$$

(In practice, of course, you probably do not explicitly write out all the steps listed above.)

Now suppose ϕ is a differentiable function of two variables x and y (so $\phi = \phi(x, y)$), while both x and y are differentiable functions of a single variable t (so $x = x(t)$ and $y = y(t)$). Then the composite function $\phi(x(t), y(t))$ is a differentiable function of t, and its derivative can be computed using a chain rule typically encountered later in the study of calculus; namely,

$$\frac{d}{dt}[\phi(x(t), y(t))] = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} \quad . \tag{7.1}$$

In practice, it is usually easier to compute this derivative by simply replacing the x and y in the formula for $\phi(x, y)$ with the corresponding formulas $x(t)$ and $y(t)$, and then computing that formula of t to compute the above derivative. Still, this chain rule (and other chain rules involving functions of several variables) can be quite useful in more advanced applications. Our particular interest is in the corresponding chain rule for computing

$$\frac{d}{dx}[\phi(x, y(x))] \quad ,$$

which we can obtain from equation (7.1) by simply letting $x = t$. Then

$$\frac{dx}{dt} = 1 \quad , \quad y = y(t) = y(x) \quad , \quad \frac{dy}{dt} = \frac{dy}{dx} \quad ,$$

and equation (7.1) reduces to

$$\frac{d}{dx}[\phi(x, y(x))] = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} \quad . \tag{7.2}$$

For brevity, we will henceforth refer to this formula as *chain rule (7.2)* (not very original, but better that constantly repeating "the chain rule described in equation (7.2)").

Don't forget the difference between $^{d}/_{dx}$ and $^{\partial}/_{\partial x}$. If $\phi = \phi(x, y)$, then

$$\frac{d\phi}{dx} = \text{the derivative of } \phi(x, y) \text{ assuming } x \text{ is the variable and } y \text{ is a function of } x\,.$$

while

$$\frac{\partial\phi}{\partial x} = \text{the derivative of } \phi(x, y) \text{ assuming } x \text{ is the variable and } y \text{ is a constant.}$$

!►Example 7.2: *Assume* y *is some function of* x *(i.e.,* $y = y(x)$ *) and*

$$\phi(x, y) = y^2 + x^2 y \quad .$$

Then

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left[y^2 + x^2 y\right] = 2xy \quad , \quad \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left[y^2 + x^2 y\right] = 2y + x^2 \quad ,$$

and, by chain rule (7.2),

$$\frac{d}{dx}\left[\phi(x, y(x))\right] = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 2xy + \left[2y + x^2\right]\frac{dy}{dx} \quad .$$

If, for example, $y = \sin(x)$ *, then the above becomes*

$$\frac{d}{dx}\left[\phi(x, y(x))\right] = 2x\sin(x) + \left[2\sin(x) + x^2\right]\cos(x) \quad .$$

On the other hand, if $y = y(x)$ *is some unknown function, then, after replacing* ϕ *with its formula, we simply have*

$$\frac{d}{dx}\left[y^2 + x^2 y\right] = 2xy + \left[2y + x^2\right]\frac{dy}{dx} \quad . \tag{7.3}$$

In our use of chain rule (7.2), y will be an unknown function of x, and the right side of equation (7.2) will correspond to one side of whatever differential equation is being considered.

7.2 The Exact Form, Defined

Let $\mathcal{R}$ be some region in the XY–plane. We will say that a first-order differential equation is in *exact form* (on $\mathcal{R}$) if and only if both of the following hold:

1. The differential equation is written in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{7.4a}$$

where $M(x, y)$ and $N(x, y)$ are known functions of x and y.

and

2. There is a differentiable function $\phi = \phi(x, y)$ on $\mathcal{R}$ such that

$$\frac{\partial\phi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial\phi}{\partial y} = N(x, y) \tag{7.4b}$$

at every point in $\mathcal{R}$.

We will refer to the above $\phi(x, y)$ as a *potential function* for the differential equation.[1]

In practice, the region $\mathcal{R}$ is often either the entire XY–plane or a significant portion of it. There are a few technical issues regarding this region, but we can deal with these issues later. In the meantime, little harm will be done by not explicitly stating the region. Just keep in mind that if we say an certain equation is in exact form, then it is in exact form on some region $\mathcal{R}$, and that the graph of any solution $y = y(x)$ derived will be restricted being a curve in that region.

[1] Referring to ϕ as a "potential function" comes from the theory of conservative vector fields. In fact, it is not common terminology in other differential equation texts. Most other texts just refer to this function as "ϕ".

!►Example 7.3: *Consider the differential equation*

$$2xy + \left[2y + x^2\right]\frac{dy}{dx} = 0 \quad . \tag{7.5}$$

This equation is in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

with

$$M(x,y) = 2xy \qquad \text{and} \qquad N(x,y) = 2y + x^2 \quad .$$

Moreover, if we glance back at example 7.2, we immediately see that, letting

$$\phi(x,y) = y^2 + x^2y \quad ,$$

we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left[y^2 + x^2y\right] = 2xy = M(x,y)$$

and

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left[y^2 + x^2y\right] = 2y + x^2 = N(x,y)$$

everywhere in the XY–plane. So equation (7.5) is in exact form (and we can take $\mathcal{R}$ to be the entire XY–plane).

In the next several sections, we will discuss how to determine when a differential equation is in exact form, how to convert one not in exact form into exact form, how to find a potential function, and how to solve the differential equation using a potential function. However, we will develop the material backwards, starting with solving a differential equation given a known potential function, and working our way towards dealing with equations not in exact form. This ends up being the natural (and least confusing) way to develop the material.

But first, a few general comments regarding exact forms and potential functions:

1. Just being written as

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

does not guarantee that a differential equation is in exact form; there still might not be a $\phi(x,y)$ with

$$\frac{\partial\phi}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = N(x,y) \quad .$$

For example, we will discover that there is no $\phi(x,y)$ satisfying

$$\frac{\partial\phi}{\partial x} = 3y + 3y^3 \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = xy^2 - x \quad .$$

So

$$3y + 3y^3 + \left[xy^2 - x\right]\frac{dy}{dx} = 0$$

is not in exact form.

2. A single differential equation will have several potential functions. In particular, adding any constant to a potential function yields another potential function. After all, if

$$\frac{\partial \phi_0}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial \phi_0}{\partial y} = N(x, y)$$

and

$$\phi_1(x, y) = \phi_0(x, y) + c$$

for some constant c, then, since any derivative of any constant is zero,

$$\frac{\partial \phi_0}{\partial x} = \frac{\partial \phi_1}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial \phi_0}{\partial y} = \frac{\partial \phi_1}{\partial y} = N(x, y) \quad .$$

Moreover, as you will verify in exercises 7.2 and 7.3 (starting on page 154), any differential equation that can be written in one exact form will actually have several exact forms, all corresponding to different (but related) potential functions.

Because of the uniqueness results concerning solutions to first-order differential equations, it does not matter which of these potential functions is used to solve the equation. Thus, in the following, we will only worry about finding *a* potential function for any given differential equation.

3. Many authors refer to an equation in exact form as an *exact equation*. We are avoiding this terminology because, unlike linear and separable differential equations, the "exactness" of an equation can be destroyed by legitimate rearrangements of that equation. For example,

$$\frac{dy}{dx} + x^2 y = e^{2x}$$

is a linear differential equation whether it is written as above or written as

$$\frac{dy}{dx} = e^{2x} - x^2 y \quad .$$

Consider, on the other hand,

$$2xy + \left[2y + x^2\right] \frac{dy}{dx} = 0 \quad .$$

As we saw in example 7.3, this differential equation is in exact form (i.e., is an "exact equation"). However, if we rewrite it as

$$\frac{dy}{dx} = -\frac{2xy}{2y + x^2} \quad ,$$

we have an equation that is not in exact form (i.e., is not an "exact equation").

7.3 Solving Equations in Exact Form Using a Known Potential Function

Observe that, if $\phi = \phi(x, y)$ is any sufficiently differentiable function of x and y, and

$$\frac{\partial\phi}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = N(x, y) \quad ,$$

then the differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

can be rewritten as

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \quad .$$

Chain rule (7.2) then tells us that the left side of this equation is just the derivative of $\phi(x, y(x))$. So this last equation can be written more concisely as

$$\frac{d}{dx}[\phi(x, y)] = 0$$

(with $y = y(x)$). Not only is this concise, it is easily integrated:

$$\int \frac{d}{dx}[\phi(x, y)]\, dx = \int 0\, dx$$

$$\hookrightarrow \qquad \phi(x, y) = c \quad .$$

Think about this last equation for a moment. It describes the relation between x and $y = y(x)$ assuming y satisfies the differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad .$$

In other words, the equation $\phi(x, y) = c$ is an implicit solution to the differential equation for which it is a potential function.

All this is important enough to be restated as a theorem.

Theorem 7.1 (importance of a potential function)

Let $\phi(x, y)$ be a potential function for a given first-order differential equation. Then that differential equation can be written as

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \quad .$$

This, in turn, can be rewritten as

$$\frac{d}{dx}[\phi(x, y)] = 0 \qquad \textit{with} \quad y = y(x) \quad ,$$

which can be integrated,

$$\int \frac{d}{dx}[\phi(x, y)]\, dx = \int 0\, dx \quad ,$$

to obtain the implicit solution

$$\phi(x, y) = c$$

where c is an arbitrary constant.

The above theorem contains all the steps for finding an implicit solution to a differential equation that can be put in exact form, provided you have a corresponding potential function. Of course, you have probably already noticed that this theorem can be shortened to the following:

Corollary 7.2
Let $\phi(x, y)$ be a potential function for a given first-order differential equation. Then

$$\phi(x, y) = c$$

is an implicit solution to that differential equation.

In solving at least the first few differential equations in the exercises at the end of this chapter, you should use all the steps described in theorem 7.1 simply to reinforce your understanding of why $\phi(x, y) = c$. Then feel free to cut out the intermediate steps (i.e., use the corollary). And, of course, don't forget to see if the implicit solution can be solved for y in terms of x, yielding an explicit solution.

!►*Example 7.4:* *Consider the differential equation*

$$2xy + \left[2y + x^2\right]\frac{dy}{dx} = 0 \quad .$$

From example 7.3, we know this is in exact form and has corresponding potential function

$$\phi(x, y) = y^2 + x^2 y \quad .$$

Since

$$\frac{\partial}{\partial x}\left[y^2 + x^2 y\right] = 2xy \qquad \text{and} \qquad \frac{\partial}{\partial y}\left[y^2 + x^2 y\right] = 2y + x^2$$

Our differential equation can be rewritten as

$$\frac{\partial}{\partial x}\left[y^2 + x^2 y\right] + \left(\frac{\partial}{\partial y}\left[y^2 + x^2 y\right]\right)\frac{dy}{dx} = 0 \quad .$$

By chain rule (7.2), this reduces to

$$\frac{d}{dx}\left[y^2 + x^2 y\right] = 0 \qquad \text{with} \quad y = y(x) \quad .$$

Integrating this,

$$\int \frac{d}{dx}\left[y^2 + x^2 y\right] dx = \int 0\, dx \quad ,$$

yields the implicit solution

$$y^2 + x^2 y = c \quad .$$

Rewriting this as

$$y^2 + x^2 y - c = 0$$

and then solving for y provides the explicit solution

$$y = \frac{-x^2 \pm \sqrt{x^4 + 4c}}{2} \quad .$$

Finding the Potential Function

Let use now consider a more difficult problem: Finding a potential function, $\phi(x, y)$, for a given differential equation that has been written in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad .$$

This means finding a function $\phi(x, y)$ satisfying both

$$\frac{\partial \phi}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N(x, y) \quad .$$

A relatively straightforward procedure for finding this ϕ will be outlined in a moment. But first, let us make a few observations regarding this pair of partial differential:

1. We are not assuming the given differential equation is in exact form, so there might not be a solution to this above pair of partial differential equations. Our method will find ϕ if it exists (i.e., if the equation is in exact form), and will lead to an obviously impossible equation otherwise.

2. Because the above pair of equations are partial differential equations, we must treat x and y as independent variables. Do *not* view y as a function of x in solving for ϕ.

3. If you are acquainted with methods for "recovering the potential for a conservative vector field", then you will recognize the following as one of those methods.

Now, here is the procedure:

Basic Procedure for Finding a Potential Function

Assume we have a differential equation in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad ,$$

For purposes of illustration, we will use the differential equation

$$2xy + 2 + \left[x^2 + 4\right]\frac{dy}{dx} = 0 \quad .$$

To find a potential function $\phi(x, y)$ for this differential equation (if it exists), do the following:

1. Identify the formulas for $M(x, y)$ and $N(x, y)$, and, using these formulas, write out the pair of partial differential equations to be solved,

$$\frac{\partial \phi}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N(x, y) \quad .$$

(In doing so, we are naively assuming the given differential equation is in exact form.)

For our example,

$$\underbrace{2xy + 2}_{M(x,y)} + [\underbrace{x^2 + 4}_{N(x,y)}]\frac{dy}{dx} = 0 \quad .$$

So the pair of partial differential equations to be solved is

$$\frac{\partial \phi}{\partial x} = 2xy + 2 \qquad \textit{and} \qquad \frac{\partial \phi}{\partial y} = x^2 + 4 \quad .$$

2. Integrate the first equation in the pair with respect to x, treating y as a constant.

$$\int \frac{\partial \phi}{\partial x}\,dx = \int M(x,y)\,dx \quad .$$

In computing the integral of M with respect to x, you get a "constant of integration". Keep in mind that this "constant" is based on y being a constant, and may thus depend on the value of y. Consequently, this "constant of integration" is actually some yet unknown function of y — call it $h(y)$.

For our example,

$$\int \frac{\partial \phi}{\partial x}\,dx = \int [2xy + 2]\,dx$$

$$\hookrightarrow \qquad \phi(x,y) = x^2y + 2x + h(y) \quad .$$

(Observe that this step yields a formula for $\phi(x,y)$ involving one yet unknown function of y only. We now just have to determine $h(y)$ to know the formula for $\phi(x,y)$.)

3. Replace ϕ in the other partial differential equation,

$$\frac{\partial \phi}{\partial y} = N(x,y) \quad ,$$

with the formula just derived for $\phi(x,y)$, and compute the partial derivative. Keep in mind that, because $h(y)$ is a function of y only,

$$\frac{\partial}{\partial y}[h(y)] = h'(y) \quad .$$

Then algebraically solve the resulting equation for $h'(y)$.

For our example:

$$\frac{\partial \phi}{\partial y} = N(x,y)$$

$$\hookrightarrow \qquad \frac{\partial}{\partial y}\left[x^2y + 2x + h(y)\right] = x^2 + 4$$

$$\hookrightarrow \qquad x^2 + h'(y) = x^2 + 4$$

$$\hookrightarrow \qquad h'(y) = 2 \quad .$$

4. Look at the formula just obtained for $h'(y)$. Because $h'(y)$ is a function of y only, its formula must involve only the variable y, no x may appear.

If the x's do not cancel out, then we have an impossible equation. This means the naive assumption made in the first step was wrong; the given differential equation was not in exact form, and there is no $\phi(x,y)$ satisfying the desired pair of partial differential equations. In this case, *Stop! Go no further in this procedure!*

On the other hand, if the previous step yields

$$h'(y) = \text{a formula of } y \text{ only (no } x\text{'s)} \quad ,$$

then integrate both sides of this equation with respect to y to obtain $h(y)$. (Because $h(y)$ does not depend on x, the constant of integration here will truly be a constant, not a function of the other variable.)

For our example, the last step yielded

$$h'(y) = 2 \quad .$$

The right side does not contain x *, so we can continue and integrate to obtain*

$$h(y) = \int h(y)\,dy = \int 2\,dy = 2y + c_1 \quad .$$

(We will later see that the constant c_1 is not that important.)

5. Combine the formula just obtained for h with the formula obtained for ϕ in step 2.

   In our example, combining the results from step 2 and the last step above yields

$$\begin{aligned} \phi(x, y) &= x^2 y + 2x + h(y) \\ &= x^2 y + 2x + 2y + c_1 \end{aligned}$$

   where c_1 *is an arbitrary constant.*

   (Because of the arbitrary constant from the integration of $h'(y)$, the formula obtained actually describes all possible ϕ's satisfying the desired pair of partial differential equations. If you look at our discussion above, it should be clear that this formula will always be of the form

$$\phi(x, y) = \phi_0(x, y) + c_1$$

   where $\phi_0(x, y)$ is a particular formula and c_1 is an arbitrary constant. But, as noted earlier, we only need to find one potential function $\phi(x, y)$. So you can set c_1 equal to your favorite constant, 0, or keep it arbitrary and see what happens.)

If the above procedure does yield a formula $\phi(x, y)$, then it immediately follows that the given equation is in exact form and $\phi(x, y)$ is a potential function for the given differential equation. But don't forget that the goal is usually to solve the given differential equation,

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad .$$

The function $\phi = \phi(x, y)$ is *not* that solution. It is a function such that the differential equation can be rewritten as

$$\frac{d}{dx}\left[\phi(x, y)\right] = 0 \quad .$$

Integrating this yields the implicit solution

$$\phi(x, y) = c$$

from which, if the implicit solution is not too complicated, we can obtain an explicit solution $y = y(x)$.

!►*Example 7.5:* *Consider solving*

$$2xy + 2 + \left[x^2 + 4\right]\frac{dy}{dx} = 0 \quad .$$

As just illustrated, this equation is in exact form, and has a potential function

$$\phi(x, y) = x^2 y + 2x + 2y + c_1$$

where c_1 *can be any constant. So the differential equation can be rewritten as*

$$\frac{d}{dx}\left[x^2 y + 2x + 2y + c_1\right] = 0 \quad ,$$

which integrates to

$$x^2 y + 2x + 2y + c_1 = c_2 \quad .$$

This is an implicit solution for the differential equation. Solving this for y *is easy, and (after letting* $c = c_2 - c_1$ *) gives us the explicit solution*

$$y = \frac{c - 2x}{x^2 + 2} \quad .$$

Note that, in the last example, the constants arising from the integration of $h'(y)$ and $d\phi/dx$ were combined at the end. It is easy to see that this will always be possible.

Other Ways to Find a Potential Function

There are other ways to find a function $\phi(x, y)$ satisfying both

$$\frac{\partial \phi}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N(x, y) \quad .$$

Two, in particular, are worth mentioning:

1. The first is the obvious modification of the one already given in which the roles of $\partial\phi/\partial x = M$ and $\partial\phi/\partial y = N$ are interchanged. That is, instead of

   integrating $\partial\phi/\partial x = M$ with respect to x, and then plugging the result into $\partial\phi/\partial y = N$,

   you

   integrate $\partial\phi/\partial y = N$ with respect to y, and then plug the result into $\partial\phi/\partial x = M$.

   The integration of $\partial\phi/\partial y = N$ will yield some formula involving x, y and an unknown function of x, $g(x)$. Plugging this formula into $\partial\phi/\partial x = M$ should yield a ordinary differential equation for $g(x)$ which does not contain y. If the y's do not cancel out, the desired ϕ does not exist. Otherwise, $g(x)$ can be obtained by integration, and then combined with the formula just obtained for $\phi(x, y)$.

   This is usually the preferred method when $\int N(x, y)\,dy$ is much easier to compute than $\int M(x, y)\,dx$. Indeed, it usually is a good idea to scan these two integrals and, if one looks much easier to compute, compute that one, and plug the result into the partial differential equation corresponding to the other integral. Don't forget to check to see if equation resulting from that just involves the appropriate variable.

2. The other method is one often attempted by beginners who do not understand why it should not be used: First independently integrate both $\partial\phi/\partial x = M$ and $\partial\phi/\partial y = N$. Then stare at the two different formulas obtained for $\phi(x, y)$ (each involving a different unknown function) and try to guess what single formula for $\phi(x, y)$ (without unknown functions) matches the results from the two integrations.

Fight any temptation to take this approach. Yes, with a little luck and skill, you can get ϕ this way. But it is usually more work, it doesn't easily warn you when $\phi(x, y)$ does not exist, and is more likely to result in errors. Why use a method involving two two integrations and two unknown functions when you can use a method involving just one one integration and one unknown function with a straightforward way to determine that one function?

7.4 Testing for Exactness — Part I

The procedure just discussed for finding a potential function ϕ for

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{7.6}$$

does not tell us whether such a ϕ even exists until step 4, after a possibly tricky integration. Fortunately, there is a simple test that that can often tell us when seeking that ϕ would be futile. This test is based on the fact that, for any sufficiently differentiable $\phi(x, y)$,

$$\frac{\partial^2 \phi}{\partial y\, \partial x} = \frac{\partial^2 \phi}{\partial x\, \partial y} \quad .$$

Now let $\mathcal{R}$ be any region in the XY–plane on which ϕ is sufficiently differentiable and satisfies

$$\frac{\partial \phi}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N(x, y) \quad .$$

Then, at every point in $\mathcal{R}$,

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[\frac{\partial \phi}{\partial x}\right] = \frac{\partial^2 \phi}{\partial y\, \partial x} = \frac{\partial^2 \phi}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial y}\right] = \frac{\partial N}{\partial x} \quad .$$

So, for equation (7.6) to be in exact form over $\mathcal{R}$, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{7.7}$$

at every point in $\mathcal{R}$. If this equation does not hold, differential equation (7.6) is not in exact form over that region — no corresponding potential function ϕ exists.

What we have not shown is that equality (7.7) necessarily implies that differential equation (7.6) is in exact form. In fact, equality (7.7) does imply that, given any point in $\mathcal{R}$, the equation is in exact form over *some* subregion of $\mathcal{R}$ containing that point. Unfortunately, showing that and describing those regions takes more development than is appropriate here. For now, let us just say that, in practice, the equality (7.7) implies that differential equation (7.6) is "probably" in exact form over the given region, and it is worthwhile to seek a corresponding potential function ϕ via the method outlined earlier.

Let us summarize what has just been derived, accepting the term "suitably differentiable" as simply meaning that the necessary partial derivatives can be computed:

Theorem 7.3 (test for probable exactness)
Let $M(x,y)$ and $N(x,y)$ be two suitably differentiable functions of two variables over a region $\mathcal{R}$ in the XY–plane, and consider the differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad .$$

1. *If*
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad \text{on} \quad \mathcal{R} \quad ,$$
then the above differential equation is not in exact form on $\mathcal{R}$.

2. *If*
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{on} \quad \mathcal{R} \quad ,$$
then the above differential equation might be in exact form on $\mathcal{R}$. It is worth seeking a corresponding potential function.

!▶*Example 7.6:* *Consider the differential equation*

$$3y + 3y^3 + \left[xy^2 - x\right]\frac{dy}{dx} = 0 \quad .$$

Here
$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[3y + 3y^3\right] = 3 + 9y^2$$

and
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[xy^2 - x\right] = y^2 - 1 \quad .$$

So
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad ,$$

telling us that the given differential equation is not in exact form over any region.

!▶*Example 7.7:* *Consider the differential equation*

$$-\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}\frac{dy}{dx} = 0 \quad .$$

Here
$$M(x,y) = -\frac{y}{x^2+y^2} \qquad \text{and} \qquad N(x,y) = \frac{x}{x^2+y^2}$$

are well defined and differentiable everywhere on the XY–plane except $(x,y) = (0,0)$. *So let's take $\mathcal{R}$ to be the entire XY–plane with the origin removed. Computing the partial derivatives, we get*

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[-\frac{y}{x^2+y^2}\right] = -\frac{1(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad .$$

So these two partial derivatives are equal throughout $\mathcal{R}$, and our test for probable exactness tells us that the given differential equation might be in exact form on $\mathcal{R}$ — it is worthwhile to try to find a corresponding potential function.

For many, the test described above in theorem 7.3 will suffice. Those who wish a more complete test should jump to section 7.7 starting on page 150 (where we will also finish solving the differential equation in example7.7).

7.5 Exact Equations: A Summary

To review:

If you suspect that a given differential equation,

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad ,$$

is in exact form, then you can quickly check for at least probable exactness by computing $\partial M/\partial y$ and $\partial N/\partial x$ and seeing if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad .$$

If the two partial derivatives are equal, then follow the procedure for finding a potential function $\phi(x, y)$ outlined on pages 136 to 138.

If that procedure is successful and yields a $\phi(x, y)$, then finish solving the given differential equation using the fact that the differential equation can be rewritten as

$$\frac{d}{dx}[\phi(x, y)] = 0 \quad ,$$

the integration of which yields the implicit solution

$$\phi(x, y) = 0 \quad .$$

If this equation can be solved for y in terms of x, do so.

If the given differential equation is not in exact form, then there is a possibility that it can be put into an exact form using appropriate "integrating factors". We will discuss these next.

By the way, don't forget that these equations may be solvable by other means. For example, the equation used to illustrate the procedure for finding ϕ was a linear differential equation, and could have been solved a bit more quickly using the methods from chapter 5.

7.6 Converting Equations to Exact Form

Basic Notions

Obviously, the first step to converting a given first-order differential equation to exact form is to get it into the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad .$$

Then apply the test for (probable) exactness. With luck, the test result will be positive. More likely, it will not.

To see how we might further convert our equation to exact form, it may help to recall why we want the exact form. It is so that the left side of the differential equation can be identified as an ordinary derivative of some formula of x and $y(x)$,

$$\frac{d}{dx}\phi(x, y(x)) \quad .$$

We had a similar situation with linear equations. Given a linear equation

$$\frac{dy}{dx} + p(x)y = f(x) \quad ,$$

we found that, *after multiplying it by an integrating factor* μ to get

$$\mu\frac{dy}{dx} + \mu py = \mu f \quad ,$$

we could identify the equation's left side as a complete derivative of a formula of x and $y(x)$, namely,

$$\frac{d}{dx}[\mu(x)y(x)] \quad .$$

The same idea can be applied to convert an equation not in exact form to one that is in exact form.

!►*Example 7.8:* *Consider the differential equation*

$$3y + 3y^3 + \left[xy^2 - x\right]\frac{dy}{dx} = 0 \quad .$$

In example 7.6, we saw that this equation is not in exact form. But look what happens after we multiply through by $\mu = x^2y^{-2}$,

$$x^2y^{-2}\left(3y + 3y^3 + \left[xy^2 - x\right]\frac{dy}{dx} = 0\right) \quad .$$

We get

$$\underbrace{3x^2y^{-1} + 3x^2y}_{M_{new}(x,y)} + \Big[\underbrace{x^3 - x^3y^{-2}}_{N_{new}(x,y)}\Big]\frac{dy}{dx} = 0$$

with

$$\frac{\partial M_{new}}{\partial y} = \frac{\partial}{\partial y}\left[3x^2y^{-1} + 3x^2y\right] = -3x^2y^{-2} + 3x^2 = 3x^2 - 3x^2y^{-2}$$

and

$$\frac{\partial N_{new}}{\partial x} = \frac{\partial}{\partial x}\left[x^3 - x^3 y^{-2}\right] = 3x^2 - 3x^2 y^{-2} \quad .$$

So

$$\frac{\partial M_{new}}{\partial y} = \frac{\partial N_{new}}{\partial x} \quad ,$$

telling us that the equation is now (probably) in exact form (over any region where y never equals 0).

We will refer to any nonzero function $\mu = \mu(x, y)$ as an *integrating factor* for a first-order differential equation

$$M + N\frac{dy}{dx} = 0$$

if and only if multiplying that equation through by μ ,

$$\mu M + \mu N\frac{dy}{dx} = 0 \quad ,$$

yields a differential equation in exact form.[2] This integrating factor may be a function of x or of y or of both x and y . Notice that, because

$$\underbrace{\mu M}_{\text{“new” } M} + \underbrace{\mu N}_{\text{“new” } N}\frac{dy}{dx} = 0$$

is exact, our test for exactness tells us that

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N] \quad .$$

Finding Integrating Factors

General Approach

Finding an integrating factor for an equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

starts with the requirement just derived,

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N] \quad . \tag{7.8}$$

Remember, M and N will be known formulas of x and y . So equation (7.8) is a differential equation in which the unknown function is our integrating factor, μ . Unfortunately, it is a rather nontrivial partial differential equation (unlike the partial differential equations in section 7.2), and a complete discussion of how to solve it for μ is beyond the scope of any introductory differential equations course. Fortunately, there are some common cases where this partial differential equation reduces to an equation we can handle. The cases we will consider are where there is an integrating factor μ that is either as function of x only, or a function of y only, or a ‘simple’ formula of x and y . In all cases, the approach is basically the same:

[2] You can show that the integrating factors found for linear equations are just special cases of the integrating factors considered here. If there is any danger of confusion, we’ll refer to the integrating factors now being discussed as the “more general” integrating factors.

1. Decide on which case you think is appropriate, and make the corresponding assumptions on μ.
2. Expand equation (7.8) by computing out the derivatives as far as possible, taking into account the assumptions made on μ.
3. See if the resulting equation can be solved for a μ satisfying the assumptions made. If so, do so. If not, start over using different assumptions on μ (unless you've run out of reasonable options).

We'll illustrate the above approach in a moment. Before that, however, a few more comments should be made:

1. Only one integrating factor is needed. So go ahead and assign convenient values to the arbitrary constants that arise in solving for μ (just as in finding an integrating factors for linear equations).
2. Once you have found an integrating factor μ, remember why you wanted it. Use it to rewrite your differential equation in exact form, and then solve the differential equation as described earlier in this chapter.
3. There are tests to determine if there are integrating factors that are functions of just x or just y. Moreover, there are formulas for μ that can be used if any of the tests are satisfied (see exercise 7.6 on page 156). DO NOT WASTE YOUR TIME TRYING TO USE THESE FORMULAS! They are hard to memorize correctly and are worth learning only if you expect to compute many, many integrating factors over a relatively short time frame. Chances are, you won't have that need, just a need to understand the basic concepts and to compute the occasional integrating factor.

Now, let's look at the common cases:

First Case: μ Being a Function of x Only

If you think the integrating factor μ could be function of x only, then assume it, and see what happens when you compute out equation (7.8). Under this assumption,

$$\mu = \mu(x) \quad , \quad \frac{\partial\mu}{\partial x} = \frac{d\mu}{dx} \quad \text{and} \quad \frac{\partial\mu}{\partial y} = 0 \quad .$$

Consequently, equation (7.8) should immediately reduce to an ordinary differential equation for μ. Moreover, since μ is supposed to be just a function of x here, this differential equation for μ should not contain any y's. Thus, if the y's do not cancel out, the assumption that μ could be just a function of x is wrong — go to the next case. But if the y's do cancel out, then solve the differential equation just obtained for a $\mu = \mu(x)$.

!►*Example 7.9:* *Consider the differential equation*

$$1 + y^3 + xy^2\frac{dy}{dx} = 0 \quad .$$

Here

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[1 + y^3\right] = 3y^2 \qquad \textit{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[xy^2\right] = y^2 \quad .$$

So $^{\partial M}/_{\partial y} \neq {}^{\partial N}/_{\partial x}$. The equation is not exact, but, with luck, we can find a function μ so that

$$\mu\big[1 + y^3\big] \;+\; \mu\big[xy^2\big]\frac{dy}{dx} \;=\; 0$$

is exact. This integrating factor μ must satisfy

$$\frac{\partial}{\partial y}\left(\mu\big[1 + y^3\big]\right) \;=\; \frac{\partial}{\partial x}\left(\mu\big[xy^2\big]\right) \quad .$$

Let's see if μ could be a function of just x . Assume $\mu = \mu(x)$. Then

$$\frac{\partial \mu}{\partial x} \;=\; \frac{d\mu}{dx} \qquad \text{and} \qquad \frac{\partial \mu}{\partial y} \;=\; 0 \quad .$$

Using this, we have

$$\frac{\partial}{\partial y}\left(\mu\big[1 + y^3\big]\right) \;=\; \frac{\partial}{\partial x}\left(\mu\big[xy^2\big]\right)$$

$$\hookrightarrow \qquad \frac{\partial \mu}{\partial y}\big[1 + y^3\big] \;+\; \mu\frac{\partial}{\partial y}\big[1 + y^3\big] \;=\; \frac{\partial \mu}{\partial x}\big[xy^2\big] \;+\; \mu\frac{\partial}{\partial x}\big[xy^2\big]$$

$$\hookrightarrow \qquad 0 \cdot \big[1 + y^3\big] \;+\; \mu\big[0 + 3y^2\big] \;=\; \frac{d\mu}{dx}\big[xy^2\big] \;+\; \mu\big[y^2\big]$$

$$\hookrightarrow \qquad 3y^2\mu \;=\; xy^2\frac{d\mu}{dx} \;+\; y^2\mu \quad .$$

The y's do cancel out, leaving us with

$$3\mu \;=\; x\frac{d\mu}{dx} \;+\; \mu \quad ,$$

which simplies to

$$\frac{d\mu}{dx} \;=\; \frac{2\mu}{x} \quad .$$

So there is an integrating factor μ that is a function of x alone. Moreover, to find such an integrating factor, we need only find one (nonzero) solution to the above simple, separable ordinary differential equation. Proceding to do so, we get

$$\int \frac{1}{\mu}\frac{d\mu}{dx}\,dx \;=\; \int \frac{2}{x}\,dx$$

$$\hookrightarrow \qquad \ln|\mu| \;=\; 2\ln|x| \;+\; c \;=\; \ln|x|^2 \;+\; c$$

$$\hookrightarrow \qquad \mu \;=\; \pm e^{\ln x^2 \,+\, c} \;=\; Ax^2 \quad .$$

Since only one nonzero integrating factor is needed, we can take $A = 1$, giving us

$$\mu(x) \;=\; x^2$$

as our integrating factor.

Unfortunately, there is always the possibility that the y's will not cancel out.

!►Example 7.10: *It is easily verified that*

$$6xy \;+\; 5\big[x^2 + y\big]\frac{dy}{dx} \;=\; 0$$

is not in exact form. To be a corresponding integrating factor, μ *must satisfy*

$$\frac{\partial}{\partial y}\big(\mu[6xy]\big) \;=\; \frac{\partial}{\partial x}\left(\mu 5\big[x^2 + y\big]\right) \quad .$$

Assume μ *is a function of* x *only. Then*

$$\frac{\partial \mu}{\partial x} \;=\; \frac{d\mu}{dx} \qquad \text{and} \qquad \frac{\partial \mu}{\partial y} \;=\; 0 \quad ,$$

and, thus,

$$\frac{\partial}{\partial y}\big(\mu[6xy]\big) \;=\; \frac{\partial}{\partial x}\left(\mu 5\big[x^2 + y\big]\right)$$

$$\hookrightarrow \qquad \frac{\partial \mu}{\partial y}[6xy] \;+\; \mu\frac{\partial}{\partial y}[6xy] \;=\; \frac{\partial \mu}{\partial x}5\big[x^2 + y\big] \;+\; \mu\frac{\partial}{\partial x}\left(5\big[x^2 + y\big]\right)$$

$$\hookrightarrow \qquad 0 \cdot 6xy \;+\; \mu[6x] \;=\; \frac{d\mu}{dx}5\big[x^2 + 5\big] \;+\; \mu[10x]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx} \;=\; \frac{10x\mu - 6x\mu}{4x^2 + 5y} \;=\; \frac{4x\mu}{4x^2 + 5y} \quad .$$

Here, the y*'s do not cancel out, as they should if our assumption that* μ *depended only on* x *were true. Hence, that assumption was wrong. This equation does not have an integrating factor that is a function of* x *only. We will have to try something else.*

Second Case: μ Being a Function of y Only

This is just like the first case, but with the roles of x and y switched. If you think the integrating factor μ could be function of y only, then assume it, and see what happens when you compute out equation (7.8). Under this assumption, $\mu = \mu(y)$,

$$\frac{\partial \mu}{\partial x} \;=\; 0 \qquad \text{and} \qquad \frac{\partial \mu}{\partial y} \;=\; \frac{d\mu}{dy} \quad .$$

Again, equation (7.8) should immediately reduce to an ordinary differential equation for μ, only this time the differential equation for μ should not contain any x's. If the x's do not cancel out, our assumption that μ could be just a function of y is wrong and we can go no further with this hope. But if the x's do cancel out, then solving the differential equation just obtained for a $\mu = \mu(y)$ yields the desired integrating factor.

!►Example 7.11: *As just seen in example 7.10,*

$$6xy \;+\; 5\big[x^2 + y\big]\frac{dy}{dx} \;=\; 0$$

does not have an integrating factor depending only on x. *So instead, let's try to find one that depends on just* y. *Assuming this,*

$$\mu \;=\; \mu(y) \qquad , \qquad \frac{\partial \mu}{\partial x} \;=\; 0 \qquad \text{and} \qquad \frac{\partial \mu}{\partial y} \;=\; \frac{d\mu}{dy} \quad .$$

Combining this with equation (7.8):

$$\frac{\partial}{\partial y}\left(\mu[6xy]\right) = \frac{\partial}{\partial x}\left(\mu\left[5x^2+5y\right]\right)$$

$$\hookrightarrow \qquad \frac{\partial\mu}{\partial y}[6xy] + \mu\frac{\partial}{\partial y}\left[6xy\right] = \frac{\partial\mu}{\partial x}\left[5x^2+5y\right] + \mu\frac{\partial}{\partial x}\left[5x^2+5y\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dy}[6xy] + \mu[6x] = 0\cdot\left[5x^2+5y\right] + \mu[10x]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx} = \frac{10x\mu - 6x\mu}{6xy} = \frac{2\mu}{3y} \quad .$$

The x's cancel out, as hoped, and we have a simple differential equation for $\mu = \mu(y)$. Solving it:

$$\int \frac{1}{\mu}\frac{d\mu}{dy}\,dy = \int \frac{2}{3y}\,dy$$

$$\hookrightarrow \qquad \ln|\mu| = \frac{2}{3}\ln|y| + c = \ln|y|^{2/3} + c$$

$$\hookrightarrow \qquad \mu = Ay^{2/3} \quad .$$

Taking $A = 1$ gives the integrating factor

$$\mu(y) = y^{2/3} \quad .$$

Third Case: μ Being a 'Simple' Function of Both Variables

Of course, it is quite possible that no function of just x or just y will be an integrating factor for a given equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad .$$

In this case, the best we can usually hope for is that there is a relatively simple function of x and y that will work as an integrating factor. This means making a 'good guess' at $\mu(x,y)$, and verifying that it satisfies equation (7.8). One 'guess' sometimes worth trying is

$$\mu(x,y) = x^\alpha y^\beta$$

where the exponents, α and β, are constants to be determined. To determine the values of the constants so that the guess works (if such constants exist), just plug this formula for μ into equation (7.8) and see if it reduces to an equation that can be solved for α and β. If so, find those values and use $\mu(x,y) = x^\alpha y^\beta$ (with the values just found for α and β) as your integrating factor. If not, keep searching (or consider dealing with the differential equation using the graphical and numerical methods we'll discuss in chapter 8).

!►*Example 7.12:* *Consider the differential equation from example 7.6,*

$$3y + 3y^3 + \left[xy^2 - x\right]\frac{dy}{dx} = 0 \quad .$$

Plugging

$$\mu(x, y) = x^{\alpha} y^{\beta}$$

into equation (7.8) for our differential equation yields:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\hookrightarrow \quad \frac{\partial}{\partial y}\left(x^{\alpha} y^{\beta}\left[3y + 3y^3\right]\right) = \frac{\partial}{\partial x}\left(x^{\alpha} y^{\beta}\left[xy^2 - x\right]\right)$$

$$\hookrightarrow \quad \frac{\partial}{\partial y}\left(3x^{\alpha} y^{\beta+1} + 3x^{\alpha} y^{\beta+3}\right) = \frac{\partial}{\partial x}\left(x^{\alpha+1} y^{\beta+2} - x^{\alpha+1} y^{\beta}\right)$$

$$\hookrightarrow \quad 3(\beta+1)x^{\alpha} y^{\beta} + 3(\beta+3)x^{\alpha} y^{\beta+2} = (\alpha+1)x^{\alpha} y^{\beta+2} - (\alpha+1)x^{\alpha} y^{\beta} \quad .$$

Combining like terms then gives

$$[3\beta + \alpha + 4]x^{\alpha} y^{\beta} + [3\beta - \alpha + 8]x^{\alpha} y^{\beta+2} = 0 \quad ,$$

which, in turn, holds if and only if

$$3\beta + \alpha + 4 = 0 \qquad \text{and} \qquad 3\beta - \alpha + 8 = 0 \quad .$$

This last pair of equations constitute a simple system of linear equations,

$$\begin{aligned} 3\beta + \alpha + 4 &= 0 \\ 3\beta - \alpha + 8 &= 0 \end{aligned}$$

which can be easily solved by any of a number of ways, yielding

$$\alpha = 2 \qquad \text{and} \qquad \beta = -2 \quad .$$

Thus, the differential equation we started with,

$$3y + 3y^3 + \left[xy^2 - x\right]\frac{dy}{dx} = 0 \quad ,$$

does have an integrating factor of the form $\mu(x, y) = x^{\alpha} y^{\beta}$ *, and it is*

$$\mu(x, y) = x^2 y^{-2}$$

(just as was used in example 7.8 on page 143).

7.7 Testing for Exactness — Part II*

Simple Connectivity and the Complete Test for Exactness

A more complete test for exactness than given in theorem 7.3 can be described if we are more careful about describing our situation. So suppose we have an equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad ,$$

and that, on some open region $\mathcal{R}$ of the XY–plane, all of the following hold:

1. The functions $M(x, y)$ and $N(x, y)$, along with the derivatives $\partial M/\partial y$ and $\partial N/\partial x$, are continuous everywhere in $\mathcal{R}$.

2. At each point (x, y) in $\mathcal{R}$,
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad .$$

That region $\mathcal{R}$ will said to be *simply connected* if each and every simple closed curve (i.e., loop) in $\mathcal{R}$ encloses only points in $\mathcal{R}$. If any simple closed curve in $\mathcal{R}$ encloses any point not in $\mathcal{R}$, then we will say that $\mathcal{R}$ is not simply connected. If you think about it, you will realize that saying a region is simply connected is just a precise way of saying that the region has no "holes". And if you think a little more about the situation, you will realize that, if our open region $\mathcal{R}$ has "holes" (i.e., is not simply connected), then it is probably because these are points where $M(x, y)$ or $N(x, y)$ or their partial derivatives fail to exist.

!►*Example 7.13:* *Again, consider the differential equation*

$$-\frac{y}{x^2 + y^2} + \frac{x}{x^2 + y^2}\frac{dy}{dx} = 0 \quad .$$

As noted in example 7.7,

$$M(x, y) = -\frac{y}{x^2 + y^2} \qquad \text{and} \qquad N(x, y) = \frac{x}{x^2 + y^2}$$

are well defined and differentiable and satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

everywhere in the region $\mathcal{R}$ consisting of the XY–plane with the origin removed. Removing this point (the origin) creates a "hole" in $\mathcal{R}$. This point is also a point not in $\mathcal{R}$ but which is enclosed by any loop in $\mathcal{R}$ around the origin.

Now we can state the full test for exactness. (Its proof will be briefly discussed at the end of this section.)

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* For the interested reader.

Theorem 7.4 (complete test for exactness)
Let $\mathcal{R}$ be a simply-connected open region in the XY–plane, and let $M(x, y)$ and $N(x, y)$ be two continuous functions on $\mathcal{R}$ whose partial derivatives are also continuous on $\mathcal{R}$. Then

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is in exact form on $\mathcal{R}$ if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

at every point in $\mathcal{R}$.

This theorem assures us that, if our region $\mathcal{R}$ is simply connected, then we can (in theory at least) use the procedure outlined on pages 136 to 138 to find the corresponding potential function $\phi(x, y)$ on $\mathcal{R}$, and from that, derive an implicit solution $\phi(x, y) = c$ to our differential equation.

Theorem 7.4 does not definitely say the differential equation is not in exact form if $\mathcal{R}$ is not simply connected. Whether the equation is or is not be in exact form over all of $\mathcal{R}$ is still uncertain. What is certain, however, is the following immediate consequence of theorem 7.4.

Corollary 7.5
Assume $M(x, y)$ and $N(x, y)$ are two continuous functions on some open region $\mathcal{R}$ of the XY–plane. Assume further that, on $\mathcal{R}$, the partial derivatives of M and N are continuous and satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad .$$

Then

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is in exact form on each simply-connected open subregion of $\mathcal{R}$.

Thus, even if our original region is not simply connected, we can at least pick any open, simply connected subregion $\mathcal{R}_1$, and (in theory at least) use the procedure outlined in section 7.3 to find a corresponding potential function $\phi_1(x, y)$ on $\mathcal{R}_1$, and from that, derive the implicit solution $\phi_1(x, y) = c$ to our differential equation, valid on subregion $\mathcal{R}_1$.

But then, why might there not be a potential function $\phi(x, y)$ valid on the entire region $\mathcal{R}$? Let's go back to an example started earlier to see.

!►*Example 7.14:* *Let us continue our consideration of the differential equation*

$$-\frac{y}{x^2 + y^2} + \frac{x}{x^2 + y^2}\frac{dy}{dx} = 0$$

from example 7.7. As just noted in the previous example, the region $\mathcal{R}$ consisting of all the XY–plane except for the origin $(0, 0)$ is not simply connected. But we can partition it into the left and right half-planes

$$\mathcal{R}_+ = \{(x, y) : x > 0\} \qquad \text{and} \qquad \mathcal{R}_- = \{(x, y) : x < 0\} \quad ,$$

which are simply connected. Theorem 7.4 assures us that our differential equation is in exact form over each of these half-planes, and, indeed, you can easy show that all the corresponding *potential functions on these regions for our differential equation are given by*

$$\phi_+(x,y) = \text{Arctan}\left(\frac{y}{x}\right) + c_+ \qquad \textit{on} \quad \mathcal{R}_+$$

and

$$\phi_-(x,y) = \text{Arctan}\left(\frac{y}{x}\right) + c_- \qquad \textit{on} \quad \mathcal{R}_-$$

where c_+ *and* c_- *are arbitrary constants.*

But could there be a potential function ϕ *on all of* $\mathcal{R}$ *corresponding to our differential equation? If so, then* ϕ *would also be a potential function over the left and right half-planes* $\mathcal{R}_+$ *and* $\mathcal{R}_-$*, and, as just noted, there would be constants* c_+ *and* c_- *such that*

$$\phi(x,y) = \text{Arctan}\left(\frac{y}{x}\right) + c_+ \qquad \textit{for} \quad x > 0$$

and

$$\phi(x,y) = \text{Arctan}\left(\frac{y}{x}\right) + c_- \qquad \textit{for} \quad x < 0 \quad .$$

Since ϕ *must be continuous everywhere except at the origin, it must, in particular, be continuous at any point on the positive* Y*–axis,* $(0,y)$ *with* $y > 0$*. So, letting* $x \to 0$ *from the positive side, we have*

$$\phi(0,y) = \lim_{x\to 0^+} \phi(x,y) = \lim_{x\to 0^+} \text{Arctan}\left(\frac{y}{x}\right) + c_+ \quad .$$

Using the substitution $t = {}^{y}\!/\!_{x}$ *and recalling the limiting values of the Arctangent function, we can rewrite the above as*

$$\phi(0,y) = \lim_{t\to +\infty} \text{Arctan}(t) + c_+ = \frac{\pi}{2} + c_+ \quad .$$

Likewise, letting $x \to 0$ *from the negative side, we have*

$$\begin{aligned} \phi(0,y) &= \lim_{x\to 0^-} \phi(x,y) \\ &= \lim_{x\to 0^-} \text{Arctan}\left(\frac{y}{x}\right) + c_- \\ &= \lim_{t\to -\infty} \text{Arctan}(t) + c_- = -\frac{\pi}{2} + c_- \quad . \end{aligned}$$

Together, the above tells us that

$$-\frac{\pi}{2} + c_- = \phi(0,1) = \frac{\pi}{2} + c_+ \quad ,$$

which, of course, means that $c_- = \pi + c_+$*. Because of the arbitrariness of the constants added to potential functions, we may, for simplicity, let* $c_+ = 0$*. Then*

$$\phi(x,y) = \begin{cases} \text{Arctan}\left(\frac{y}{x}\right) & \textit{if} \quad x > 0 \\ \text{Arctan}\left(\frac{y}{x}\right) + \pi & \textit{if} \quad x < 0 \\ \frac{\pi}{2} & \textit{if} \quad x = 0 \textit{ and } y > 0 \end{cases} \quad .$$

But look at what must now happen at a point on the negative Y*–axis, say, at* $(0,-1)$*.*

$$\lim_{x\to 0^+} \phi(x,-1) = \lim_{x\to 0^+} \text{Arctan}\left(\frac{-1}{x}\right) = \lim_{t\to -\infty} \text{Arctan}(t) = -\frac{\pi}{2}$$

and

$$\lim_{x\to 0^-} \phi(x,-1) \;=\; \lim_{x\to 0^-} \text{Arctan}\left(\frac{-1}{x}\right) \;+\; \pi \;=\; \lim_{t\to +\infty} \text{Arctan}(t) \;+\; \pi \;=\; \frac{3\pi}{2} \quad .$$

So

$$\lim_{x\to 0^+} \phi(x,-1) \;\neq\; \lim_{x\to 0^-} \phi(x,-1) \quad .$$

Thus, there are points in $\mathcal{R}$ at which $\phi(x,y)$ is not continuous, contrary to the fact that a potential function on $\mathcal{R}$ must be continuous everywhere on $\mathcal{R}$. And thus, the answer to our question "Could there be a potential function ϕ on all of $\mathcal{R}$ corresponding to our differential equation?" is "No".

The above example illustrates that, while we can partition a non-simply connected region into simply-connected subregions and then find all possible potential functions for our differential equation over each subregion, it may still be impossible to "paste together" these regions and potential functions to obtain a potential function that is well defined across all the boundaries between the partitions.

Is this truly a problem for us, whose main interest is in solving the differential equation? Not really. We can still solve the given differential equation. All we need to do is to choose our simply-connected partitions reasonably intelligently.

!►*Example 7.15:* *Let's solve the initial-value problem*

$$-\frac{y}{x^2+y^2} \;+\; \frac{x}{x^2+y^2}\frac{dy}{dx} \;=\; 0 \qquad \textit{with} \quad y(1) = 3$$

using the potential functions from the last example.

Since $(x,y) = (1,3)$ is in the right half plane, it makes sense to use ϕ_+ from the last example. Letting $c_+ = 0$, this potential function is

$$\phi_+(x,y) \;=\; \text{Arctan}\left(\frac{y}{x}\right) \qquad \textit{for} \quad x > 0 \quad .$$

So our differential equation has an implicit solution

$$\text{Arctan}\left(\frac{y}{x}\right) \;=\; C_+ \qquad \textit{for} \quad x > 0 \quad .$$

Taking the tangent of both sides, letting $A = \tan(C_+)$, and then solving for y yields the general solution

$$y \;=\; Ax \qquad \textit{for} \quad x > 0 \quad .$$

Combined with the initial condition, this is

$$3 = y(1) = A \cdot 1 \quad .$$

So $A = 3$ and the solution to our initial-value problem is

$$y \;=\; 3x \qquad \textit{for} \quad x > 0 \quad .$$

(We will leave the issue of whether we truly need to restrict x to being positive as an exercise for the interested.)

Proving Theorem 7.4

This is one theorem we will not attempt to prove. A good proof would require a review of "integrals over curves in the plane" and "Green's theorem", both of which are subjects you may recall seeing near the end of your calculus course. Moreover, if you replace the equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

with

$$\mathbf{F}(x, y) = M(x.y)\mathbf{i} + N(x, y)\mathbf{j}$$

and replace the phrase "in exact form" with "a conservative vector field", then theorem 7.4 becomes the theorem describing when a two-dimensional vector field is conservative. The proofs of these two theorems are virtually the same, and since the theorem about conservative vector fields is in any good calculus text, this author will save space and writing by directing the interested student to reviewing the relevant chapters in his/her old calculus book.

Additional Exercises

7.1. *For each choice of $\phi(x, y)$ given below, find a differential equation for which the given ϕ is a potential function, and then solve the differential equation using the given potential function.*

a. $\phi(x, y) = 3xy$

b. $\phi(x, y) = y^2 - 2x^3y$

c. $\phi(x, y) = x^2y - xy^3$

d. $\phi(x, y) = x \operatorname{Arctan}(y)$

7.2. *The following concern the differential equation*

$$\frac{dy}{dx} = \frac{1}{y} - \frac{y}{2x} \quad . \tag{7.9}$$

a. *Verify that the above differential equation can be rewritten as*

$$\left[y^2 - 2x\right] + 2xy\frac{dy}{dx} = 0 \quad ,$$

and then verify that this is an exact form for equation (7.9) by showing that

$$\phi(x, y) = xy^2 - x^2$$

is a corresponding potential function.

b. *Solve equation (7.9) using the above potential function.*

c. *Note that we can also rewrite equation (7.9) as*

$$e^{xy^2 - x^2}\left[y^2 - 2x\right] + e^{xy^2 - x^2}2xy\frac{dy}{dx} = 0 \quad .$$

Show that this is also an exact form by showing that

$$\psi(x, y) = e^{xy^2 - x^2}$$

is a corresponding potential function.

7.3. *Assume $\phi(x, y)$ is a potential function corresponding to*

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad .$$

Show that

$$\psi_1(x, y) = e^{\phi(x,y)} \qquad \text{and} \qquad \psi_2(x, y) = \sin(\phi(x, y))$$

are also potential functions for this differential equation, though corresponding to different exact forms.

7.4. *Each of the following differential equations is in exact form. Find a corresponding potential function for each, and then find a general solution to the differential equation using that potential function (even if it can be solved by simpler means).*

a. $2xy + y^2 + \left[2xy + x^2\right]\frac{dy}{dx} = 0$

b. $2xy^3 + 4x^3 + 3x^2y^2\frac{dy}{dx} = 0$

c. $2 - 2x + 3y^2\frac{dy}{dx} = 0$

d. $1 + 3x^2y^2 + \left[2x^3y + 6y\right]\frac{dy}{dx} = 0$

e. $4x^3y + \left[x^4 - y^4\right]\frac{dy}{dx} = 0$

f. $1 + \ln|xy| + \frac{x}{y}\frac{dy}{dx} = 0$

g. $1 + e^y + xe^y\frac{dy}{dx} = 0$

h. $e^y + \left[xe^y + 1\right]\frac{dy}{dx} = 0$

7.5. *For each of the following differential equations,*

i. *verify that the equation is not in exact form,*

ii. *find an integrating factor, and*

iii. *solve the given differential equation (using the integrating factor just found).*

a. $2y^3 + \left[4x^3y^3 - 3xy^2\right]\frac{dy}{dx} = 0$

b. $y + \left[y^4 - 3x\right]\frac{dy}{dx} = 0$

c. $2x^{-1}y + \left[4x^2y - 3\right]\frac{dy}{dx} = 0$

d. $1 + [1 - x\tan(y)]\frac{dy}{dx} = 0$

e. $3y + 3y^2 + \left[2x + 4xy\right]\frac{dy}{dx} = 0$

f. $2x(y+1) \; - \; \frac{dy}{dx} \; = \; 0$

g. $1 \; + \; y^4 \; + \; xy^3\frac{dy}{dx} \; = \; 0$

h. $4xy \; + \; \left[3x^2+5y\right]\frac{dy}{dx} \; = \; 0$ *for* $y > 0$

i. $6 \; + \; 12x^2y^2 \; + \; \left[7x^3y + \frac{x}{y}\right]\frac{dy}{dx} \; = \; 0$

7.6. *The following problems concern the differential equation*

$$M(x,y) \; + \; N(x,y)\frac{dy}{dx} \; = \; 0 \quad . \tag{7.10}$$

Assume M *and* N *are continuous and have continuous partial derivatives, and let* P *and* Q *be the functions given by*

$$P \; = \; \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} \qquad \text{and} \qquad Q \; = \; \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} \quad .$$

a. *Show that, if* P *is a function of* x *only (so all the* y*'s cancel out), then*

$$\mu(x) \; = \; e^{\int P(x)\,dx}$$

is an integrating factor for equation (7.10).

b. *Show that, if* Q *is a function of* y *only (so all the* x*'s cancel out), then*

$$\mu(y) \; = \; e^{\int Q(x)\,dx}$$

is an integrating factor for equation (7.10).