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The Starting Point: Basic Concepts and Terminology

Let us begin our study of "differential equations" with a few basic questions — questions that any beginner should ask:

What are "differential equations"?

What can we do with them? Solve them? If so, what do we solve for? And how?

and, of course,

What good are they, anyway?

In this chapter, we will try to answer these questions (along with a few you would not yet think to ask), at least well enough to begin our studies. With luck we will even raise a few questions that cannot be answered now, but which will justify continuing our study. In the process, we will also introduce and examine some of the basic concepts, terminology and notation that will be used throughout this book.

1.1 Differential Equations: Basic Definitions and Classifications

A *differential equation* is an equation involving some function of interest along with a few of its derivatives. Typically, the function is unknown, and the challenge is to determine what that function could possibly be.

Differential equations can be classified either as "ordinary" or as "partial". An *ordinary differential equation* is a differential equation in which the function in question is a function of only one variable. Hence, its derivatives are the "ordinary" derivatives encountered early in calculus. For the most part, these will be the sort of equations we'll be examining in this text. For example,

$$\frac{dy}{dx} = 4x^3$$

$$\frac{dy}{dx} + \frac{4}{x}y = x^2$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 65\cos(2x)$$

$$4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + [4x^2 - 1]y = 0$$

and

$$\frac{d^4y}{dx^4} = 81y$$

are some differential equations that we will later deal with. In each, y denotes a function that is given by some, yet unknown, formula of x . Of course, there is nothing sacred about our choice of symbols. We will use whatever symbols are convenient for the variable and function to be determined.[1,2]

A *partial differential equation* is a differential equation in which the function of interest depends on two or more variables. Consequently, the derivatives of this function are the partial derivatives developed in the later part of most calculus courses.[3] Because the methods for studying partial differential equations often involve solving ordinary differential equations, it is wise to first become reasonably adept at dealing with ordinary differential equations before tackling partial differential equations.

As already noted, this text is mainly concerned with ordinary differential equations. So let us agree that, unless otherwise indicated, the phrase "differential equation" in this text means "ordinary differential equation". If you wish to further simplify the phrasing to "DE" or even to something like "Diffy-Q", go ahead. This author, however, will not be so informal.

Differential equations are also classified by their "order". The *order* of a differential equation is simply the order of the highest order derivative explicitly appearing in the equation. The equations

$$\frac{dy}{dx} = 4x^3 \quad , \quad \frac{dy}{dx} + \frac{4}{x}y = x^2 \quad \text{and} \quad y\frac{dy}{dx} = -9.8x$$

are all first-order equations. So is

$$\frac{dy}{dx} + 3y^2 = y\left(\frac{dy}{dx}\right)^4 \quad ,$$

despite the appearance of the higher powers — dy/dx is still the highest *order* derivative in this equation, even if it is multiplied by itself a few times.

The equations

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 65\cos(2x) \quad \text{and} \quad 4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + [4x^2 - 1]y = 0$$

are second-order equations, while

$$\frac{d^3y}{dx^3} = e^{4x} \quad \text{and} \quad \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = x^2$$

[1] On occasion, we may write "$y = y(x)$" to explicitly indicate that, in some expression, y denotes a function given by some formula of x with $y(x)$ denoting that "formula of x". More often, it will simply be understood that y is a function given by some formula of whatever variable appears in our expressions.

[2] Of course, there is nothing sacred about using y and x to denote the function and variable. We can (and will) use whatever symbols are convenient. In particular, it is standard to use t instead of x for the variable when the variable corresponds to time.

[3] A brief introduction to partial derivatives is given in section 3.7 for those who are interested and haven't yet seen partial derivatives.

are third-order equations.

?►*Exercise 1.1:* *What is the order of each of the following equations?*

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 7y = \sin(x)$$

$$\frac{d^5y}{dx^5} - \cos(x)\frac{d^3y}{dx^3} = y^2$$

$$\frac{d^5y}{dx^5} - \cos(x)\frac{d^3y}{dx^3} = y^6$$

$$\frac{d^{42}y}{dx^{42}} = \left(\frac{d^3y}{dx^3}\right)^2$$

In practice, higher-order differential equations are usually more difficult to solve than lower-order equations. This, of course, is not an absolute rule. There are some very difficult first-order equations, as well as some very easily solved twenty-seventh-order equations.

Solutions: The Basic Notions*

Any function that satisfies a given differential equation is called a *solution* to that differential equation. "Satisfies the equation", means that, if you plug the function into the differential equation and compute the derivatives, then the result is an equation that is true no matter what real value we replace the variable with. And if that resulting equation is not true for some real values of the variable, then that function is not a solution to that differential equation.

!►*Example 1.1:* *Consider the differential equation*

$$\frac{dy}{dx} - 3y = 0 \quad .$$

If, in this differential equation, we let $y(x) = e^{3x}$ *(i.e., if we replace* y *with* e^{3x} *), we get*

$$\frac{d}{dx}\left[e^{3x}\right] - 3e^{3x} = 0$$

$$\hookrightarrow \qquad 3e^{3x} - 3e^{3x} = 0$$

$$\hookrightarrow \qquad 0 = 0 \quad ,$$

which certainly is true for every real value of x *. So* $y(x) = e^{3x}$ *is a solution to our differential equation.*

On the other hand, if we let $y(x) = x^3$ *in this differential equation, we get*

$$\frac{d}{dx}\left[x^3\right] - 3x^3 = 0$$

$$\hookrightarrow \qquad 3x^2 - 3x^3 = 0$$

$$\hookrightarrow \qquad 3x^2(1-x) = 0 \quad ,$$

* Warning: The discussion of "solutions" here is rather incomplete so that we can get to the basic, intuitive concepts quickly. We will refine our notion of "solutions" in section 1.3 starting on page 15.

which is true only if $x = 0$ or $x = 1$. But our interest is not in finding values of x that make the equation true, but in finding functions *of x (i.e., $y(x)$) that make the equation true for* all *values of x. So $y(x) = x^3$ is* not *a solution to our differential equation. (And it makes no sense, whatsoever, to refer to either $x = 0$ or $x = 1$ as solutions, here.)*

Typically, a differential equation will have many different solutions. Any formula (or set of formulas) that describes all possible solutions is called a *general solution* to the equation.

!►*Example 1.2:* *Consider the differential equation*

$$\frac{dy}{dx} = 6x \quad .$$

All possible solutions can be obtained by just taking the indefinite integral of both sides,

$$\int \frac{dy}{dx}\,dx = \int 6x\,dx$$

$$\hookrightarrow \qquad y(x) + c_1 = 3x^2 + c_2$$

$$\hookrightarrow \qquad y(x) = 3x^2 + c_2 - c_1$$

where c_1 and c_2 are arbitrary constants. Since the difference of two arbitrary constants is just another arbitrary constant, we can replace the above $c_2 - c_1$ with a single arbitrary constant c and rewrite our last equation more succinctly as

$$y(x) = 3x^2 + c \quad .$$

This formula for y describes all possible solutions to our original differential equation — it is a general solution *to the differential equation in this example. To obtain an individual solution to our differential equation, just replace c with any particular number. For example, respectively letting $c = 1$, $c = -3$, and $c = 827$ yield the following three solutions to our differential equation:*

$$3x^2 + 1 \quad , \quad 3x^2 - 3 \quad \text{and} \quad 3x^2 + 827 \quad .$$

As just illustrated, general solutions typically involve arbitrary constants. In many applications, we will find that the values of these constants are not truly arbitrary but are fixed by additional conditions imposed on the possible solutions (so, in these applications at least, it would be more accurate to refer to the "arbitrary" constants in the general solutions of the differential equations as "yet undetermined" constants).

Normally, when given a differential equation and no additional conditions, we will want to determine all possible solutions to the given differential equation. Hence, "solving a differential equation" often means "finding a general solution" to that differential equation. That will be the default meaning of the phrase "solving a differential equation" in this text. Notice, however, that the resulting "solution" is not a single function that satisfies the differential equation (which is what we originally defined "a solution" to be), but is a formula describing all possible functions satisfying the differential equation (i.e., a "general solution"). Such ambiguity often arises in everyday language and we'll just have to live with it. Simply remember that, in practice, the phrase "a solution to a differential equation" can refer either to any single function that satisfies the differential equation,

or

any formula describing all the possible solutions (more correctly called a general solution).

In practice, it is usually clear from the context just which meaning of the word "solution" is being used. On occasions where it might not be clear, or when we wish to be very precise, it is standard to call any single function satisfying the given differential equation a *particular solution*. So, in the last example, the formulas

$$3x^2 + 1 \quad , \quad 3x^2 - 3 \quad \text{and} \quad 3x^2 + 827$$

describe particular solutions to

$$\frac{dy}{dx} = 6x \quad .$$

Initial-Value Problems

One set of auxiliary conditions that often arises in applications is a set of "initial values" for the desired solution. This is a specification of the values of the desired solution and some of its derivatives at a single point. To be precise, an N^{th}*-order set of initial values* for a solution y consists of an assignment of values to

$$y(x_0) \quad , \quad y'(x_0) \quad , \quad y''(x_0) \quad , \quad y'''(x_0) \quad , \quad \ldots \quad \text{and} \quad y^{(N-1)}(x_0)$$

where x_0 is some fixed number (in practice, x_0 is often 0) and N is some nonnegative integer.[4] Note that there are exactly N values being assigned and that the highest derivative in this set is of order $N - 1$.

We will find that N^{th}-order sets of initial values are especially appropriate for N$^{\text{th}}$-order differential equations. Accordingly, the term N^{th}*-order initial-value problem* will always mean a problem consisting of

1. an N^{th}-order differential equation, and
2. an N^{th}-order set of initial values.

For example,

$$\frac{dy}{dx} - 3y = 0 \qquad \text{with} \quad y(0) = 4$$

is a first-order initial-value problem. "$dy/dx - 3y = 0$" is the first-order differential equation, and "$y(0) = 4$" is the first-order set of initial values. On the other hand, the third-order differential equation

$$\frac{d^3y}{dx^3} + \frac{dy}{dx} = 0$$

along with the third-order set of initial conditions

$$y(1) = 3 \quad , \quad y'(1) = -4 \quad \text{and} \quad y''(1) = 10$$

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[4] Remember, if $y = y(x)$, then

$$y' = \frac{dy}{dx} \quad , \quad y'' = \frac{d^2y}{dx^2} \quad , \quad y''' = \frac{d^3y}{dx^3} \quad , \quad \ldots \quad \text{and} \quad y^{(k)} = \frac{d^ky}{dx^k} \quad .$$

We will use the 'prime' notation for derivatives when the d/dx notation becomes cumbersome.

makes up a third-order initial-value problem.

A *solution* to an initial-value problem is a solution to the differential equation that also satisfies the given initial values. The usual approach to solving such a problem is to first find the general solution to the differential equation (via any of the methods we'll develop later), and then determine the values of the 'arbitrary' constants in the general solution so that the resulting function also satisfies each of the given initial values.

!►Example 1.3: *Consider the initial-value problem*

$$\frac{dy}{dx} = 6x \qquad \text{with} \quad y(1) = 8 \quad .$$

From example 1.2 we know that the general solution to the above differential equation is

$$y(x) = 3x^2 + c$$

where c *is an arbitrary constant. Combining this formula for* y *with the requirement that* $y(1) = 8$, *we have*

$$8 = y(1) = 3 \cdot 1^2 + c = 3 + c \quad ,$$

which, in turn, requires that

$$c = 8 - 3 = 5 \quad .$$

So the solution to the initial-value problem is given by

$$y(x) = 3x^2 + c \qquad \text{with} \quad c = 5 \quad ;$$

that is,

$$y(x) = 3x^2 + 5 \quad .$$

By the way, the terms "initial values", "initial conditions", and "initial data" are essentially synonymous and, in practice, are used interchangeably.

1.2 Why Care About Differential Equations? Some Illustrative Examples

Perhaps the main reason to study differential equations is that they naturally arise when we attempt to mathematically describe "real-world" processes that vary with, say, time or position. Let us look at one well-known process: the falling of some object towards the earth. To illustrate some of the issues involved, we'll develop two slightly different sets of mathematical descriptions for the same process.

By the way, any collection of equations and formulas describing some process is called a *(mathematical) model* of the process, and the process of developing a mathematical model is called, unsurprisingly, *modeling*.

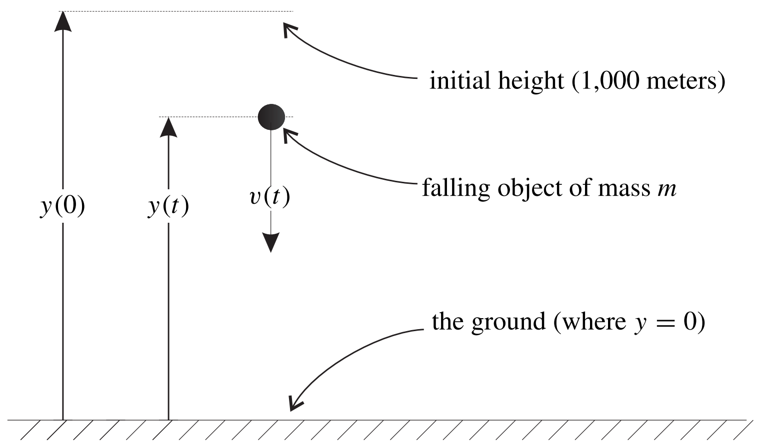

Figure 1.1: Rough sketch of a falling object of mass m.

The Situation to Be Modeled:

Let us concern ourselves with the vertical position and motion of an object dropped from a plane at a height of 1,000 meters. Since it's just being dropped, we may assume its initial downward velocity is 0 meters per second. The precise nature of the object — whether it's a falling marble, a frozen duck (live, unfrozen ducks don't usually fall) or some other familiar falling object — is not important at this time. Visualize it as you will.

The first two things one should do when developing a model is to sketch the process (if possible) and to assign symbols to quantities that may be relevant. A crude sketch of the process is in figure 1.1 (I've sketched the object as a ball since a ball is easy to sketch). Following ancient traditions, let's make the following symbolic assignments:

$$\begin{aligned}
m &= \text{the mass (in grams) of the object} \\
t &= \text{time (in seconds) since the object was dropped} \\
y(t) &= \text{vertical distance (in meters) between the object and the ground at time } t \\
v(t) &= \text{vertical velocity (in meters/second) of the object at time } t \\
a(t) &= \text{vertical acceleration (in meters/second}^2\text{) of the object at time } t
\end{aligned}$$

Where convenient, we will use y, v and a as shorthand for $y(t)$, $v(t)$ and $a(t)$. Remember that, by the definition of velocity and acceleration,

$$v = \frac{dy}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt} = \frac{d^2y}{dt^2} \quad .$$

From our assumptions regarding the object's position and velocity at the instant it was dropped, we have that

$$y(0) = 1{,}000 \qquad \text{and} \qquad \left.\frac{dy}{dt}\right|_{t=0} = v(0) = 0 \quad . \tag{1.1}$$

These will be our initial values. (Notice how appropriate it is to call these the "initial values" — $y(0)$ and $v(0)$ are, indeed, the initial position and velocity of the object.)

As time goes on, we expect the object to be falling faster and faster downwards, so we expect both the position and velocity to vary with time. Precisely how these quantities vary with time might be something we don't yet know. However, from Newton's laws, we do know

$$F = ma$$

where F is the sum of the (vertically acting) forces on the object. Replacing a with either the corresponding derivative of velocity or position, this equation becomes

$$F = m\frac{dv}{dt} \tag{1.2}$$

or, equivalently,

$$F = m\frac{d^2y}{dt^2} \quad . \tag{1.2$'$}$$

If we can adequately describe the forces acting on the falling object (i.e., the F), then the velocity, $v(t)$, and vertical position, $y(t)$, can be found by solving the above differential equations, subject to the initial conditions in line (1.1).

The Simplest Falling Object Model

The Earth's gravity is the most obvious force acting on our falling object. Checking a convenient physics text, we find that the force of the Earth's gravity acting on an object of mass m is given by

$$F_{\text{grav}} = -gm \qquad \text{where} \qquad g = 9.8 \quad \left(\text{meters/second}^2\right) \quad .$$

Of course, the value for g is an approximation and assumes that the object is not too far above the Earth's surface. It also assumes that we've chosen "up" to be the positive direction (hence the negative sign).

For this model, let us suppose the Earth's gravity, F_{grav} , is the only significant force involved. Assuming this (and keeping in mind that we are measuring distance in meters and time in seconds), we have

$$F = F_{\text{grav}} = -9.8m$$

in the " $F = ma$ " equation. In particular, equation (1.2$'$) becomes

$$-9.8m = m\frac{d^2y}{dt^2} \quad .$$

The mass conveniently divides out, leaving us with

$$\frac{d^2y}{dt^2} = -9.8 \quad .$$

Taking the indefinite integral with respect to t of both sides of this equation yields

$$\int \frac{d^2y}{dt^2}\,dt = \int -9.8\,dt$$

$$\hookrightarrow \qquad \int \frac{d}{dt}\left(\frac{dy}{dt}\right) dt = \int -9.8\,dt$$

$$\hookrightarrow \qquad \frac{dy}{dt} + c_1 = -9.8t + c_2$$

$$\hookrightarrow \qquad \frac{dy}{dt} = -9.8t + c$$

where c_1 and c_2 are the "arbitrary constants of integration" and $c = c_2 - c_1$. This gives us our formula for ${}^{dy}/_{dt}$ up to an unknown constant c. But recall that the initial velocity is zero,

$$\left.\frac{dy}{dt}\right|_{t=0} = v(0) = 0 \quad .$$

On the other hand, setting t equal to zero in the formula just derived for ${}^{dy}/_{dt}$ yields

$$\left.\frac{dy}{dt}\right|_{t=0} = -9.8 \cdot 0 + c \quad .$$

Combining these two expressions for $y'(0)$ yields

$$0 = \left.\frac{dy}{dt}\right|_{t=0} = -9.8 \cdot 0 + c \quad .$$

Thus, $c = 0$ and our formula for ${}^{dy}/_{dt}$ reduces to

$$\frac{dy}{dt} = -9.8t \quad .$$

Again, we have a differential equation that is easily solved by integrating both sides with respect to t,

$$\int \frac{dy}{dt}\,dt = \int -9.8t\,dt$$

$$\hookrightarrow \qquad y(t) + C_1 = -9.8\left[\frac{1}{2}t^2\right] + C_2$$

$$\hookrightarrow \qquad y(t) = -4.9t^2 + C$$

where, again, C_1 and C_2 are the "arbitrary constants of integration" and $C = C_2 - C_1$.[5] Combining this last equation with the initial condition for $y(t)$ (from line (1.1)), we get

$$1{,}000 = y(0) = -4.9 \cdot 0^2 + C \quad .$$

Thus, $C = 1{,}000$ and the vertical position (in meters) at time t is given by

$$y(t) = -4.9t^2 + 1{,}000 \quad .$$

[5] Note that slightly different symbols are being used to denote the different constants. This is highly recommended to prevent confusion when (and if) we ever review our computations.

A Better Falling Object Model

The above model does not take into account the resistance of the air to the falling object — a very important force if the object is relatively light or has a parachute. Let us add this force to our model. That is, for our " $F = ma$ " equation, we'll use

$$F = F_{\text{grav}} + F_{\text{air}}$$

where F_{grav} is the force of gravity discussed above, and F_{air} is the force due to the air resistance acting on this particular falling body.

Part of our problem now is to determine a good way of describing F_{air} in terms relevant to our problem. To do that, let us list a few basic properties of air resistance that should be obvious to anyone who has stuck their hand out of a car window:

1. The force of air resistance does not depend on the position of the object, only on the relative velocity between it and the surrounding air. So, for us, F_{air} will just be a function of v, $F_{\text{air}} = F_{\text{air}}(v)$. (This assumes, of course, that the air is still — no up- or downdrafts — and that the density of the air remains fairly constant throughout the distance this object falls.)

2. This force is zero when the object is not moving, and its magnitude increases as the speed increases (remember, speed is the magnitude of the velocity). Hence, $F_{\text{air}}(v) = 0$ when $v = 0$, and $|F_{\text{air}}(v)|$ gets bigger as $|v|$ gets bigger.

3. Air resistance acts *against* the direction of motion. This means that the direction of the force of air resistance is opposite to the direction of motion. Thus, the sign of $F_{\text{air}}(v)$ will be opposite that of v.

While there are many formulas for $F_{\text{air}}(v)$ that would satisfy the above conditions, common sense suggests that we first use the simplest. That would be

$$F_{\text{air}}(v) = -\gamma v$$

where γ is some positive value. The actual value of γ will depend on such parameters as the object's size, shape, and orientation, as well as the density of the air through which the object is moving. For any given object, this value could be determined by experiment (with the aid of the equations we will soon derive).

?►*Exercise 1.2:* *Convince yourself that*

a: *this formula for $F_{air}(v)$ does satisfy the above three conditions, and*

b: *no simpler formula would work.*

We are now ready to derive the appropriate differential equations for our improved model of a falling object. The total force is given by

$$F = F_{\text{grav}} + F_{\text{air}} = -9.8m - \gamma v \quad .$$

Since this formula explicitly involves v instead of dy/dt, let us use the " $F = ma$ " equation from line (1.2) on page 10,

$$F = m\frac{dv}{dt} \quad .$$

Combining the last two equations,

$$m\frac{dv}{dt} = F = -9.8m - \gamma v \quad .$$

Cutting out the middle and dividing through by the mass gives the slightly simpler equation

$$\frac{dv}{dt} = -9.8 - \kappa v \qquad \text{where} \quad \kappa = \frac{\gamma}{m} \quad . \tag{1.3}$$

Remember that γ , m and, hence, κ are positive constants, while $v = v(t)$ is a yet unknown function that satisfies the initial condition $v(0) = 0$. After solving this initial-value problem for $v(t)$, we could then find the corresponding formula for height at time t , $y(t)$, by solving the simple initial-value problem

$$\frac{dy}{dt} = v(t) \qquad \text{with} \quad y(0) = 1{,}000 \quad .$$

Unfortunately, we cannot solve equation (1.3) by simply integrating both sides with respect to t ,

$$\int \frac{dv}{dt}\,dt = \int [-9.8 - \kappa v]\,dt \quad .$$

The first integral is not a problem. By the relation between derivatives and integrals, we still have

$$\int \frac{dv}{dt}\,dt = v(t) + c_1$$

where c_1 is an arbitrary constant. It's the other side that is a problem. Since κ is a constant, but $v = v(t)$ is an unknown function of t , the best we can do with the righthand side is

$$\int [-9.8 - \kappa v]\,dt = -\int 9.8\,dt - \kappa \int v(t)\,dt = -9.8t + c_2 - \kappa \int v(t)\,dt \quad .$$

Again, c_2 is an arbitrary constant. However, since $v(t)$ is an unknown function, its integral is simply another unknown function of t . Thus, letting $c = c_2 - c_1$ and "integrating the equation" simply gives us

$$v(t) = -9.8t + c - (\kappa \cdot \textit{some unknown function of } t\,) \quad ,$$

which is not very helpful.

Fortunately, this is a text on differential equations, and methods for solving equations such as equation (1.3) will be discussed in chapters 4 and 5. But there's no need to rush things. The main goal here is to simply to see how differential equations arise in applications. Of course, now that we have equation (1.3), we also have a good reason to continue on and learn how to solve it.

By the way, if we replace v in equation (1.3) with ${}^{dy}/{}_{dt}$, we get the second-order differential equation

$$\frac{d^2y}{dt^2} = -9.8 - \kappa\frac{dy}{dt} \quad .$$

This can be integrated, yielding

$$\frac{dy}{dt} = -9.8t - \kappa y + c$$

where c is an arbitrary constant. Again, this is first-order differential equation that we cannot solve until we delve more deeply into the various methods for solving these equations. And if, in this last equation, we again use the fact that $v = {}^{dy}/_{dt}$, all we get is

$$v = -9.8t - \kappa y + c \tag{1.4}$$

which is another not-very-helpful equation relating the unknown functions $v(t)$ and $y(t)$.[6]

Summary of Our Models and the Related Initial Value Problems

For the first model of a falling body, we had the second-order differential equation

$$\frac{d^2y}{dt^2} = -9.8 \quad .$$

along with the initial conditions

$$y(0) = 1{,}000 \qquad \text{and} \qquad y'(0) = 0 \quad .$$

In other words, we had a second-order initial-value problem. This problem, as we saw, was rather easy to solve.

For the second model, we still had the initial conditions

$$y(0) = 1{,}000 \qquad \text{and} \qquad y'(0) = 0 \quad ,$$

but we found it a little more convenient to write the differential equation as

$$\frac{dv}{dt} = -9.8 - \kappa v \qquad \text{where} \quad \frac{dy}{dt} = v$$

and κ was some positive constant. There are a couple of ways we can view this collection of equations. First of all, we could simply replace the v with ${}^{dy}/_{dt}$ and say we have the second-order initial problem

$$\frac{d^2y}{dt^2} = -9.8 - \kappa\frac{dy}{dt}$$

with

$$y(0) = 1,000 \qquad \text{and} \qquad y'(0) = 0 \quad .$$

Alternatively, we could (as was actually suggested) view the problem as two successive first-order problems:

$$\frac{dv}{dt} = -9.8 - \kappa v \qquad \text{with} \quad v(0) = 0 \quad ,$$

followed by

$$\frac{dy}{dt} = v(t) \qquad \text{with} \quad y(0) = 1{,}000 \quad .$$

The first of these two problems can be solved using methods we'll develop later. And once we have the solution, $v(t)$, to that, the second can easily be solved by integration.

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[6] well, not completely useless — see exercise 1.10 b on page 21.

Though, ultimately, the two ways of viewing our second model are equivalent, there are advantages to the second: It is conceptually simple, and it makes it a little easier to use solution methods that will be developed relatively early in this text. It also leads us to finding $v(t)$ before even considering $y(t)$. Moreover, it is probably the velocity of landing, not the height of landing, that most concerns a falling person with (or without) a parachute. Indeed, if we are lucky, the solution to the first, $v(t)$, may tell us everything we are interested in, and we won't have to deal with the initial-value problem for y at all.

Finally, I should mention that, together, the two equations

$$\frac{dv}{dt} = -9.8 - \kappa v \qquad \text{and} \qquad \frac{dy}{dt} = v$$

form a "system of differential equation". That is, they comprise a set of differential equations involving unknown functions that are related to each other. This is an especially simple system since it can be solved by successively solving the individual equations in the system. Much more complicated systems can arise that are not so easily solved, especially when modeling physical systems consisting of many components, each of which can be modeled by a differential equation involving several different functions (as in, say, a complex electronic circuit). Dealing with these sorts of systems will have to wait until we've become reasonably adept at dealing with individual differential equations.

1.3 More on Solutions

Intervals of Interest

When discussing a differential equation and its solutions, we should include a specification of the interval over which the solution(s) is (are) to be valid. The choice of this interval, which we may call the *interval of solution*, the *interval of the solution's validity*, or, simply, the *interval of interest*, may be based on the problem leading to the differential equation, on mathematical considerations, or, to a certain extent, on the whim of the person presenting the differential equation. One thing we will insist on, in this text at least, is that **solutions must be continuous over this interval**.

!►*Example 1.4:* *Consider the equation*

$$\frac{dy}{dx} = \frac{1}{(x-1)^2} \quad .$$

Integrating this gives

$$y(x) = c - \frac{1}{x-1}$$

where c *is an arbitrary constant. No matter what value* c *is, however, this function cannot be continuous over any interval containing* $x = 1$ *because* $(x-1)^{-1}$ *"blows up" at* $x = 1$ *. So we will only claim that our solutions are valid over intervals that do not include* $x = 1$ *. In particular, we have valid (continuous) solutions to this differential equation over the intervals* $[0,1)$, $(-\infty,1)$, $(1,\infty)$ *, and* $(2,5)$ *; but not over* $(0,2)$ *or* $(0,1]$ *or* $(-\infty,\infty)$.

Why should we make such an issue of continuity? Well consider, if a function is not continuous at a point, then its derivatives do not exist at that point — and without the derivatives existing, how can we claim that the function satisfies a particular differential equation?

Another reason for requiring continuity is that the differential equations most people are interested in are models for "real-world" phenomena, and real-world phenomena are normally continuous processes while they occur — the temperature of an object does not instantaneously jump by fifty degrees nor does the position of an object instantaneously change by three kilometers. If the solutions do "blow up" at some point, then

1. some of the simplifying assumptions made in developing the model are probably no longer valid, or
2. a catastrophic event is occurring in our process at that point, or
3. both.

Whatever is the case, it would be foolish to use the solution derived to predict what is happening beyond the point where "things blow up". That should certainly be considered a point where the validity of the solution ends.

Sometimes, it's not the mathematics that restricts the interval of interest, but the problem leading to the differential equation. Consider the simplest falling object model discussed earlier. There we had an object start falling from an airplane at $t = 0$ from a height of 1,000 meters. Solving the corresponding initial-value problem, we obtained

$$y(t) = -4.9t^2 + 1{,}000$$

as the formula for the height above the earth at time t. Admittedly, this satisfies the differential equation for all t, but, in fact, it only gives the height of the object from the time it starts falling, $t = 0$, to the time it hits the ground, T_{hit}.[7] So the above formula for $y(t)$ is a valid description of the position of the object only for $0 \leq t \leq T_{\text{hit}}$; that is, $[0, T_{\text{hit}}]$ is the interval of interest for this problem. Any use of this formula to predict the position of the object at a time outside the interval $[0, T_{\text{hit}}]$ is just plain foolish.

In practice, the interval of interest is often not explicitly given. This may be because the interval is implicitly described in the problem, or because determining this interval is part of the problem (e.g., determining where the solutions must "blow up"). It may also be because the person giving the differential equation is lazy or doesn't care what interval is used because the issue at hand is to find formulas that hold independently of the interval of interest.

In this text, if no interval of interest is given or hinted at, assume it to be any interval that makes sense. Often, this will be the entire real line, $(-\infty, \infty)$.

Solutions Over Intervals

In introducing the concept of the "interval of interest", we have implicitly refined our notion of "a (particular) solution to a differential equation". Let us make that refinement explicit: A *solution* to a differential equation over an interval of interest is a function that is both continuous and satisfies the differential equation over the given interval.

[7] T_{hit} is computed in exercise 1.9 on page 21.

Recall that the *domain* of a function is the set of all numbers that can be plugged into the function. Naturally, if a function is a solution to a differential equation over some interval, then that function's domain must include that interval.[8]

Since we've refined our definition of particular solutions, we should make the corresponding refinement to our definition of a general solution: A *general solution* to a differential equation over an interval of interest is a formula or set of formulas describing all possible particular solutions over that interval.

Describing Particular Solutions

Let us get somewhat technical for a moment. Suppose we have a solution y to some differential equation over some interval of interest. Remember, we've defined y to be a "function". If you look up the basic definition of "function" in your calculus text, you'll find that, strictly speaking, y is a mapping of one set of numbers (the domain of y) onto another set of numbers (the range of y). This means that, for each value x in the function's domain, y assigns a corresponding number which we usually denote $y(x)$ and call "the value of y at x". If we are lucky, this function, y, is described by some formula, say, x^2. That is, the value of $y(x)$ can be determined for each x in the domain by the equation

$$y(x) = x^2 \quad .$$

Strictly speaking, the function y, it's value at x (i.e., $y(x)$), and any formula describing how to compute $y(x)$ are different things. In everyday usage, however, the fine distinctions between these concepts are often ignored, and we say things like

$$\textit{consider the function } x^2 \qquad \text{or} \qquad \textit{consider } y = x^2$$

instead of the more correct statement

$$\textit{consider the function } y \textit{ where } y(x) = x^2 \textit{ for each } x \textit{ in the domain of } y .$$

For our purposes, "everyday usage" will usually suffice and we won't worry that much about the differences between y, $y(x)$, and a formula describing y. This will save ink and paper, simplify the English, and, frankly, make it easier to follow many of our computations.

In particular, when we seek a particular solution to a differential equation, we will usually be quite happy to find a convenient formula describing the solution. We will then probably mildly abuse terminology by referring to that formula as "the solution". Please keep in mind that, in fact, any such formula is just one *description* of the solution — a very useful description since it tells you how to compute $y(x)$ for every x in the interval of interest. But other formulas can also describe the same function. For example, you can easily verify that

$$x^2 \quad , \quad (x+3)(x-3)+9 \quad \text{and} \quad \int_{t=0}^{x} 2t\,dt$$

are all formulas describing the same function on the real line.

There will also be differential equations for which we simply cannot find a convenient formula describing the desired solution (or solutions). On those occasions we will have to find some

[8] In theory, it makes sense to restrict the domain of a solution to the interval of interest so that irrelevant questions regarding the behavior of the function off the interval have no chance of arising. At this point of our studies, let us just be sure that a function serving as a solution makes sense at least over whatever interval we have interest in.

alternative way to describe our solutions. Some of these will involve using the differential equations to sketch approximations to the graphs of their solutions. Other alternative descriptions will involve formulas that approximate the solutions and allow us to generate lists of values approximating a solution at different points. These alternative descriptions may not be as convenient or as accurate as explicit formulas for the solutions, but they will provide usable information about the solutions.

Additional Exercises

1.3. *For each differential equation given below, three choices for a possible solution* $y = y(x)$ *are given. Determine whether each choice is or is not a solution to the given differential equation. (In each case, assume the interval of interest is the entire real line* $(-\infty, \infty)$ *.)*

a. $\dfrac{dy}{dx} = 3y$

i. $y = e^{3x}$ ***ii.*** $y = x^3$ ***iii.*** $y = \sin(3x)$

b. $x\dfrac{dy}{dx} = 3y$

i. $y = e^{3x}$ ***ii.*** $y = x^3$ ***iii.*** $y = \sin(3x)$

c. $\dfrac{d^2y}{dx^2} = 9y$

i. $y = e^{3x}$ ***ii.*** $y = x^3$ ***iii.*** $y = \sin(3x)$

d. $\dfrac{d^2y}{dx^2} = -9y$

i. $y = e^{3x}$ ***ii.*** $y = x^3$ ***iii.*** $y = \sin(3x)$

e. $x\dfrac{dy}{dx} - 2y = 6x^4$

i. $y = x^4$ ***ii.*** $y = 3x^4$ ***iii.*** $y = 3x^4 + 5x^2$

f. $\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 2y = 0$

i. $y = \sin(x)$ ***ii.*** $y = x^3$ ***iii.*** $y = e^{x^2}$

g. $\dfrac{d^2y}{dx^2} + 4y = 12x$

i. $y = \sin(2x)$ ***ii.*** $y = 3x$ ***iii.*** $y = \sin(2x) + 3x$

h. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$

i. $y = e^{3x}$ ***ii.*** $y = xe^{3x}$ ***iii.*** $y = 7e^{3x} - 4xe^{3x}$

1.4. *For each initial-value problem given below, three choices for a possible solution* $y = y(x)$ *are given. Determine whether each choice is or is not a solution to the given initial-value problem.*

a. $\dfrac{dy}{dx} = 4y$ *with* $y(0) = 5$

i. $y = e^{4x}$ **ii.** $y = 5e^{4x}$ **iii.** $y = e^{4x} + 1$

b. $x\dfrac{dy}{dx} = 2y$ *with* $y(2) = 20$

i. $y = x^2$ **ii.** $y = 10x$ **iii.** $y = 5x^2$

c. $\dfrac{d^2y}{dx^2} - 9y = 0$ *with* $y(0) = 1$ *and* $y'(0) = 9$

i. $y = 2e^{3x} - e^{-3x}$ **ii.** $y = e^{3x}$ **iii.** $y = e^{3x} + 1$

d. $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 36x^6$ *with* $y(1) = 1$ *and* $y'(1) = 12$

i. $y = 10x^3 - 9x^2$ **ii.** $y = 3x^6 - 2x^2$ **iii.** $y = 3x^6 - 2x^3$

1.5. *For the following, let*

$$y = \sqrt{x^2 + c}$$

where c *is an arbitrary constant.*

a. *Verify that this* y *is a solution to*

$$\frac{dy}{dx} = \frac{x}{y}$$

no matter what value c *is.*

b. *What value should* c *be so that the above* y *satisfies the initial condition*

i. $y(0) = 3$? **ii.** $y(2) = 3$?

c. *Using your results for the above, give a solution to each of the following initial-value problems:*

i. $\dfrac{dy}{dx} = \dfrac{x}{y}$ *with* $y(0) = 3$

ii. $\dfrac{dy}{dx} = \dfrac{x}{y}$ *with* $y(2) = 3$

1.6. *For the following, let*

$$y = y(x) = Ae^{x^2} - 3$$

where A *is an arbitrary constant.*

a. *Verify that this* y *is a solution to*

$$\frac{dy}{dx} - 2xy = 6x$$

no matter what value A *is.*

b. *In fact, it can be verified (using methods that will be developed later) that the above formula for y is a general solution to the above differential equation. Using this fact, finish solving each of the following initial-value problems:*

i. $\frac{dy}{dx} - 2xy = 6x$ *with* $y(0) = 1$

ii. $\frac{dy}{dx} - 2xy = 6x$ *with* $y(1) = 0$

1.7. *For the following, let*

$$y = A\cos(2x) + B\sin(2x)$$

where A and B are arbitrary constants.

a. *Verify that this y is a solution to*

$$\frac{d^2y}{dx^2} + 4y = 0$$

no matter what values A and B are.

b. *Again, it can be verified that the above formula for y is a general solution to the above differential equation. Using this fact, finish solving each of the following initial-value problems:*

i. $\frac{d^2y}{dx^2} + 4y = 0$ *with* $y(0) = 3$ *and* $y'(0) = 8$

ii. $\frac{d^2y}{dx^2} + 4y = 0$ *with* $y(0) = 0$ *and* $y'(0) = 1$

1.8. *It was stated (on page 7) that "N^{th}-order sets of initial values are especially appropriate for N^{th}-order differential equations." The following problems illustrate one reason this is true. In particular, they demonstrate that, if y satisfies some N^{th}-order initial-value problem, then it automatically satisfies particular higher-order sets of initial values. Because of this, specifying the initial values for $y^{(m)}$ with $m \geq N$ is unnecessary and may even lead to problems with no solutions.*

a. *Assume y satisfies the first-order initial-value problem*

$$\frac{dy}{dx} = 3xy + x^2 \qquad \text{with} \qquad y(1) = 2 \quad .$$

i. *Using the differential equation along with the given value for $y(1)$, determine what value $y'(1)$ must be.*

ii. *Is it possible to have a solution to*

$$\frac{dy}{dx} = 3xy + x^2$$

that also satisfies both $y(1) = 2$ and $y'(1) = 4$? (Give a reason.)

iii. *Differentiate the given differential equation to obtain a second-order differential equation. Using the equation obtained along with the now known values for $y(1)$ and $y'(1)$, find the value of $y''(1)$.*

iv. *Can we continue and find $y'''(1)$, $y^{(4)}(1)$, $\ldots$?*

b. *Assume* y *satisfies the second-order initial-value problem*

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0 \qquad \text{with} \qquad y(0) = 3 \quad \text{and} \quad y'(0) = 5 \quad .$$

i. *Find the value of* $y''(0)$ *and of* $y'''(0)$

ii. *Is it possible to have a solution to*

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$$

that also satisfies all of the following:

$$y(0) = 3 \quad , \quad y'(0) = 5 \quad \text{and} \quad y'''(0) = 0 \quad ?$$

1.9. *Consider the simplest model we developed for a falling object (see page 10). In that, we derived*

$$y(t) = -4.9t^2 + 1{,}000$$

as the formula for the height y *above ground of some falling object at time* t *.*

a. *Find* T_{hit} *, the time the object hits the ground.*

b. *What is the velocity of the object when it hits the ground?*

c. *Suppose that, instead of being dropped at* $t = 0$ *, the object is tossed up with an initial velocity of 2 meters per second. If this is the only change to our problem, then:*

i. *How does the corresponding initial-value problem change?*

ii. *What is the solution* $y(t)$ *to this initial value problem?*

iii. *What is the velocity of the object when it hits the ground?*

1.10. *Consider the "better" falling object model (see page 12), in which we derived the differential equation*

$$\frac{dv}{dt} = -9.8 - \kappa v \tag{1.5}$$

for the velocity. In this, κ *is some positive constant used to describe the air resistance felt by the falling object.*

a. *This differential equation was derived assuming the air was still. What differential equation would we have derived if, instead, we had assumed there was a steady updraft of 2 meters per second?*

b. *Recall that, from equation (1.5) we derived the equation*

$$v = -9.8t - \kappa y + c$$

relating the velocity v *to the distance above ground* y *and the time* t *(see page 14). In the following, you will show that it, along with experimental data, can be used to determine the value of* κ *.*

i. *Determine the value of the constant of integration,* c *, in the above equation using the given initial values (i.e.,* $y(0) = 1{,}000$ *and* $v(0) = 0$ *).*

ii. *Suppose that, in an experiment, the object was found to hit the ground at* $t = T_{hit}$ *with a speed of* $v = v_{hit}$. *Use this, along with the above equation, to find* κ *in terms of* T_{hit} *and* v_{hit}.

1.11. *For the following, let*

$$y = Ax + Bx \ln|x|$$

where A *and* B *are arbitrary constants.*

a. *Verify that this* y *is a solution to*

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \qquad \text{on the intervals } (0, \infty) \text{ and } (-\infty, 0) \quad ,$$

no matter what values A *and* B *are.*

b. *Again, we will later be able to show that the above formula for* y *is a general solution for the above differential equation. Given this, find the solution to the above differential equation satisfying* $y(1) = 3$ *and* $y'(1) = 8$.

c. *Why should your answer to* ***1.11 b*** *not be considered a valid solution to*

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

over the entire real line, $(-\infty, \infty)$?