12

Infinite Series

Many of the functions we will be dealing with later are described using "infinite series", often either power series or series of eigenfunctions for certain self-adjoint operators ("Fourier series" are the best-known examples of eigenfunction series). You were probably introduced to infinite series in calculus, where their main use is in providing fodder for seemingly meaningless convergence tests. With luck, you later saw real applications of infinite series in later courses.

In this set of notes, we will quickly review the basic theory of infinite series, and further develop those aspects that we will find most useful later. With luck, we may even stumble across an application.

12.1 Introduction

Recall that an *infinite series* (often shortened to *series*) is mathspeak for a summation with an infinite number of terms,

$$\sum_{k=\gamma}^{\infty} u_k = u_\gamma + u_{\gamma+1} + u_{\gamma+2} + u_{\gamma+3} + \cdots$$

The γ is a fixed integer that varies from series to series. The u_k's can be anything that can be added together — numbers, vectors, matrices, functions, … . We will be most interested in infinite series of numbers and of functions. Some examples are:

$$3 + .1 + .04 + .001 + .0005 + \cdots \qquad (\pi)$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad \text{(the harmonic series)}$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \qquad \text{(the alternating harmonic series)}$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots \qquad \text{(a geometric series)}$$

$$\sum_{k=0}^{\infty} \left(\frac{i}{3}\right)^k = \left(\frac{i}{3}\right)^0 + \left(\frac{i}{3}\right)^1 + \left(\frac{i}{3}\right)^2 + \left(\frac{i}{3}\right)^3 + \cdots \qquad \text{(another geometric series)}$$

$$\sum_{k=2}^{\infty} 3^k = 3^2 + 3^3 + 3^4 + 3^5 + \cdots \qquad \text{(yet another geometric series)}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = x^0 + x^1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots \qquad \text{(a power series)}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \sin(k\pi x) = \sin(\pi x) + \frac{1}{4}\sin(2\pi x) + \frac{1}{9}\sin(3\pi x) + \frac{1}{16}\sin(4\pi x) + \cdots$$

(a Fourier sine series)

$$\sum_{k=1}^{\infty} \frac{1}{k!} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^2 + \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^3 + \cdots \qquad \text{(a power series of matrices)}$$

(12.1)

Suppose we have a particular infinite series $\sum_{k=\gamma}^{\infty} u_k$. For each integer $N \geq \gamma$, we will refer to S_N, defined by

$$S_N = \sum_{k=\gamma}^{N} u_k = u_\gamma + u_{\gamma+1} + u_{\gamma+2} + \cdots + u_N \quad ,$$

as the corresponding N^{th} *partial sum* of the series.[1] Naturally, we want to be able to say

$$\sum_{k=\gamma}^{\infty} u_k = \lim_{N\to\infty} S_N \quad ,$$

which, of course, requires that the limit exists (in some sense — exactly what the limit means may depend on the application). If this limit exists (in the desired sense), then

1. $\sum_{k=\gamma}^{\infty} u_k$ actually adds up to something (namely $\lim_{N\to\infty} S_N$). This thing it adds up to is called the *sum* of the series, and is also denoted by $\sum_{k=\gamma}^{\infty} u_k$.

2. We say the series is *convergent*.

If the limit does not exist, then

1. $\sum_{k=\gamma}^{\infty} u_k$ does not add up to something, and we cannot treat this summation as representing anything other than a probably useless expression.

2. We say the series is *divergent*.

It should be obvious that the convergence or divergence of an infinite series does not depend on its first few terms. After all, if $\sum_{k=\gamma}^{\infty} u_k$ is any infinite series and η is an integer greater than γ, then

$$\sum_{k=\gamma}^{\infty} u_k = \sum_{k=\gamma}^{\eta-1} u_k + \sum_{k=\eta}^{\infty} u_k$$

and, since the summation from $k = \gamma$ to $k = \eta - 1$ is a simple finite sum,

$$\sum_{k=\gamma}^{\infty} u_k \quad \text{converges} \qquad \iff \qquad \sum_{k=\eta}^{\infty} u_k \quad \text{converges} \quad .$$

[1] Alternatively, you can let S_N be the sum of the first N terms. It doesn't really matter.

We should also recall that series can be added together and multiplied by scalars in a natural manner. Consider the general case where

$$\sum_{k=\gamma}^{\infty} A_k \qquad \text{and} \qquad \sum_{k=\gamma}^{\infty} B_k$$

are two series whose terms are all taken from the same vector space (e.g., all the A_k's and B_k's are numbers, or all are functions with a common domain, or all are matrices of the same size) and α and β are any two scalars. Since the partial sums are finite, we have

$$\sum_{k=\gamma}^{N} [\alpha A_k + \beta B_k] \;=\; \sum_{k=\gamma}^{N} \alpha A_k \;+\; \sum_{k=\gamma}^{N} \beta B_k \;=\; \alpha \sum_{k=\gamma}^{N} A_k \;+\; \beta \sum_{k=\gamma}^{N} B_k \quad .$$

Thus,

$$\begin{aligned} \lim_{N\to\infty} \sum_{k=\gamma}^{N} [\alpha A_k + \beta B_k] \;&=\; \lim_{N\to\infty} \left[\alpha \sum_{k=\gamma}^{N} A_k \;+\; \beta \sum_{k=\gamma}^{N} B_k \right] \\ &=\; \alpha \lim_{N\to\infty} \sum_{k=\gamma}^{N} A_k \;+\; \beta \lim_{N\to\infty} \sum_{k=\gamma}^{N} B_k \quad . \end{aligned}$$

So if

$$\sum_{k=\gamma}^{\infty} A_k \qquad \text{and} \qquad \sum_{k=\gamma}^{\infty} B_k$$

are convergent, so is

$$\sum_{k=\gamma}^{\infty} [\alpha A_k + \beta B_k] \quad ,$$

and we have

$$\sum_{k=\gamma}^{\infty} [\alpha A_k + \beta B_k] \;=\; \alpha \sum_{k=\gamma}^{\infty} A_k \;+\; \beta \sum_{k=\gamma}^{\infty} B_k \quad .$$

Of course, omitting the B_k's in the above would have shown that, for any nonzero scalar α,

$$\sum_{k=\gamma}^{\infty} A_k \quad \text{is convergent} \qquad \iff \qquad \sum_{k=\gamma}^{\infty} \alpha A_k \quad \text{is convergent}$$

with

$$\sum_{k=\gamma}^{\infty} \alpha A_k \;=\; \alpha \sum_{k=\gamma}^{\infty} A_k \quad .$$

12.2 Series of Numbers
Basics on Convergence

In this section, we'll assume the u_k's in any given $\sum_{k=\gamma}^{\infty} u_k$ are numbers (real or complex). Recall that,

$$\sum_{k=\gamma}^{\infty} u_k \quad \text{converges} \qquad \Longrightarrow \qquad |u_k| \to 0 \text{ as } k \to \infty \quad .$$

Conversely,

$$|u_k| \not\to 0 \text{ as } k \to \infty \qquad \Longrightarrow \qquad \sum_{k=\gamma}^{\infty} u_k \quad \text{diverges} \quad .$$

However, it is possible for $\sum_{k=\gamma}^{\infty} u_k$ to diverge even though $|u_k| \to 0$ as $k \to \infty$. The classic example is the harmonic series, $\sum_{k=1}^{\infty} 1/k$, which diverges (easily verified via the integral test).

Our sense of convergence/divergence can be refined as follows:

1. $\sum_{k=\gamma}^{\infty} u_k$ *converges* if and only if $\lim_{N\to\infty} \sum_{k=\gamma}^{N} u_k$ exists as a finite number.
2. $\sum_{k=\gamma}^{\infty} u_k$ *diverges* if and only if it doesn't converge.
3. $\sum_{k=\gamma}^{\infty} u_k$ *converges absolutely* if and only if $\lim_{N\to\infty} \sum_{k=\gamma}^{N} |u_k|$ exists as a finite number.
4. $\sum_{k=\gamma}^{\infty} u_k$ *converges conditionally* if and only if the series converges, but does not converge absolutely; i.e.,

$$\lim_{N\to\infty} \sum_{k=\gamma}^{N} u_k \quad \text{exits as a finite number,}$$

but

$$\lim_{N\to\infty} \sum_{k=\gamma}^{N} |u_k| \quad \text{does not.}$$

It's worth noting that an absolutely convergent series is also merely convergent. This is "easily verified" (see the appendix to this section, starting on page 12–14). Consequently, if a series is not convergent (i.e., is divergent) then it cannot be absolutely convergent.

It is also worth remembering that, if $\sum_{k=\gamma}^{\infty} u_k$ converges, then the error in using the N^{th} partial sum instead of the entire series is the "tail end" of the series,

$$\text{Error}_N = \sum_{k=\gamma}^{\infty} u_k - \sum_{k=\gamma}^{N} u_k = \sum_{k=N+1}^{\infty} u_k \quad .$$

An almost trivial, yet occasionally useful, consequence is that

$$\sum_{k=\gamma}^{\infty} u_k \quad \text{converges} \qquad \Longrightarrow \qquad \sum_{k=N+1}^{\infty} u_k \to 0 \text{ as } N \to \infty \quad .$$

When dealing with series, it is sometimes useful to remember that the 'triangle inequality',

$$|a + b| \le |a| + |b| \quad ,$$

holds whenever a and b are any two real or complex numbers.[2] This inequality is easily extended to sums with many terms:

$$\begin{aligned}\left|u_\gamma + u_{\gamma+1} + u_{\gamma+2} + \cdots + u_N\right| &\leq \left|u_\gamma\right| + \left|u_{\gamma+1} + u_{\gamma+2} + \cdots + u_N\right| \\ &\leq \left|u_\gamma\right| + \left|u_{\gamma+1}\right| + \left|u_{\gamma+2} + \cdots + u_N\right| \\ &\vdots \\ &\leq \left|u_\gamma\right| + \left|u_{\gamma+1}\right| + \left|u_{\gamma+2}\right| + \ + \cdots \left|u_N\right| \quad .\end{aligned}$$

That is,

$$\left|\sum_{k=\gamma}^{N} u_k\right| \leq \sum_{k=\gamma}^{N} |u_k| \quad .$$

Letting $N \to \infty$,

$$\left|\sum_{k=\gamma}^{\infty} u_k\right| \leq \sum_{k=\gamma}^{\infty} |u_k| \quad .$$

This holds whether or not the series is absolutely convergent; however; if the series is not absolutely convergent, then the right-hand side will be $+\infty$. In particular, in applying the triangle inequality to the expression given above for the error in using the N^{th} partial sum instead of the entire series, we get

$$|\text{Error}_N| = \left|\sum_{k=N+1}^{\infty} u_k\right| \leq \sum_{k=N+1}^{\infty} |u_k| \quad .$$

If a series converges absolutely, it is because the u_k's are shrinking to zero "fast enough" as $k \to \infty$ to ensure that

$$\sum_{k=\gamma}^{\infty} |u_k| < \infty$$

(some tests for determining when the terms are shrinking "fast enough" will be discussed later). A conditionally convergent series does not have its terms shrinking fast enough to ensure convergence. Instead, a conditionally convergent series converges because of a fortuitous pattern of cancellations in the summation.

!►*Example 12.1:* *Consider the partial sums of the alternating harmonic series*

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = \underbrace{\overbrace{\underbrace{1}_{S_1} - \frac{1}{2}}^{S_2} + \frac{1}{3}}_{S_3} - \frac{1}{4} + \cdots \quad .$$

If we plot these partial sums on the real line (see figure 12.1), it becomes clear that this series converges to some value between whatever last two S_N's are plotted. In particular, this value must be between $1/2$ and 1.

[2] The validity of this inequality can easily be seen if you treat a and b as complex numbers, and consider the sides of the possible triangles with corners a, b and $a+b$ in the complex plane. This also explains why it's called the triangle inequality.

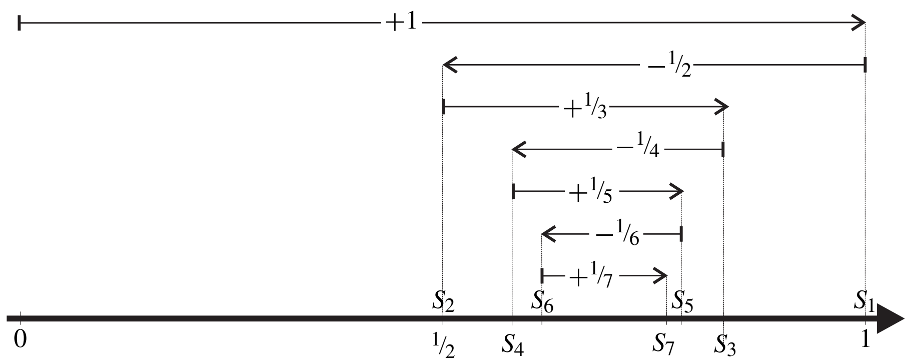

Figure 12.1: Constructing the first seven partial sums — S_1 , S_1 , S_1 , ... and S_7 — of the alternating harmonic series,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots \quad .$$

Note how each term partially cancels out the previous term.

Extending these observations yields

Theorem 12.1 (The Alternating Series Test for Convergence)
Assume $\sum_{k=\gamma}^{\infty} u_k$ is an alternating series; that is,

$$\sum_{k=\gamma}^{\infty} u_k = \pm \left\{ \left| u_\gamma \right| - \left| u_{\gamma+1} \right| + \left| u_{\gamma+2} \right| - \left| u_{\gamma+3} \right| + \cdots \right\} \quad .$$

Suppose, further, that the terms "steadily decrease to zero"; that is,

$$u_k \rightarrow 0 \qquad \text{as} \qquad k \rightarrow \infty$$

with

$$|u_k| > |u_{k+1}| > |u_{k+2}| > \cdots$$

for all k's bigger than some fixed integer K . Then $\sum_{k}^{\infty} u_k$ converges at least conditionally. Moreover, the error in using the N^{th} partial sum for the entire series is less than the first term neglected,

$$\left| \sum_{k=\gamma}^{\infty} u_k - \sum_{k=\gamma}^{N} u_k \right| \leq |u_{N+1}|$$

(provided we choose $N > K$).

Unfortunately, when the sum of a series depends on "a fortuitous pattern of cancellations", changing that pattern changes the sum. This means that simply rearranging the terms in a conditionally convergent series can change the value to which it adds up — or even yield a series that no longer converges.

!►*Example 12.2:* *Consider the alternating harmonic series*

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad ,$$

which, in example 12.1, we saw converged to some value less than 1 *. Cleverly moving each negative term further down the series, we get*

$$\begin{aligned} 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \cdots \\ &> 1 + \frac{1}{3} + \frac{1}{6} - \frac{1}{2} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} - \frac{1}{4} + \cdots \\ &> 1 + \frac{1}{2} - \frac{1}{2} + \frac{3}{12} - \frac{1}{4} + \cdots \\ &\geq 1 \quad . \end{aligned}$$

Thus, we can get an infinite series exactly the same terms as in the alternating series (just rearranged), but which adds up to something greater than 1 *(if it converges at all).*

It can be shown that rearranging the terms in an absolutely convergent series does not change the value to which it adds up. Thus, if you plan to rearrange a series to simplify computations, make sure it is an absolutely convergent series.

The Geometric Series

Definition

Any series of the form

$$\sum_{k=\gamma}^{\infty} ar^k$$

where a and r are fixed (nonzero) numbers[3] is called a *geometric series*, with

$$\sum_{k=0}^{\infty} r^k = r^0 + r^1 + r^2 + r^3 + \cdots$$

being the *basic geometric series*. Note that

$$\begin{aligned} \sum_{k=\gamma}^{\infty} ar^k &= ar^{\gamma} + ar^{\gamma+1} + ar^{\gamma+2} + ar^{\gamma+3} + \cdots \\ &= ar^{\gamma}\left[r^0 + r^1 + r^2 + r^3 + \cdots\right] \\ &= ar^{\gamma} \sum_{k=0}^{\infty} r^k \quad . \end{aligned}$$

Convergence of the Geometric Series

First note that[4]

$$\lim_{k\to\infty} |r|^k = \begin{cases} \infty & \text{if } |r| > 1 \\ 1 & \text{if } |r| = 1 \\ 0 & \text{if } |r| < 1 \end{cases} \quad .$$

[3] We insist on r being nonzero to avoid triviality when $\gamma \geq 0$ and to avoid division by 0 when $\gamma < 0$!

[4] Verify this, yourself, if it isn't obvious.

So, clearly, $\sum_{k=\gamma}^{\infty} ar^k$ will diverge if $|r| \geq 1$ (since the terms won't shrink to zero).

Now consider the partial sums of the basic geometric series,

$$\begin{aligned} S_N &= \sum_{k=0}^{N} r^k = r^0 + r^1 + r^2 + r^3 + \cdots + r^N \\ &= 1 + r^1 + r^2 + r^3 + \cdots + r^N \end{aligned}$$

when $|r| < 1$. Multiplying by r,

$$\begin{aligned} rS_N &= r\left[1 + r^1 + r^2 + r^3 + \cdots + r^N\right] \\ &= r^1 + r^2 + r^3 + r^4 + \cdots + r^{N+1} \quad , \end{aligned}$$

and subtracting this from S_N:

$$\begin{aligned} S_N &= 1 + r^1 + r^2 + r^3 + \cdots + r^N \\ -rS_N &= -\left[r^1 + r^2 + r^3 + r^4 + \cdots + r^{N+1}\right] \\ \hline (1-r)S_N &= 1 - r^{N+1} \end{aligned}$$

Dividing through by $1 - r$ then gives a simple formula for computing the N^{th} partial sum,

$$S_N = \frac{1 - r^{N+1}}{1 - r} \quad . \tag{12.2}$$

This formula does not actually require that $|r| < 1$. But it is important that $|r| < 1$ if we want $N \to \infty$, because then

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1 - r^{N+1}}{1 - r} = \frac{1 - 0}{1 - r} \quad ,$$

which is a finite number. Thus, if $|r| < 1$, the basic geometric series converges and we have a simple formula for its sum:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} \quad . \tag{12.3}$$

More generally,

Theorem 12.2 (Convergence of the Geometric Series)
The geometric series

$$\sum_{k=\gamma}^{\infty} ar^k$$

converges if and only if $|r| < 1$. *Moreover, when* $|r| < 1$,

$$\sum_{k=\gamma}^{\infty} ar^k = ar^\gamma \sum_{k=0}^{\infty} r^k = ar^\gamma \left[\frac{1}{1 - r}\right] \quad .$$

!▶ ***Example 12.3:***

$$\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

and

$$\sum_{k=2}^{\infty} 8 \left(\frac{1}{3}\right)^k = 8 \left(\frac{1}{3}\right)^2 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = 8 \left(\frac{1}{3}\right)^2 \left[\frac{3}{2}\right] = \frac{4}{3} \quad .$$

However,

$$\sum_{k=0}^{\infty} 3^k$$

diverges.

Convergence Tests for Other Series

The Idea of Bounded Partial Sums

Unfortunately, we usually cannot find such nice formulas as (12.2) for the partial sums of our series, and so, determining whether a given series converges can be a challenge. That's one reason you spent so much time in your calculus class on "tests for convergence". Of greatest interest to us are the comparison and ratio tests. First, though, we really should note a very basic test, the "bounded partial sums" test for series of nonnegative real numbers.

Theorem 12.3 (Bounded Partial Sums Test)
Suppose all the terms in

$$\sum_{k=\gamma}^{\infty} u_k$$

are nonnegative *real numbers (i.e., $u_k \geq 0$ for each $k \geq \gamma$). Then this series converges if and only if there is a finite value M bounding every partial sum,*

$$S_N = \sum_{k=\gamma}^{N} u_k \leq M \qquad \text{for every} \quad N \geq \gamma \quad . \tag{12.4}$$

PROOF: As usual, let S_N denote the N^{th} partial sum. Keeping in mind that the terms are all nonnegative real numbers, we see that

$$S_{N+1} = \sum_{k=\gamma}^{N+1} u_k = \sum_{k=\gamma}^{N} u_k + u_{N+1} = S_N + \underbrace{u_{N+1}}_{\geq 0} \geq S_N \geq 0 \quad .$$

So the partial sums form a "never decreasing" sequence,

$$0 \leq S_\gamma \leq S_{\gamma+1} \leq S_{\gamma+2} \leq S_{\gamma+3} \leq \cdots \leq S_N \leq S_{N+1} \leq \cdots \quad .$$

Clearly, if $\sum_{k=\gamma}^{\infty} u_k$ converges, then we can choose

$$M = \sum_{k=\gamma}^{\infty} u_k \quad \left(= \lim_{N\to\infty} S_N \right) \quad .$$

On the other hand, if (12.4) holds, then the partial sums form a bounded, nondecreasing sequence,

$$S_\gamma \leq S_{\gamma+1} \leq S_{\gamma+2} \leq S_{\gamma+3} \leq \cdots \leq S_N \leq S_{N+1} \leq \cdots \leq M \quad .$$

It is then easily seen that any such sequence must converge to some value S_∞ less than or equal to M. Thus, our series converges with

$$\sum_{k=\gamma} u_k = \lim_{k\to\infty} S_n = S_\infty \quad .$$

(If the convergence of the S_N's is not obvious to you, let I be the set of all real numbers that are less than or equal to at least one of the partial sums,

$$I = \{x \in \mathbb{R} : x \leq S_N \text{ for some } N\} \quad .$$

Note that I is a subinterval of $(-\infty, M]$ containing 0. So, for some real number S_∞ in the interval $[0, M]$, we have

$$I = (-\infty, S_\infty) \qquad \text{or} \qquad I = (-\infty, S_\infty] \quad .$$

Either way, if we choose any $\varepsilon > 0$ and set $x_\varepsilon = S_\infty - \varepsilon$, then x_ε is in I and, because of the way we constructed this interval, there must be an N_ε such that

$$S_\infty - \varepsilon = x_\varepsilon \leq S_{N_\varepsilon} \quad .$$

By the choice of x_ε and the fact that the partial sums form a nondecreasing sequence, we now have, for each choice of $\varepsilon > 0$, a corresponding N_ε such that

$$|S_\infty - S_N| \leq \varepsilon \qquad \text{whenever} \quad N > N_\varepsilon \quad .$$

This, you should recall, is the definition for

$$\lim_{N\to\infty} S_N$$

existing and S_∞.) ∎

The Comparison and Ratio Tests

The trick to using the "bounded partial sums test" is in finding the value M (or showing that none exists). Many of the more widely used tests are really the bounded partial sums test along with a moderately clever way of finding that M or showing it doesn't exist. For example, if you already have a convergent series of nonnegative real numbers, then you can use the sum of that series as your M for verifying the convergence of 'smaller' series. That yields the well-known comparison test.

Theorem 12.4 (Comparison Test)
Suppose

$$\sum_{k=\gamma_1}^{\infty} a_k \qquad \text{and} \qquad \sum_{k=\gamma_2}^{\infty} b_k$$

are two infinite series such that, for some integer K *,*

$$0 \leq a_k \leq b_k \quad \text{for each} \quad k \geq K \quad .$$

Then,

$$\sum_{k=\gamma_2}^{\infty} b_k \text{ converges} \quad \Longrightarrow \quad \sum_{k=\gamma_1}^{\infty} a_k \text{ converges.}$$

$$\sum_{k=\gamma_1}^{\infty} a_k \text{ diverges} \quad \Longrightarrow \quad \sum_{k=\gamma_2}^{\infty} b_k \text{ diverges.}$$

Combining either test above with the appropriate geometric series then yields the ratio test.

Theorem 12.5 (Ratio Test)
Let $\sum_{k=\gamma}^{\infty} u_k$ *be an infinite series, and suppose we can find an integer* K *and a positive value* r *such that either*

$$\left|\frac{u_{k+1}}{u_k}\right| \leq r < 1 \quad \text{for each} \quad k \geq K$$

or

$$\left|\frac{u_{k+1}}{u_k}\right| \geq r \geq 1 \quad \text{for each} \quad k \geq K \quad .$$

Then:

$$r < 1 \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} u_k \quad \text{converges (in fact, converges absolutely).}$$

$$r \geq 1 \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} u_k \quad \text{diverges.}$$

PROOF: If $r \geq 1$, then

$$\left|\frac{u_{k+1}}{u_k}\right| \geq r \geq 1 \quad \Longrightarrow \quad |u_{k+1}| \geq |u_k| > 0 \quad \text{for each} \quad k \geq K \quad .$$

Hence we cannot have $u_k \to 0$ as $k \to \infty$, which means the series must diverge.

On the other hand, if

$$\left|\frac{u_{k+1}}{u_k}\right| \leq r < 1 \quad \text{for each} \quad k \geq K \quad ,$$

then

$$|u_{k+1}| \leq r\,|u_k| \quad \text{for each} \quad k \geq K \quad .$$

Thus:

$$|u_{K+1}| \leq r\,|u_K|$$
$$|u_{K+2}| \leq r\,|u_{K+1}| \leq r \cdot r\,|u_K| \leq r^2\,|u_K|$$

$$|u_{K+3}| \;\leq\; r\,|u_{K+2}| \;\leq\; r \cdot r^2\,|u_K| \;\leq\; r^3\,|u_K|$$
$$\vdots$$

Consequently, using what we know about the geometric series (and the fact that $0 \leq r < 1$),

$$\begin{aligned}
\sum_{k=K}^{\infty} |u_k| &= |u_K| + |u_{K+1}| + |u_{K+2}| + |u_{K+3}| + \cdots \\
&\leq |u_K| + r\,|u_K| + r^2\,|u_K| + r^3\,|u_K| + \cdots \\
&= \sum_{k=K}^{\infty} |u_K|\,r^k \\
&= |u_K|\,r^K \left[\frac{1}{1-r}\right] \quad .
\end{aligned}$$

Either the bounded partial sums test or the comparison test can clearly be invoked, assuring us that $\sum_{k=\gamma}^{\infty} |u_k|$ converges. ■

The Limit Comparison and Limit Ratio Tests

The limit comparison test is a variation of the comparison that, when applicable, is often easier to use than the original test. Basically, it involves computing a limit of a ratio to see if the original comparison test could be invoked and what the result of that test would be. To see how this works, assume we have two infinite series

$$\sum_{k=\gamma_1}^{\infty} a_k \qquad \text{and} \qquad \sum_{k=\gamma_2}^{\infty} b_k \quad ,$$

and that the limit

$$\lim_{k\to\infty} \left|\frac{a_k}{b_k}\right|$$

exists as a finite, nonzero value. Then there are two other finite real numbers c and C such that

$$0 \;<\; c \;<\; \lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| \;<\; C \;<\; \infty \quad .$$

By the nature of "limits", this means there is an integer K such that

$$0 \;<\; c \;<\; \left|\frac{a_k}{b_k}\right| \;<\; C \;<\; \infty \qquad \text{whenever} \quad k \geq K \quad .$$

By algebra, then, we have

$$0 \;<\; c\,|b_k| \;<\; |a_k| \;<\; C\,|b_k| \qquad \text{whenever} \quad k \geq K \quad .$$

By basic algebra and the comparison test, it follows that

$$\sum_{k=\gamma_2}^{\infty} c\,|b_k| \;\text{ converges} \quad \Longrightarrow \quad \sum_{k=\gamma_2}^{\infty} |b_k| \;\text{ converges} \quad \Longrightarrow \quad \sum_{k=\gamma_1}^{\infty} |a_k| \;\text{ converges} \quad ,$$

while

$$\sum_{k=\gamma_1}^{\infty} |a_k| \text{ diverges} \quad \Longrightarrow \quad \sum_{k=\gamma_2}^{\infty} C\,|b_k| \text{ diverges} \quad \Longrightarrow \quad \sum_{k=\gamma_2}^{\infty} |b_k| \text{ diverges} \quad .$$

These results, along with observing what of these results can be salvaged when

$$\lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| = 0 \qquad \text{or} \qquad \lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| = \infty \quad ,$$

can be summarized as follows:

Theorem 12.6 (limit comparison test)
Suppose

$$\sum_{k=\gamma_1}^{\infty} a_k \qquad \text{and} \qquad \sum_{k=\gamma_2}^{\infty} b_k$$

are two infinite series such that

$$\lim_{k\to\infty} \left|\frac{a_k}{b_k}\right|$$

exists as either a finite number or as $+\infty$ *. Then:*

$$\lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| < \infty \quad \text{and} \quad \sum_{k=\gamma_2}^{\infty} |b_k| \text{ converges} \qquad \Longrightarrow \qquad \sum_{k=\gamma_1}^{\infty} |a_k| \text{ converges} \quad .$$

while

$$\lim_{k\to\infty} \left|\frac{a_k}{b_k}\right| > 0 \quad \text{and} \quad \sum_{k=\gamma_1}^{\infty} |b_k| \text{ diverges} \qquad \Longrightarrow \qquad \sum_{k=\gamma_2}^{\infty} |a_k| \text{ diverges} \quad .$$

In a similar manner, if

$$\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right|$$

exists and is not 1 , then you can easily verify that the ratio test can be applied to the series $\sum_{k=\gamma}^{\infty} u_k$ using a value of r between this limit and 1 . If you think about it, that means you don't really have to find r , and you can simplify the ratio test (in this case) to

Theorem 12.7 (Limit Ratio Test)
Let $\sum_{k=\gamma}^{\infty} u_k$ *be an infinite series for which*

$$\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right|$$

exists (either as a finite number or as $+\infty$ *). Then*

$$\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| < 1 \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} u_k \quad \text{converges (in fact, converges absolutely).}$$

$$\lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| > 1 \quad \Longrightarrow \quad \sum_{k=\gamma}^{\infty} u_k \quad \text{diverges.}$$

(If the limit is 1 *, there is no conclusion.)*

Other Tests for Convergence

For some other important convergence tests, such as the integral test and the root test, see the first section of the text by Arfken, Weber and Harris or, better yet, see your old calculus text. By the way, some nifty extensions of the the comparison test are given on pages 8 and 9 of Arfken, Weber and Harris.

Appendix: On the Convergence of Absolutely Convergence Series

Let us look at that claim made earlier that any absolutely convergent series must be a convergent series. That is, we claimed

$$\sum_{k=\gamma}^{\infty} |u_k| \quad \text{is convergent} \qquad \Longrightarrow \qquad \sum_{k=\gamma}^{\infty} u_k \quad \text{is convergent} \qquad .$$

This can be easily verified by splitting the series into convenient 'subseries' and considering those subseries.

Let us start by considering the case where each u_k is a real number and assume

$$\sum_{k=\gamma}^{\infty} |u_k| \quad \text{is convergent} \qquad .$$

We'll split $\sum_{k=\gamma}^{\infty} u_k$ into its positive and negative parts by setting

$$a_k = \begin{cases} |u_k| & \text{if} \quad u_k \geq 0 \\ 0 & \text{if} \quad u_k < 0 \end{cases} \qquad \text{and} \qquad b_k = \begin{cases} 0 & \text{if} \quad u_k \geq 0 \\ |u_k| & \text{if} \quad u < 0 \end{cases}$$

for each $k \geq \gamma$. Trivially, we have

$$0 \leq a_k \leq |u_k| \qquad \text{and} \qquad 0 \leq b_k \leq |u_k| \qquad \text{for each} \quad k \geq \gamma \quad .$$

Since $\sum_{k=\gamma}^{\infty} |u_k|$ converges, the comparison assures us that

$$\sum_{k=\gamma}^{\infty} a_k \qquad \text{and} \qquad \sum_{k=\gamma}^{\infty} b_k \quad \text{both also converge.}$$

Hence, as noted much earlier,

$$\sum_{k=\gamma}^{\infty} [a_k - b_k] \quad \text{converges and equals} \quad \sum_{k=\gamma}^{\infty} a_k - \sum_{k=\gamma}^{\infty} b_k \quad . \tag{12.5}$$

But, for each $k \geq \gamma$,

$$a_k - b_k = \left\{ \begin{matrix} |u_k| - 0 & \text{if} \quad u_k \geq 0 \\ 0 - |u_k| & \text{if} \quad u_k < 0 \end{matrix} \right\} = \left\{ \begin{matrix} u_k & \text{if} \quad u_k \geq 0 \\ u_k & \text{if} \quad u < 0 \end{matrix} \right\} = u_k \quad .$$

So statement (12.5) can be restated as

$$\sum_{k=\gamma}^{\infty} u_k \quad \text{converges and equals} \quad \sum_{k=\gamma}^{\infty} a_k - \sum_{k=\gamma}^{\infty} b_k \quad .$$

verifying our claim when the terms are real numbers.

To verify the claim when the terms are complex numbers, simply split the series and its terms into real and imaginary parts, and apply the above.

12.3 Infinite Series of Functions

Functions as Terms

Let us now expand our studies to infinite series whose terms are functions. Such a series will be written, generically, as

$$\sum_{k=\gamma}^{\infty} u_k(x) \quad .$$

Each $u_k(x)$ is a function of x (of course other symbols for the variable may be used), and we will assume all these functions are defined on some common domain $\mathcal{D}$. This domain can be a subset of either the real line, $\mathbb{R}$, or the complex plane, $\mathbb{C}$.

Initially, we will mainly be interested in power series — series for which there are constants x_0 and a bunch of a_k's such that

$$u_k(x) = a_k(x - x_0)^k \qquad \text{for} \quad k = 0, 1, 2, 3, \dots \quad .$$

If $x_0 = 0$ and all the a_k's are the same value a, then the series reduces to a well-known geometric series

$$\sum_{k=0}^{\infty} ax^k \quad .$$

Later, we will find ourselves very interested in series in which the $u_k(x)$'s are given by such functions as sines, cosines, Bessel functions, Hermite polynomials, and so forth.

Convergence and Error Functions

Suppose we have

$$\sum_{k=\gamma}^{\infty} u_k(x)$$

and a region $\mathcal{D}$ on which each $u_k(x)$ is defined. Typically, $\mathcal{D}$ is a subinterval of $\mathbb{R}$ when we are considering functions of a real variable, and is a two-dimensional subregion of $\mathbb{C}$ when we are considering functions of a complex variable (in which case we usually use z instead of x as the variable).

The basic notion of convergence of this series of functions (called "pointwise convergence") naturally depends on the convergence of the series of numbers obtained by replacing the variable with specific values from $\mathcal{D}$. We say that

$$\sum_{k=\gamma}^{\infty} u_k(x) \quad \textit{converges pointwise} \text{ on } \mathcal{D}$$

if and only if

$$\sum_{k=\gamma}^{\infty} u_k(x_0) \quad \text{converges (as a series of numbers) for each } x_0 \in \mathcal{D} .$$

!►Example 12.4: *We know the geometric series* $\sum_{k=0}^{\infty} z^k$ *converges whenever* $|z| < 1$. *So this series converges pointwise on any set of points that lies completely within the unit circle* *about* 0 *in the complex plane. In particular,* $\sum_{k=0}^{\infty} x^k$ *converges pointwise on the interval* $(-1, 1)$ *. It also converges pointwise on any subinterval of* $(-1, 1)$ *. It does not, however, converge pointwise on, say,* $(-1, 1]$ *since* $\sum_{k=0}^{\infty} x^k$ *diverges when* $x = 1$ *.*

For much of our work, pointwise convergence will not be sufficient. To help describe a stronger type of convergence, let's look at the error in using the N^{th} partial sum in place of the entire series,

$$\begin{aligned}\mathcal{E}_N(x) &= \left|\text{error in using } \sum_{k=\gamma}^{N} u_k(x) \text{ in place of } \sum_{k=\gamma}^{\infty} u_k(x)\right| \\ &= \left|\sum_{k=\gamma}^{\infty} u_k(x) - \sum_{k=\gamma}^{N} u_k(x)\right| \\ &= \left|\sum_{k=N+1}^{\infty} u_k(x)\right| \quad .\end{aligned}$$

IF $\sum_{k=\gamma}^{\infty} u_k(x)$ converges pointwise on $\mathcal{D}$, then $\mathcal{E}_N(x)$ is a well-defined function on $\mathcal{D}$, and, clearly,

$$\mathcal{E}_N(x) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty \qquad \text{for each} \quad x \in \mathcal{D} \quad .$$

However, the rate at which $\mathcal{E}_N(x) \rightarrow 0$ may depend wildly on the choice of x. To identify cases where $\mathcal{E}_N(x)$ is not so poorly behaved on the region of interest, let $\mathcal{E}_{N,\max}$ be the greatest error in using the N^{th} partial sum in place of the entire series on $\mathcal{D}$; that is

$$\mathcal{E}_{N,\max} = \max\{\mathcal{E}_N(x) : x \in \mathcal{D}\} \quad .$$

We do allow this value to be infinite.[5] Observe that $\mathcal{E}_N(x)$ is a function of x, while $\mathcal{E}_{N,max}$ is a single number that "uniformly bounds" the possible values of $\mathcal{E}_N(x)$. Accordingly, we then say that

$$\sum_{k=\gamma}^{\infty} u_k(x) \quad \textit{converges uniformly} \text{ on } \mathcal{D}$$

if and only if

$$\mathcal{E}_{N,max} \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty \quad .$$

Observe that

$$\sum_{k=\gamma}^{\infty} u_k(x) \text{ converges uniformly on } \mathcal{D} \qquad \Longrightarrow \qquad \sum_{k=\gamma}^{\infty} u_k(x) \text{ converges pointwise on } \mathcal{D} .$$

Before looking at an example that makes everything clear, let me tell you about a famous, yet simple, test for uniform convergence.

---

[5] Strictly speaking, we should refer to $\mathcal{E}_{N,\max}$ as the least upper bound on the error in using the N^{th} partial sum in place of the entire series on $\mathcal{D}$, and we should have used "sup" instead of "max". This is because the actual maximum may not exist, while the least upper bound always will. For example, if

$$\{\mathcal{E}_N(x) : x \in \mathcal{D}\} = [0, 4) \quad ,$$

then we will use $\mathcal{E}_{N,\max} = 4$ even though there is no x in $\mathcal{D}$ for which $\mathcal{E}_N(x) = 4$.

Theorem 12.8 (Weierstrass M Test)
Let

$$\sum_{k=\gamma}^{\infty} u_k(x)$$

be a series with all the $u_k(x)$*'s being functions defined on some domain* $\mathcal{D}$ *. Suppose, further, that, for each* k *, we can find a finite number* M_k *such that*

1. $|u_k(x)| \leq M_k$ *for every* $x \in \mathcal{D}$ *, and*

2. $\sum_{k=\gamma}^{\infty} M_k$ *converges.*

Then

$$\sum_{k=\gamma}^{\infty} u_k(x) \quad \text{converges uniformly on } \mathcal{D} .$$

Basically, the Weierstrass M test says that, if you can bound a series of functions over some domain by a single convergent series of numbers, then that series of functions will converge uniformly.

PROOF: First of all, for each $N \geq \gamma$ and each $x \in \mathcal{D}$,

$$\sum_{k=\gamma}^{N} |u_k(x)| \leq \sum_{k=\gamma}^{N} M_k \leq \sum_{k=\gamma}^{\infty} M_k \quad .$$

Since the last sum is convergent, it bounds the indicated partial sums, which tells us that $\sum_{k=\gamma}^{\infty} u_k(x)$ converges (absolutely) for each x in $\mathcal{D}$. In other words, $\sum_{k=\gamma}^{\infty} u_k(x)$ converges at least pointwise on $\mathcal{D}$.

To verify that the convergence is uniform, we first observe that

$$\mathcal{E}_N(x) = \left| \sum_{k=N+1}^{\infty} u_k(x) \right| \leq \sum_{k=N+1}^{\infty} |u_k(x)| \leq \sum_{k=N+1}^{\infty} M_k \qquad \text{for each} \quad x \in \mathcal{D} \quad .$$

This, along with the convergence of $\sum_{k=\gamma}^{\infty} M_k$, gives

$$\mathcal{E}_{N,max} = \max \{\mathcal{E}_N(x) : x \in \mathcal{D}\} \leq \left[\sum_{k=N+1}^{\infty} M_k \right] \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty \quad ,$$

verifying the uniform convergence of our original series. ∎

Now, for examples, let us consider the basic geometric power series

$$\sum_{k=0}^{\infty} u_k(x) = \sum_{k=0}^{\infty} x^k$$

on various subintervals of $\mathbb{R}$.

!► ***Example 12.5:*** *Consider*

$$\sum_{k=0}^{\infty} u_k(x) = \sum_{k=0}^{\infty} x^k \quad \text{on the interval} \quad \mathcal{D} = \left[-\frac{1}{2}, \frac{1}{2}\right] \quad .$$

Here, for each k and each x,

$$|u_k(x)| = \left|x^k\right| = |x|^k \leq \left(\frac{1}{2}\right)^k \leftarrow \text{use this as } M_k \quad .$$

Since $\left|1/2\right| < 1$, we know

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

is a convergent geometric series. The Weierstrass M test then assures us that

$$\sum_{k=0}^{\infty} x^k \quad \text{converges uniformly on} \quad \left[-\frac{1}{2}, \frac{1}{2}\right] \quad .$$

In fact, for every positive integer N and each x in this interval, a bound for the error in using the N^{th} partial sum instead of the entire series is easily computed:

$$\begin{aligned}
\mathcal{E}_N(x) = \left|\sum_{k=0}^{\infty} x^k - \sum_{k=\gamma}^{N} x^k\right| &= \left|\sum_{k=N+1}^{\infty} x^k\right| \\
&\leq \sum_{k=N+1}^{\infty} \left|x^k\right| \\
&\leq \sum_{k=N+1}^{\infty} \left(\frac{1}{2}\right)^k \leftarrow \text{this is } \mathcal{E}_{N,max} \\
&= \left(\frac{1}{2}\right)^{N+1} \left[\frac{1}{1-\frac{1}{2}}\right] = \left(\frac{1}{2}\right)^N \quad .
\end{aligned}$$

(It's worth noting that, if the series is not a geometric series, it is very unlikely that we can find the exact value of $\mathcal{E}_{N,max}$.)

!► ***Example 12.6:*** *Now consider the same series*

$$\sum_{k=0}^{\infty} u_k(x) = \sum_{k=0}^{\infty} x^k \quad ,$$

but on the interval

$$\mathcal{D} = (-1, 1) \quad .$$

From example 12.4, we know this series does converge pointwise on the given interval. What about uniform convergence?

In the previous example, we were able to choose the M_k's for the Weierstrass M test by finding the maximum value of each term over the interval. Attempting the same thing here, where x can be any value between -1 and 1, yields

$$|u_k(x)| = \left|x^k\right| = |x|^k \leq 1^k = 1 \leftarrow \text{use this as } M_k \quad .$$

Clearly, we cannot choose any smaller values for the M_k*'s. But*

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + \cdots$$

is a divergent series. So we cannot appeal to the Weierstrass M test.

Fortunately, since $\sum_{k=0}^{\infty} x^k$ *is a geometric series, we can completely compute the error terms. For each* x *in the interval,*

$$\mathcal{E}_N(x) = \left| \sum_{k=N+1}^{\infty} x^k \right| = \left| \frac{x^{N+1}}{1-x} \right| \quad .$$

Unfortunately, for each positive integer N *,*

$$\lim_{x \to 1^-} \mathcal{E}_N(x) = \lim_{x \to 1^-} \frac{x^{N+1}}{1-x} = \frac{1}{0} = +\infty \quad .$$

And so, on $(-1, 1)$

$$\mathcal{E}_{N,max} = \max \{ \mathcal{E}_N(x) : x \in (-1, 1) \} = +\infty \quad .$$

Hence, of course

$$\mathcal{E}_{N,max} \not\to 0 \quad \text{as} \quad N \to \infty \quad ,$$

which means that our geometric series $\sum_{k=0}^{\infty} x^k$ *does NOT converge uniformly on* $(-1, 1)$*.*

?►*Exercise 12.1:* *Again, consider the basic geometric series*

$$\sum_{k=0}^{\infty} x^k \quad .$$

a: *Graph (roughly)* $\mathcal{E}_N(x)$ *on* $(-1, 1)$ *for an arbitrary choice of* N*. At least get the general shape of the function on the interval, as well as the behavior of the function near the points* $x = 1$*,* $x = 0$*, and* $x = -1$*.*

b: *What happens to the graph of* $\mathcal{E}_N(x)$ *as* $N \to \infty$*?*

c: *Does* $\sum_{k=0}^{\infty} x^k$ *does converge uniformly on* $(-1, 0]$*.*

There are ways of testing for uniform convergence other than the Weierstrass M test. Abel's test is worth mentioning (see theorem 12.10 on page 12–22). Unfortunately an explanation of why Abel's test works is not for the faint of mathematical heart (see section 12.4 on page 12–22).

Also, if the series ends up being an alternating series on some interval, then you can often use the error estimate from the alternating series test (theorem 12.1 on page 12–6) to find an upper bound on $\mathcal{E}_N(x)$ over the interval for each N. Showing that this upper bound shrinks to zero as $N \to \infty$ then shows that the series converges uniformly on that interval. This is especially useful for power series whose coefficients shrink to zero too slowly to ensure absolute convergence at the endpoints of their "intervals of convergence".

?►Exercise 12.2: *Using the error estimate from the alternating series test, show that*

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} x^k$$

converges uniformly on $[0, 1]$.

?►Exercise 12.3: *Problem 1.2.1-a (page 24) of Arfken, Weber and Harris.*

The Importance of Uniform Convergence
In Approximations

First of all, if

$$\sum_{k=\gamma}^{\infty} u_k(x) \quad \text{is uniformly convergent on } \mathcal{D} ,$$

then it can be uniformly approximated to any degree of accuracy by an appropriate partial sum. That is, for any desired maximum error $\varepsilon > 0$, there is a corresponding N_ε such that, for every $N \geq N_\varepsilon$,

$$\left| \text{error in using } \sum_{k=\gamma}^{N} u_k(x) \text{ for } \sum_{k=\gamma}^{\infty} u_k(x) \right| = \mathcal{E}_N(x) \leq \varepsilon \qquad \text{for every} \quad x \in \mathcal{D} \quad .$$

All one has to do is choose N_ε so that

$$\mathcal{E}_{N,max} \leq \varepsilon \qquad \text{whenever} \quad N \geq N_\varepsilon \quad ,$$

which, in theory, is certainly possible since, for a uniformly convergent series,

$$\mathcal{E}_{N,max} \rightarrow 0 \text{ as } N \rightarrow \infty \quad .$$

(Of course, knowing that N_ε exists is quite different from actually being able to find N_ε .)

On the other hand, if

$$\sum_{k=\gamma}^{\infty} u_k(x) \quad \text{is not uniformly convergent on } \mathcal{D} ,$$

then the above does not hold. Partial sums cannot "uniformly approximate" the entire series to any desired degree of accuracy. No matter how many terms you pick for a partial sum, there will always be some values of x for which that partial sum is a lousy approximation of $\sum_{k=\gamma}^{\infty} u_k(x)$.

In the Calculus

Suppose $\sum_{k=\gamma}^{\infty} u_k(x)$ converges pointwise on $\mathcal{D}$. Then we can define a function f on $\mathcal{D}$ via

$$f(x) = \sum_{k=\gamma}^{\infty} u_k(x) \qquad \text{for each} \quad x \in \mathcal{D} \quad .$$

Indeed, we may have derived the $u_k(x)$'s to obtain such a series expansion for $f(x)$. (In solving partial differential equations, these u_k's will usually be "eigenfunctions" for some self-adjoint differential operator on a vector space of functions.)

The next theorem is the big theorem on integrating and differentiating uniformly convergent series. We will use it repeatedly (and often without comment) when solving partial differential equations.

Theorem 12.9 (uniform convergence and calculus)

Suppose $\sum_{k=\gamma}^{\infty} u_k(x)$ converges uniformly on an interval $\mathcal{I}$, and that each $u_k(x)$ is a smooth function on the interval (i.e., each u_k and its derivative is continuous on $\mathcal{I}$). Let

$$f(x) = \sum_{k=\gamma}^{\infty} u_k(x) \qquad \text{for each} \quad x \in \mathcal{I} \quad .$$

Then:

1. *$f(x)$ is continuous on $\mathcal{I}$.*
2. *$f(x)$ can be integrated "term by term" on $\mathcal{I}$. More precisely, if $[a,b]$ is a subinterval of $\mathcal{I}$, then*

$$\int_a^b f(x)\,dx = \int_a^b \sum_{k=\gamma}^{\infty} u_k(x)\,dx = \sum_{k=\gamma}^{\infty} \int_a^b u_k(x)\,dx \quad .$$

3. *If the corresponding series of derivatives, $\sum_{k=\gamma}^{\infty} {u_k}'(x)$ also converges uniformly on $\mathcal{I}$, then f can be differentiated "term by term",*

$$f'(x) = \frac{d}{dx}\sum_{k=\gamma}^{\infty} u_k(x) = \sum_{k=\gamma}^{\infty} \frac{d}{dx} u_k(x) = \sum_{k=\gamma}^{\infty} {u_k}'(x) \quad ,$$

PROOF: Trust me. Or take a course in real analysis.

A similar theorem holds for series uniformly convergent on a region in the complex plane.

It should be pointed out that, if you do not have uniform convergence, you cannot assume the integration and differentiation formulas in the above theorem. For example, it can be shown that the series

$$\sum_{k=1}^{\infty} \frac{1}{k} \sin(k\pi x)$$

converges pointwise (but not uniformly) to a "sawshaped" function on $\mathbb{R}$. However,

$$\sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{1}{k}\sin(k\pi x)\right] = \sum_{k=1}^{\infty} \pi \cos(k\pi x)$$

"blows up" at odd integer values of x.

?►*Exercise 12.4:* *Using a computer mathematics package such as Mathematica or Maple, sketch the 25th partial sum of each of the above two series. Do your sketches seem to verify the claims just made?*

12.4 'Optional' Addendum for Section 12.3: Abel's Test

The next theorem is what Arfken, Weber and Harris call *Abel's test*.[6] It is a subtle result, and we will prove it by employing a remarkably clever construction usually attributed to the early nineteenth-century mathematician Niels Abel.

Theorem 12.10 (Abel's Test)
Let $\phi_1, \phi_2, \phi_3, \ldots$ be a sequence of functions on an interval $\mathcal{I}$ such that, for some finite value M and $k = 1, 2, 3, \ldots$,

$$0 \leq \phi_{k+1}(x) \leq \phi_k(x) \leq M \qquad \text{for all } x \text{ in } \mathcal{I} \quad .$$

Suppose further that $a_1,\ a_2,\ a_3,\ \ldots$ is a sequence of numbers such that $\sum_{k=1}^{\infty} a_k$ converges. Then

$$\sum_{k=1}^{\infty} a_k\, \phi_k(x)$$

converges uniformly on $\mathcal{I}$.

The proof that follows was lifted (with some necessary alterations) from the proof of lemma 16.3 in *Foundations of Fourier Analysis* by Howell . Be warned: I've 'cleaned up' this proof only a little for these notes. And before proving it, I need to state a basic fact about the convergence of series that, up to now, we've not needed:

Lemma 12.11
Let $\sum_{k=1}^{\infty} b_k$ be a series of numbers. Assume that, for each $\epsilon > 0$, there is a corresponding integer N_ϵ such that

$$\left|\sum_{k=N+1}^{K} b_k\right| \leq \epsilon \qquad \text{whenever} \quad N_\epsilon \leq N < K < \infty \quad .$$

Then $\sum_{k=1}^{\infty} b_k$ converges, and, for each $\epsilon > 0$,

$$\left|\sum_{k=1}^{\infty} b_k - \sum_{k=1}^{N} b_k\right| = \left|\sum_{k=N+1}^{\infty} b_k\right| \leq \epsilon \qquad \text{whenever} \quad N \geq N_\epsilon \quad .$$

Basically, this lemma gives you permission to say

Since

$$\left|\sum_{k=N+1}^{K} b_k\right| \leq \epsilon \qquad \text{whenever} \quad N_\epsilon \leq N \leq K < \infty \quad ,$$

we have

$$\left|\sum_{k=N+1}^{\infty} b_k\right| = \lim_{K\to\infty}\left|\sum_{k=N+1}^{K} b_k\right| \leq \epsilon \qquad \text{whenever} \quad N_\epsilon \leq N \quad .$$

[6] Actually, it's not quite the same test as other authors call "Abel's test", but it is related to them.

Naively, this may seem obvious. It's not quite as simple as that. Still, I won't spend space here to prove this lemma. Accept it, or take an introductory course in real analysis. It should be part of the "completeness of the real number system" discussion.

Now, here's the proof of theorem 12.10:

PROOF (Abel's test): Let a "maximum allowed error" $\epsilon > 0$ be chosen.

Since $\sum_{k=1}^{\infty} a_k$ is convergent, there is an integer N_ϵ such that

$$\left|\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{N} a_k\right| = \left|\sum_{k=N+1}^{\infty} a_k\right| < \frac{\epsilon}{2M} \qquad \text{whenever} \quad N_\epsilon \leq N < \infty \quad . \tag{12.6}$$

Now, pick any integer N with $N \geq N_\epsilon$, and, for each integer k greater than N, let

$$A_k = \sum_{j=N+1}^{k} a_j \quad .$$

Observe that $A_{N+1} = a_{N+1}$ and that, for $k = N+2, N+3, N+4, \ldots$,

$$a_k + A_{k-1} = A_k$$

and

$$|A_k| = \left|\sum_{j=N+1}^{k} a_j\right| = \left|\sum_{j=N+1}^{\infty} a_j - \sum_{j=k+1}^{\infty} a_j\right|$$

$$\leq \left|\sum_{j=N+1}^{\infty} a_j\right| + \left|\sum_{j=k+1}^{\infty} a_j\right| \leq \frac{\epsilon}{2M} + \frac{\epsilon}{2M} = \frac{\epsilon}{M} \quad . \tag{12.7}$$

Here is the clever bit: Pick any x in the interval $\mathcal{I}$ and any $M > N$. For the sake of brevity in the following calculations, let ψ_k denote $\phi_k(x)$. Observe that

$$\begin{aligned}
\sum_{k=N+1}^{M} a_k\,\phi_k(x) &= \sum_{k=N+1}^{M} a_k\,\psi_k \\
&= a_{N+1}\,\psi_{N+1} + \sum_{k=N+2}^{M} a_k \psi_k \\
&= a_{N+1}\,\psi_{N+1} + \sum_{k=N+2}^{M} (\underbrace{a_k + A_{k-1}}_{=A_k} - A_{k-1})\psi_k \\
&= A_{N+1}\,\psi_{N+1} + \sum_{k=N+2}^{M} (A_k - A_{k-1})\psi_k \\
&= A_{N+1}\psi_{N+1} + (A_{N+2} - A_{N+1})\psi_{N+2} \\
&\qquad + (A_{N+3} - A_{N+2})\psi_{N+3} + \cdots + (A_M - A_{M-1})\psi_M
\end{aligned}$$

$$= A_{N+1}(\psi_{N+1} - \psi_{N+2}) + A_{N+2}(\psi_{N+2} - \psi_{N+3}) + A_{N+3}(\psi_{N+3} - \psi_{N+4}) + \cdots + A_{M-1}(\psi_{M-1} - \psi_M) + A_M\psi_M$$

$$= A_M\psi_M + \sum_{k=N+1}^{M-1} A_k(\psi_k - \psi_{k+1}) \quad .$$

This, along with inequality (12.7), gives

$$\left|\sum_{k=N+1}^{M} a_k\psi_k\right| \leq |A_M|\,|\psi_M| + \sum_{k=N+1}^{M-1} |A_k|\,|\psi_k - \psi_{k+1}|$$

$$\leq \frac{\epsilon}{M}\left[|\psi_M| + \sum_{k=N+1}^{M-1} |\psi_k - \psi_{k+1}|\right] \quad . \tag{12.8}$$

Remember, $0 \leq \phi_{k+1}(x) \leq \phi_k(x) \leq 1$ for each positive integer k, and ψ_k is just shorthand for $\phi(x)$. So $|\psi_M| = \psi_M$ and

$$|\psi_k - \psi_{k+1}| = |\phi_k(x) - \phi_{k+1}(x)| = \phi_k(x) - \phi_{k+1}(x) = \psi_k - \psi_{k+1} \quad .$$

Plugging this into inequality (12.8) gives us

$$\left|\sum_{k=N+1}^{M} a_k\psi_k\right| \leq \frac{\epsilon}{M}\left[\psi_M + \sum_{k=N+1}^{M-1} (\psi_k - \psi_{k+1})\right] \quad .$$

But, since

$$\sum_{k=1}^{N-1}(\psi_k - \psi_{k+1}) = (\psi_{N+1} - \psi_{N+2}) + (\psi_{N+2} - \psi_{N+3}) + (\psi_{N+3} - \psi_{N+4}) + \cdots + (\psi_{M-1} - \psi_M)]$$

$$= \psi_{N+1} - \psi_M \quad ,$$

and $\psi_{N+1} = \phi_{N+1}(x) \leq M$, our last inequality reduces to

$$\left|\sum_{k=N+1}^{M} a_k\psi_k\right| \leq \frac{\epsilon}{M}\left[(\psi_{N+1} - \psi_M) + \psi_M\right] \leq \frac{\epsilon}{M}\psi_{N+1} \leq \epsilon \quad .$$

That is, for each x in $\mathcal{I}$

$$\left|\sum_{k=N+1}^{M} a_k\phi_k(x)\right| \leq \epsilon \qquad \text{whenever} \quad N_\epsilon \leq N < M < \infty \quad .$$

Lemma 12.11 now assures us that $\sum_{k=1}^{\infty} a_k\phi_k(x)$ converges, and that, no matter which x in $\mathcal{I}$ we choose,

$$\left|\sum_{k=1}^{\infty} a_k\phi_k(x) - \sum_{k=1}^{N} a_k\phi_k(x)\right| \leq \epsilon \qquad \text{whenever} \quad N \geq N_\epsilon \quad .$$

And that tells us the convergence is uniform. ∎

12.5 Power Series
Basics

Let us now restrict ourselves to power series. Letting x_0 be any fixed point on the real line or the complex plane, the general form for a *power series about* x_0 is

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where the a_k's are constants. Depending on the application or mood of the instructor, the variable, x can be real or complex. In practice, the "center", x_0, is often 0, in which case the series is

$$\sum_{k=0}^{\infty} a_k x^k \quad .$$

Even if $x_0 \neq 0$, we can almost always treat $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ as being

$$\sum_{k=0}^{\infty} a_k x^k \quad \text{shifted by } x_0 \quad .$$

Radii of Convergence

The convergence issues for power series are greatly simplified because of the following remarkable lemma:

Lemma 12.12
Suppose we have a power series

$$\sum_{k=0}^{\infty} a_k x^k$$

and a (nonzero) value r. Then

$$\sum_{k=0}^{\infty} a_k r^k \text{ converges} \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k x^k \text{ converges absolutely whenever } |x| < |r| \quad ,$$

while

$$\sum_{k=0}^{\infty} a_k r^k \text{ diverges} \quad \Longrightarrow \quad \sum_{k=0}^{\infty} a_k x^k \text{ diverges whenever } |r| < |x| \quad .$$

PROOF: First, assume

$$\sum_{k=0}^{\infty} a_k r^k \quad \text{converges} \quad \text{and} \quad |x| < |r| \quad .$$

Then we must have

$$\left|a_k r^k\right| \to 0 \quad \text{as} \quad k \to \infty \quad ,$$

which, in turn means that, for some integer K ,

$$\left|a_k r^k\right| \leq 1 \qquad \text{for each} \quad k \geq K \quad .$$

Thus

$$\left|a_k x^k\right| = \left|a_k r^k \cdot \frac{x^k}{r^k}\right| = \left|a_k r^k\right| \left|\frac{x}{r}\right|^k \leq \left|\frac{x}{r}\right|^k \qquad \text{for each} \quad k \geq K \quad .$$

Moreover, since $|x| < |r|$, we have that $|x/r| < 1$. Hence, the geometric series

$$\sum_{k=0}^{\infty} \left|\frac{x}{r}\right|^k$$

converges. The above and the comparison test then tells us that

$$\sum_{k=0}^{\infty} \left|a_k x^k\right|$$

also converges, verifying the first claim.

The second claim actually follows from the first: Assume

$$\sum_{k=0}^{\infty} a_k r^k \text{ diverges} \quad \text{and} \quad |r| < |x| \quad .$$

By what we just verified (with the roles of r and x reversed), we know

$$\sum_{k=0}^{\infty} a_k x^k \text{ converges} \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_k r^k \text{ converges,}$$

contrary to the assumption. Hence, under the given assumptions, $\sum_{k=0}^{\infty} a_k x^k$ cannot converge; it must diverge. ∎

Keep in mind that a power series $\sum_{k=0}^{\infty} a_k x^k$ must converge or diverge at each point of $[0, \infty)$. Applying the above lemma, we find that there are exactly three possibilities:

1. The series diverges for every r in $(0, \infty)$. Then the series converges for no x in $\mathbb{R}$ or $\mathbb{C}$ except 0 .

2. The series converges for at least one $x = r_c$ in $(0, \infty)$, and diverges for at least one $x = r_d$ in $(0, \infty)$. In this case, the lemma tells us that the series converges absolutely whenever $|x| < r_c$ and diverges whenever $r_d < |x|$. By seeking the largest possible r_c and smallest possible r_d , we eventually discover a single positive value R , with the series converging absolutely whenever $|x| < R$, and diverging whenever $R < |x|$.

3. The series converges for every x in $(0, \infty)$. Then the lemma assures us that the series converges absolutely at every x in $\mathbb{R}$ and $\mathbb{C}$.

All this and "shifting by x_0" give us the following major result:

Theorem 12.13 (convergence of power series)
Given a power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \quad ,$$

there is an R *— which is either* 0*, a positive value, or* $+\infty$ *— such that*

$$|x - x_0| < R \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_k (x - x_0)^k \text{ converges absolutely.}$$

$$|x - x_0| > R \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_k (x - x_0)^k \text{ diverges.}$$

(The convergence at each x *where* $|x - x_0| = R$ *must be checked separately.)*

The R from the theorem is called the *radius of convergence* for the series. If x denotes a complex variable, then the set of all x in the complex plane satisfying $|x - x_0| < R$ is a disk of radius R centered at x_0. That is why we call R a "radius". Note that, whether x is a real or complex variable,

$$R = 0 \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_k (x - x_0)^k \text{ converges only if } x = x_0 \text{ (which is trivial).}$$

$$R = \infty \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_k (x - x_0)^k \text{ converges for all } x \text{ (which is very nice).}$$

Suppose, for the moment, that x denotes a real variable. If you think about it briefly, you'll see that the above theorem assures us that the set of all x's for which the series converges is either the trivial interval

$$[x_0, x_0] \qquad \text{when} \quad R = 0$$

or the interval

$$(-\infty, \infty) \qquad \text{when} \quad R = \infty$$

or one of the following intervals:

$$(x_0 - R, x_0 + R) \quad , \quad [x_0 - R, x_0 + R) \quad , \quad (x_0 - R, x_0 + R] \quad \text{or} \quad [x_0 - R, x_0 + R] \quad .$$

Whichever it is, the interval of all x's for which the series converges is called the *interval of convergence* for the series. It can be found by

1. determining R, and then, if R is neither 0 nor ∞,
2. testing the series for convergence at each endpoint of $(x_0 - R, x_0 + R)$.

It is important to know the radius of convergence R for a given power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

so that we know when the series can be treated as a valid function and when the series is rubbish. One way to find R is via the limit ratio test. By that test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}(x-x_0)^{k+1}}{a_k(x-x_0)^k}\right| < 1 \quad\Longrightarrow\quad \sum_{k=0}^{\infty} a_k(x-x_0)^k \text{ converges absolutely.}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}(x-x_0)^{k+1}}{a_k(x-x_0)^k}\right| > 1 \quad\Longrightarrow\quad \sum_{k=0}^{\infty} a_k(x-x_0)^k \text{ diverges.}$$

Clearly, the above limit can only equal 1 if $|x - x_0| = R$. Thus, to find the radius of convergence R for

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k \quad ,$$

set

$$\lim_{k\to\infty}\left|\frac{a_{k+1}R^{k+1}}{a_k R^k}\right| = 1$$

and solve for R.[7] This test requires that the appropriate limits exist. If they don't, the test is useless, and you have to try something else — possibly a similar approach using the root test.

!►Example 12.7: *Consider the power series*

$$\sum_{k=0}^{\infty} \frac{2^k}{1+k} x^k \quad .$$

Here, the center is $x_0 = 0$*, and*

$$a_n = \frac{2^n}{1+n} \quad .$$

As noted above, the radius of convergence R *must satisfy*

$$1 = \lim_{k\to\infty}\left|\frac{a_{k+1}R^{k+1}}{a_k R^k}\right| \quad .$$

That is,

$$1 = \lim_{k\to\infty} \frac{\dfrac{2^{k+1}}{1+[k+1]}R^{k+1}}{\dfrac{2^k}{1+k}R^k} = \lim_{k\to\infty}\frac{1+k}{2+k}2R = 2R\lim_{k\to\infty}\frac{1+k}{2+k} = 2R\cdot 1 \quad .$$

Solving this for R *gives us our radius of convergence,*

$$R = \frac{1}{2} \quad .$$

So the above power series converge absolutely on the interval $(-{}^1\!/_2, {}^1\!/_2)$.

Now let's check the convergence at the endpoints of this interval.

---

[7] Yes, you can simplify this to

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| \quad ,$$

but I find it easier to remember the ratio test and derive the R than to remember whether it is ${}^{a_k}\!/_{a_{k+1}}$ or ${}^{a_{k+1}}\!/_{a_k}$.

At $x = {}^1\!/_2$,

$$\sum_{k=0}^{\infty} \frac{2^k}{1+k} x^k = \sum_{k=0}^{\infty} \frac{2^k}{1+k} \left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty} \frac{1}{1+k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \quad .$$

This we recognize as being the harmonic series, which we know diverges. So our power series diverges at the endpoint $x = {}^1\!/_2$.

At $x = -{}^1\!/_2$,

$$\sum_{k=0}^{\infty} \frac{2^k}{1+k} x^k = \sum_{k=0}^{\infty} \frac{2^k}{1+k} \left(\frac{-1}{2}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{1+k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad .$$

This we recognize as being the alternating harmonic series, which we know converges. So our power series converges at the endpoint $x = {}^1\!/_2$.

Hence, the interval of convergence for our power series is $[-{}^1\!/_2, {}^1\!/_2)$.

(Note: If we had not obtained "well-known" series at $x = \pm{}^1\!/_2$, *then we would had to use one or more of the "tests for convergence" discussed in section 12.2.)*

While on the subject of radii of convergence, here is a lemma relating the radius of convergence for any power series to the radius of convergence for the series' "term-by-term derivative". We will need it for the "calculus of power series".

Lemma 12.14 (radius of convergence for differentiated power series)
The radii of convergence for the two power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} k\, a_k (x - x_0)^{k-1}$$

are equal.

PROOF: For simplicity, we will let $x_0 = 0$ and just consider the convergence for the two power series

$$\sum_{k=0}^{\infty} a_k x^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} k\, a_k x^{k-1}$$

using the limit comparison test (theorem 12.6 on page 12–13). The lemma will then follow from the following by simply "shifting by x_0".

Let R be the radius of convergence for the first series, and R' the radius of convergence for the second series.

Suppose x is any point with $|x| < R'$. Then $\sum_{k=0}^{\infty} \left|k a_k x^{k-1}\right|$ converges, and, computing the limit for the limit comparison test, we get

$$\lim_{k\to\infty} \left|\frac{a_k x^k}{k a_k x^{k-1}}\right| = \lim_{k\to\infty} \frac{|x|}{k} = 0 \quad .$$

The limit comparison test then tells us that $\sum_{k=0}^{\infty} \left|a_k x^k\right|$ also converges, which is only possible if $|x| \leq R$. Hence

$$|x| < R' \quad \Longrightarrow \quad |x| \leq R \quad . \tag{12.9}$$

Now assume x is any point with $R' < |x|$, and let y be a point satisfying

$$R' < |y| < |x| \quad .$$

Then

$$\sum_{k=0}^{\infty} \left| k\, a_k y^{k-1} \right|$$

must diverge, and, computing the limit for the limit comparison test, we get

$$\lim_{k\to\infty} \left| \frac{a_k x^k}{k a_k y^{k-1}} \right| = |y| \lim_{k\to\infty} \left| \frac{x^k}{k y^k} \right| = |y| \lim_{k\to\infty} \frac{r^k}{k} \qquad \text{where} \quad r = \left| \frac{x}{y} \right| \quad .$$

By our choice of y, $r > 1$. Using this fact and L'Hôpital's rule, we can continue the computation of the above limit:

$$\lim_{k\to\infty} \left| \frac{a_k x^k}{k a_k y^{k-1}} \right| = |y| \lim_{k\to\infty} \frac{r^k}{k} = |y| \lim_{k\to\infty} \frac{\frac{d}{dk}\left[r^k\right]}{\frac{d}{dk}[k]} = |y| \lim_{k\to\infty} \frac{r^k \ln r}{1} = +\infty \quad .$$

The limit comparison test then tells us that $\sum_{k=0}^{\infty} \left| a_k x^k \right|$ also diverges, which is only possible if $R \le |x|$. Thus,

$$R' < |x| \quad \Longrightarrow \quad R \le |x| \quad . \tag{12.10}$$

Finally, let us note that lines (12.9) and (12.10), together, tells us that $R' = R$. If this is not obvious, let x be the midpoint between R' and R,

$$x = \frac{R' + R}{2} \quad .$$

Then either

$$R' < x < R \qquad \text{or} \qquad R < x < R' \qquad \text{or} \qquad R' = x = R \quad .$$

But line (12.10) rules out $R' < x < R$, and line (12.9) rules out $R < x < R'$, leaving us with

$$R' = x = R \quad .$$

∎

Letting $b_k = k\, a_k$ and re-indexing appropriately, we can change this last lemma into another lemma about series of integrated terms:

Lemma 12.15 (radius of convergence for integrated power series)
The radii of convergence for the two series

$$\sum_{k=0}^{\infty} \frac{1}{k+1} b_k (x - x_0)^{k+1} \qquad \textit{and} \qquad \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

are equal.

?►*Exercise 12.5:* *Derive lemma 12.15 from lemma 12.14.*

Uniform Convergence and the Calculus of Power Series

Since we haven't yet discussed complex differentiation or integration of parts of the complex plane, we will now mainly consider power series in which the variable is a real-value variable.

If we are going to do calculus with a power series,

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \quad ,$$

we had better figure out the intervals over which it is uniformly convergent (so we know when we can compute derivatives and integrals "term by term", as described in the theorem on uniform convergence and calculus, theorem 12.9 on page 12–21).

Let R be the radius of convergence for

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \quad .$$

To avoid triviality, assume $R > 0$. Also, choose any positive value S less than R, and consider the convergence of the power series over the interval

$$[x_0 - S, x_0 + S] \quad .$$

This will be an interval over which our power series converges uniformly. To see this, let

$$M_k = |a_k| \, S^k \qquad \text{for each} \quad k \geq \gamma \quad ,$$

and observe the following:

1. For each x in $[x_0 - S, x_0 + S]$,

$$\left| a_k (x - x_0)^k \right| = |a_k| \, |x - x_0|^k \leq |a_k| \, S^k = M_k \quad .$$

2. Since $x_0 + S$ satisfies $|(x_0 + S) - x_0| < R$, we know the series

$$\sum_{k=0}^{\infty} a_k S^k = \sum_{k=0}^{\infty} a_k ((x_0 + S) - x_0)^k \quad .$$

converges absolutely. Thus,

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} |a_k| \, S^k$$

converges.

The Weierstrass M test then assures us that

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \text{ converges uniformly on } [x_0 - s, x_0 + s] \, .$$

Since S was any real value between 0 and R, what we just derived can be expanded slightly to:

Lemma 12.16
Let R *be the radius of convergence for*

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \quad .$$

Then this series will converge uniformly on any interval $[a, b]$ *(or* (a, b) *, etc.) where*

$$x_0 - R \; < \; a \; < \; b \; < \; x_0 + R \quad .$$

Combining all we've learned about the convergence of power series with the theorem on uniform convergence and calculus (theorem 12.9), we get the following

Theorem 12.17 (calculus of power series)
Let $R > 0$ *be the radius of convergence for a power series*

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k \quad ,$$

If $R = \infty$ *, let* $\mathcal{I} = (-\infty, \infty)$ *. Otherwise let* $\mathcal{I} = (x_0 - R, x_0 + R)$ *. Define* f *on* $\mathcal{I}$ *by*

$$f(x) \; = \; \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{for each} \quad x \in \mathcal{I} \quad .$$

Then:

1. *The above power series converges uniformly on each closed subinterval of* $\mathcal{I}$ *.*

2. f *is a continuous function on* $\mathcal{I}$ *.*

3. $f(x)$ *can be integrated "term by term" on* $\mathcal{I}$ *. More precisely, if* $[a, b]$ *is any subinterval of* $\mathcal{I}$ *, then*

$$\begin{aligned} \int_a^b f(x)\, dx \; &= \; \int_a^b \sum_{k=0}^{\infty} a_k (x - x_0)^k \, dx \\ &= \; \sum_{k=0}^{\infty} \int_a^b a_k (x - x_0)^k \, dx \; = \; \sum_{k=0}^{\infty} \frac{1}{k+1} a_k \left[(b - x_0)^{k+1} - (a - x_0)^{k+1} \right] \quad , \end{aligned}$$

   and this last series converges.

4. $f(x)$ *is infinitely differentiable on* $\mathcal{I}$ *, and its derivatives can be found by differentiating the series "term by term":*

$$f'(x) \; = \; \sum_{k=0}^{\infty} \frac{d}{dx} a_k (x - x_0)^k \; = \; \sum_{k=0}^{\infty} k \, a_k (x - x_0)^{k-1}$$

$$f''(x) \; = \; \sum_{k=0}^{\infty} \frac{d^2}{dx^2} a_k (x - x_0)^k \; = \; \sum_{k=0}^{\infty} k(k-1) \, a_k (x - x_0)^{k-2}$$

$$f'''(x) = \sum_{k=0}^{\infty} \frac{d^3}{dx^3} a_k (x - x_0)^k = \sum_{k=0}^{\infty} k(k-1)(k-2)\, a_k (x - x_0)^{k-3}$$

$$\vdots$$

Moreover, each of these power series has R as its radius of convergence and converges uniformly on each closed subinterval of $\mathcal{I}$.

Be assured: Similar results hold for power series in which the variable is complex (in which case, the interval $\mathcal{I}$ is replaced by a disk of radius R about x_0 in the complex plane).

Power Series Representations of Functions

Mathematicians and physicists have long been thrilled whenever a function of interest f can be expressed as a power series about a point x_0 on some region $\mathcal{D}$,

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{for all } x \text{ in } \mathcal{D} \quad .$$

$\mathcal{D}$, of course, must be in the region — interval or disk about x_0 — in which the power series converges. We will automatically assume $\mathcal{D}$ is that region of convergence, unless there is a good reason not to.

For the moment, let us generally restrict ourselves to functions and power series of a real variable (with the understanding that almost everything we do will extend to functions and power series of complex variables).

When a function can be represented by a power series (about x_0),

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{for all } x \text{ in } \mathcal{D} \quad ,$$

we say "f is *analytic* (about x_0) on $\mathcal{D}$". There are two big advantages of dealing with such a function:

1. The value of $f(x)$ for any specific value of $x \in \mathcal{D}$ can be closely approximated by a suitable partial sum

$$\sum_{k=0}^{N} a_k (x - x_0)^k \quad ,$$

which may be more easily computed than using any other formula or definition for $f(x)$.

2. The calculus with $f(x)$ is essentially the calculus of power series. In particular, f is infinitely differentiable on any $[a, b] \subset \mathcal{D}$, with

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad ,$$

$$f'(x) = \sum_{k=0}^{\infty} k a_k (x - x_0)^{k-1} \quad ,$$

$$f''(x) = \sum_{k=0}^{\infty} k(k-1)a_k(x-x_0)^{k-2} \quad ,$$

$$\vdots$$

It's a good exercise to look at these series after writing them out in 'long' form; that is, as

$$f(x) = a_0(x-x_0)^0 + a_1(x-x_0)^1 + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots \quad ,$$

instead of

$$f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k \quad .$$

?►Exercise 12.6: *Write out the series just above for $f(x)$, $f'(x)$, etc. in long form and observe how the first few terms of each simplify. Then use your series to verify that*

$$a_0 = f(x_0) \quad , \quad a_1 = f'(x_0) \quad , \quad a_2 = \frac{1}{2}f''(x_0) \quad \ldots$$

From the results of this last exercise it follows that power series representations about a given point are unique. More precisely, if, in some region $\mathcal{D}$ containing x_0,

$$f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k \qquad \text{and} \qquad f(x) = \sum_{k=0}^{\infty} b_k(x-x_0)^k \quad ,$$

then

$$b_k = a_k \qquad \text{for} \quad k = 0, 1, 2, 3, \ldots \quad .$$

However, we can still have different power series representations for a given function about two different points,

$$f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k \qquad \text{and} \qquad f(x) = \sum_{k=0}^{\infty} b_k(x-x_1)^k \quad .$$

with

$$b_k \neq a_k \qquad \text{for} \quad k = 0, 1, 2, 3, \ldots \quad .$$

Moreover, the regions over which these two series converge may also be different. (This will be a more significant issue when our variables are complex.)

Do note that, if you already have a power series representation for some function,

$$f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k \qquad \text{for all} \quad x \in \mathcal{D} \quad ,$$

then you can get the power series representations about x_0 for all derivatives (and integrals) of f by suitably differentiating (and integrating) the known power series for $f(x)$.

Another basic method for determining if a function f is analytic about x_0 and for getting its power series about x_0, starts with the integral equation

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(s)\,ds \quad .$$

”Cleverly” integrating by parts, with

$$u = f'(s) \qquad\qquad dv = ds$$
$$du = f''(s)\,ds \qquad\qquad v = s - x$$

gives

$$\begin{aligned} f(x) - f(x_0) &= f'(s)(s-x)\Big|_{s=x_0}^{x} - \int_{x_0}^{x} f''(s)(s-x)\,ds \\ &= f'(x)\underbrace{(x-x)}_{0} - f'(x_0)\underbrace{(x_0-x)}_{-(x-x_0)} - \int_{x_0}^{x} f''(s)(s-x)\,ds \quad . \end{aligned}$$

So

$$f(x) = f(x_0) + f'(x)(x_0 - x) - \int_{x_0}^{x} f''(s)(s-x)\,ds \quad .$$

Integrating by parts again, with

$$u = f''(s) \qquad\qquad dv = (s-x)\,ds$$
$$du = f'''(s)\,ds \qquad\qquad v = \frac{1}{2}(s-x)^2$$

gives

$$\begin{aligned} f(x) &= f(x_0) + f'(x)(x_0 - x) - \left[f''(s)\frac{1}{2}(s-x)^2\Big|_{s=x_0}^{x} - \int_{x_0}^{x} f'''(s)\frac{1}{2}(s-x)^2\,ds \right] \\ &= \cdots \\ &= f(x_0) + f'(x)(x_0 - x) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{2}\int_{x_0}^{x} f'''(s)(s-x)^2\,ds \quad . \end{aligned}$$

Repeating this process until we finally see the light yields, for each positive integer N ,

$$f(x) = P_N(x) + R_N(x)$$

where

$$\begin{aligned} P_N(x) &= f(x_0) + f'(x)(x_0 - x) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{3\cdot 2}f'''(x_0)(x-x_0)^3 \\ &\qquad + \cdots + \frac{1}{N\cdot(N-1)\cdots 3\cdot 2}f^{(N)}(x_0)(x-x_0)^N \\ &= \sum_{k=0}^{N}\frac{1}{k!}f^{(k)}(x_0)(x-x_0)^k \end{aligned}$$

and

$$R_N(x) = (-1)^N\frac{1}{N!}\int_{x_0}^{x} f^{(N+1)}(s)(s-x)^N\,ds \quad .$$

You should recognize $P_N(x)$ as the N^{th} *degree Taylor polynomial about* x_0 with $R_N(x)$ being the corresponding error in using that polynomial for $f(x)$. If, for each $x \in \mathcal{D}$,

$$R_N(x) \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad ,$$

then f is analytic on $\mathcal{D}$, with

$$f(x) = \lim_{N\to\infty} P_N(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k \qquad \text{for each } x \text{ in } \mathcal{D} \quad .$$

This power series is the famous *Taylor series for* $f(x)$ *about* x_0.[8]

Let's briefly consider the problem of showing that

$$R_N(x) \to 0 \quad \text{as } N \to \infty$$

for some given function f. In practice, if you can completely compute the value of $R_N(x)$, then you already know enough about computing $f(x)$ that you don't need to find its Taylor series. More likely, you won't be able to completely compute $R_N(x)$, but you will be able to 'bound' each derivative of f; that is, for each positive integer k, you'll be able to find an $M_k(x)$ such that, for every s in the interval having endpoints x_0 and x,

$$\left| f^{(k)}(s) \right| \le M_k(x) \quad .$$

Then, if $x_0 \le x$,

$$\begin{aligned}
|R_N(x)| &= \left| (-1)^N \frac{1}{N!} \int_{x_0}^{x} f^{(N+1)}(s)(s - x)^N \, ds \right| \\
&\le \frac{1}{N!} \int_{x_0}^{x} \left| f^{(N+1)}(s)(s - x)^N \right| \, ds \\
&\le \frac{1}{N!} \int_{x_0}^{x} M_{N+1}(x)(x - s)^N \, ds \\
&= \frac{1}{N!} M_{N+1}(x) \left[\frac{-1}{N+1} (x - s)^{N+1} \Big|_{x_0}^{x} \right] \\
&= -\frac{1}{(N+1)!} M_{N+1}(x) \left[(x - x)^{N+1} - (x - x_0)^{N+1} \right] \\
&= \frac{1}{(N+1)!} M_{N+1}(x)(x - x_0)^{N+1} \quad .
\end{aligned}$$

Similar computations yield

$$|R_N(x)| \le \frac{1}{(N+1)!} M_{N+1}(x)(x_0 - x)^{N+1} \qquad \text{if} \quad x \le x_0 \quad .$$

Either way, we have

$$|R_N(x)| \le \frac{1}{(N+1)!} M_{N+1}(x) \, |x - x_0|^{N+1} \quad .$$

This gives an upper bound on each remainder term. If

$$\frac{1}{(N+1)!} M_{N+1}(x) \, |x - x_0|^{N+1} \to 0 \quad \text{as } N \to \infty \quad ,$$

then we know

$$R_N(x) \to 0 \quad \text{as } N \to \infty \quad ,$$

[8] also called the *Maclaurin series* if $x_0 = 0$.

and $f(x)$ can be represented by its Taylor series,

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k \quad .$$

At this point you should further refresh your memory regarding Taylor series, especially the "well-known" cases:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \quad ,$$

$$\sin(x) = \cdots \text{ (You figure this out.)} \quad ,$$

$$\cos(x) = \cdots \text{ (You figure this out.)} \quad ,$$

and the binomial series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \quad .$$

Rederive these series, determine the values of x for which they are valid by considering $R_N(x)$ (for the binomial series, this depends on the value of p), and play with them by doing the homework assigned. (See also the subsection *Taylor's Expansion* starting on page 25, of Arfken, Weber and Harris).

12.6 Power Series of Other Things

Suppose

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

for all real or complex values of of x, and let X be any (real or complex) thingee for which

$$\mathsf{X}^k = \underbrace{\mathsf{X} \cdot \mathsf{X} \cdots \mathsf{X}}_{k \text{ times}}$$

makes sense for each nonnegative integer k. Then we can *define* $f(\mathsf{X})$ by

$$f(\mathsf{X}) = \sum_{k=0}^{\infty} a_k \mathsf{X}^k \quad .$$

!►*Example 12.8:* *Recall that*

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

for all real x (it also holds for all complex x). Now, if $\mathbf{A}$ is any square matrix, say of dimension $n \times n$, then so is

$$\mathbf{A}^k = \underbrace{\mathbf{A} \cdot \mathbf{A} \cdots \mathbf{A}}_{k \text{ times}} \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

For $k = 0$, we naturally assume/define

$$\mathbf{A}^0 = \mathbf{I}_n \quad \text{(the } n \times n \text{ identity matrix)} \quad .$$

So, the "exponential function of square matrices" is defined by

$$\exp(\mathbf{A}) = e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k \quad .$$

This function of matrices turns out to be useful in solving certain systems of differential equations

!►*Example 12.9:* *Along the same lines, we can even have the differential operator*

$$\exp\left(\frac{d}{dx}\right) = e^{d/dx} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dx^k} \quad .$$

?►*Exercise 12.7:* *All the following concern the exponential function of square $(n \times n)$ matrices.*

a: *Let $\mathbf{A}$ be a fixed matrix, and show that*

$$\frac{d}{dt} \exp(\mathbf{A}t) = \mathbf{A} \exp(\mathbf{A}t) \quad .$$

b: *Let $\mathbf{D}$ be the diagonal matrix*

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad .$$

i: *Convince yourself that, for each nonnegative integer k,*

$$\mathbf{D}^k = \begin{bmatrix} (\lambda_1)^k & 0 & 0 & \cdots & 0 \\ 0 & (\lambda_2)^k & 0 & \cdots & 0 \\ 0 & 0 & (\lambda_3)^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (\lambda_n)^k \end{bmatrix} \quad .$$

ii: *Compute $\exp(\mathbf{D})$, simplifying it as much as possible.*

c: *Is it true that $\exp(\mathbf{A})^\dagger = \exp\left(\mathbf{A}^\dagger\right)$?*[9]

[9] Recall: $\mathbf{A}^\dagger$ = adjoint of $\mathbf{A}$ = transpose of the complex conjugate of $\mathbf{A}$.

d: *Is* $\exp(\mathbf{A})$ *Hermitian if* $\mathbf{A}$ *is Hermitian?*[10]

e: *Show that*

$$\mathbf{U}^\dagger \exp(\mathbf{A})\mathbf{U} \;=\; \exp(\mathbf{U}^\dagger \mathbf{A}\mathbf{U}) \quad .$$

whenever $\mathbf{U}$ *is unitary.*[11]

f: *Using results from the above, compute* $\exp(\mathbf{A})$ *when*

$$\mathbf{A} \;=\; \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \quad .$$

?►Exercise 12.8: *The three Pauli spin matrices are*

$$\boldsymbol{\sigma}_1 \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad \boldsymbol{\sigma}_2 \;=\; \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\sigma}_3 \;=\; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad .$$

Using these matrices:

a: *Show that, for* $k = 1,\ 2,$ *and* 3 *,*

$$(\boldsymbol{\sigma}_k)^2 \;=\; \mathbf{I}_2$$

where $\mathbf{I}_2$ *is the* 2×2 *identity matrix.*

b: *Then show that, for* $k = 1,\ 2,$ *and* 3 *,*

$$\exp(i\boldsymbol{\sigma}_k\theta) \;=\; \mathbf{I}_2 \cos\theta \;+\; i\boldsymbol{\sigma}_k \sin\theta \quad .$$

?►Exercise 12.9: *Let* r *be a fixed real number, and show that, for any analytic function* f *,*

$$\exp\left(-r\frac{d}{dx}\right) f(x) \;=\; f(x-r) \quad .$$

(Hint: $-r \;=\; [x-r] - x$ *)*

12.7 Convergence in Norm

Definition and General Commentary

We can generalize many of our notions of convergence to handle any infinite series

$$\sum_{k=\gamma}^{\infty} u_k$$

[10] Remember: $\mathbf{A}$ is Hermitian $\Longleftrightarrow$ $\mathbf{A}$ is self adjoint $\Longleftrightarrow$ $\mathbf{A}^\dagger = \mathbf{A}$.

[11] Remember: $\mathbf{U}$ is unitary $\Longleftrightarrow$ $\mathbf{U}^\dagger = \mathbf{U}^{-1}$.

for which all the u_k's are elements of some vector space $\mathcal{V}$ having an inner product $\langle \cdot \mid \cdot \rangle$ and corresponding norm $\|\cdot\|$.[12] Given such a series, we say that it converges (in that norm) if and only if there is a S in $\mathcal{V}$ such that

$$\lim_{N\to\infty} \left\| S - \sum_{k=\gamma}^{N} u_k \right\| = 0 \quad .$$

This S, naturally, is called the sum of the series and we normally write

$$S = \sum_{k=\gamma}^{\infty} u_k \quad .$$

If there is no such S in $\mathcal{V}$, then the series diverges.

It should be noted that the sort of series convergence already discussed are just special cases of norm convergence. When our series was just a series of numbers, these numbers are just elements of the vector space $\mathbb{R}$ or $\mathbb{C}$, and the norm was the usual "absolute value" norm,

$$\|u\| = |u| \quad .$$

When we were discussing the uniform convergence of functions on some domain $\mathcal{D}$, then the vector space was the set of all "reasonable" functions on $\mathcal{D}$ (with the exact meaning of "reasonable" dependent on the choice of the $u_k(x)$'s), and the norm was "maximum value" norm,

$$\|u\| = \max\{|u(x)| : x \in \mathcal{D}\} \quad .$$

(Again, this really should be the "least upper bound" and we really should call it the "sup" norm.)

?►*Exercise 12.10:* *Convince yourself of the validity of the above claims.*

Of course, using the standard inner products and norms for traditional vectors and for matrices, we can extend much of our discussion regarding series of numbers to series of traditional vectors and series of matrices. Later, when dealing with Fourier series and solving partial differential equations, we will be especially interested in infinite series of functions defined over some region $\mathcal{D}$ using the "energy" inner product and norm

$$\langle f \mid g \rangle = \int_{\mathcal{D}} f^*(x) g(x)\, dx$$

and

$$\|f\| = \sqrt{\langle f \mid f \rangle} = \sqrt{\int_{\mathcal{D}} |f(x)|^2 \, dx} \quad ,$$

or even using a "weighted energy" inner product and norm

$$\langle f \mid g \rangle = \int_{\mathcal{D}} f^*(x) g(x) w(x)\, dx$$

[12] Remember: $\|u\| = \sqrt{\langle u \mid u \rangle}$. You may want to review the basics about inner products and norms in section 3.3 of our notes.

and

$$\|f\| = \sqrt{\langle f \mid f \rangle} = \sqrt{\int_{\mathcal{D}} |f(x)|^2 \, w(x) \, dx}$$

where w is some positive-valued function on $\mathcal{D}$. (When dealing with these integral norms, we will also extend what we did with "orthogonal sets" and "self-adjoint operators".)

?►Exercise 12.11: *Let our vector space be the set of all continuous functions on the interval* $[-1, 1]$ *with the standard energy inner product and norm mentioned above. Let* f *be the analytic function on* $[-1, 1]$ *given by some power series*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

whose radius of convergence R *is larger than* 1. *Remember, this means that the series converges uniformly (to* f *) on* $[-1, 1]$. *Now show that this series also converges in norm (using the standard energy norm).*

Some Basic Inequalities

The Inequalities

At this point, it may be worthwhile to note a couple of relations that possibly should have been mentioned earlier when we discussed general inner products and norms. (Proofs of these inequalities are given in the next subsection for those interested.)

1. *(The Schwarz Inequaltiy)* Recall that, if $\mathbf{u}$ and $\mathbf{v}$ are two traditional vectors having an angle of θ between them, then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos(\theta) \quad .$$

Thus,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \quad .$$

This is a special case of the *Schwarz inequality*

$$|\langle u \mid v \rangle| \leq \|u\| \, \|v\| \quad ,$$

which holds whenever u and v are elements of any vector space having any inner product $\langle \cdot \mid \cdot \rangle$.

2. *(The Triangle Inequality)* The classical triangle inequality is that, for any two real or complex numbers u and v,

$$|u + v| \leq |u| + |v| \quad .$$

Combining this with the Schwarz inequality allows us to derive the general triangle inequality

$$\|u + v\| \leq \|u\| + \|v\| \quad ,$$

which holds whenever u and v are elements of any vector space having any inner product $\langle \cdot \mid \cdot \rangle$.

The general triangle inequality can be extended to sums with infinitely many terms just as we extended the classical triangle inequality. That is, we have

$$\left\| \sum_{k=\gamma}^{\infty} u_k \right\| \le \sum_{k=\gamma}^{\infty} \|u_k\|$$

in general. In some cases this can be used to show that, if $\sum_{k=\gamma}^{\infty} \|u_k\|$ converges as a series of numbers, then $\sum_{k=\gamma}^{\infty} \|u_k\|$ converges in norm. However, this is not always the case, and sometimes when you do have "convergence", the convergence is to something *not* in the vector space you started with. For example, a series of infinitely differentiable functions may converge to a function that is not infinitely differentiable (a common occurrence with Fourier series).

We will discuss these and other issues as necessary later, when the need arises.

Proving the Inequalities

For all the following, assume u and v are some arbitrary elements in some arbitrary vector space $\mathcal{V}$ having an inner product $\langle \cdot | \cdot \rangle$.

The easiest way to verify the Schwarz inequality,

$$|\langle u \mid v \rangle| \le \|u\| \, \|v\| \quad ,$$

is probably through clever algebra. First observe that this inequality is certainly true if either u or v is the zero element. Now assume neither is the zero element, and, for convenience, let A and B be the scalars

$$A = \|v\|^2 \qquad \text{and} \qquad B = \langle v \mid u \rangle \quad .$$

Note that $A^* = A$ and that, by the properties of inner products,

$$B = \langle u \mid v \rangle^* \qquad \text{and} \qquad B^* = \langle u \mid v \rangle \quad .$$

Furthermore (applying the properties and recalling our conventions),

$$\begin{aligned} 0 &\le \|Au - Bv\|^2 \\ &= \langle Au - Bv \mid Au - Bv \rangle \\ &= \langle Au \mid Au \rangle - \langle Au \mid Bv \rangle - \langle Bv \mid Au \rangle + \langle Bv \mid Bv \rangle \\ &= A^*A \langle u \mid u \rangle - A^*B \langle u \mid v \rangle - B^*A \langle v \mid u \rangle + B^*B \langle v \mid v \rangle \\ &= A^2 \|u\|^2 - ABB^* - B^*AB + B^*BA \\ &= A \left(\|u\|^2 A - |B|^2 \right) \quad . \end{aligned}$$

Cutting out the middle and recalling what A and B are yields

$$0 \le \|v\|^2 \left(\|u\|^2 \|v\|^2 - |\langle u \mid v \rangle|^2 \right) \quad .$$

Since $\|v\|$ is nonzero here, the last inequality must mean that

$$0 \le \|u\|^2 \|v\|^2 - |\langle u \mid v \rangle|^2 \quad .$$

That is

$$|\langle u \mid v \rangle|^2 \le \|u\|^2 \|v\|^2 \quad .$$

Taking the square root of both sides then yields the Schwarz inequality.

Verifying the general triangle inequality is now easy:

$$\begin{aligned}
\|u+v\|^2 &= \langle\, u+v \mid u+v \,\rangle \\
&= \langle\, u \mid u \,\rangle + \langle\, u \mid v \,\rangle + \langle\, v \mid u \,\rangle + \langle\, v \mid v \,\rangle \\
&= \|u\|^2 + \langle\, u \mid v \,\rangle + \langle\, v \mid u \,\rangle + \|v\|^2 \\
&\leq \|u\|^2 + |\langle\, u \mid v \,\rangle| + |\langle\, v \mid u \,\rangle| + \|v\|^2 \\
&\leq \|u\|^2 + \|u\|\,\|v\| + \|u\|\,\|v\| + \|v\|^2 \\
&\leq \|u\|^2 + 2\,\|u\|\,\|v\| + \|v\|^2 \\
&= (\|u\|+\|v\|)^2 \quad .
\end{aligned}$$

Cut out the middle and take the square root of both sides then yields

$$\|u+v\| \;\leq\; \|u\|+\|v\| \quad ,$$

which is the claimed triangle inequality.