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Power Series and Differential Equations: The Method of Frobenius*

It's all well and good to be able to find power series representations for functions you know via the standard computations for Taylor series. Even better is to be able to find power series representations for functions you don't know, say, solutions to certain nasty differential equations that cannot be solved by elementary methods. That is where the method of Frobenius is useful. It is a procedure for finding power series formulas for solutions to a great many differential equations of interest in mathematical physics. Actually, these formulas are often not true power series, but "modified power series". To be precise, the series obtained are power series multiplied by relatively simple functions.

The basic method is described and illustrated in the first section. This will give you the tools to start practicing the method. After that, we will discuss when the method is applicable and what sort of solutions, in general, the method yields.

13.1 The Basic Method of Frobenius (Illustrated)

Preliminaries

Let us assume we have a differential equation of the form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0$$

where a , b , and c are "suitable functions". Just what "suitable" means will be discussed later. For now, we will assume they are "suitable" rational functions (i.e., polynomials divided by polynomials) such as in *Bessel's equation of order* $1/2$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left[1 - \frac{1}{4x^2}\right]y = 0 \quad . \tag{13.1}$$

This is the equation we will solve to illustrate the method.

The basic method of Frobenius seeks solutions of the form of power series about some given point x_0 multiplied by $x - x_0$ to some power. That is, it seeks solutions of the form

$$y(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

* A more complete development of this material is at www.uah.edu/math/howell/DEtext (chap. 31–35).

where r and the a_k's are constants to be determined by the method.

The Basic Steps

Before actually starting the method, there are two "pre-steps":

Pre-step 1 Choose a value for x_0. If conditions are given for $y(x)$ at some point, then use that point for x_0. Otherwise, choose x_0 as convenient — which usually means you choose $x_0 = 0$.

For our example, we have no initial values at any point, so we will choose x_0 as simply as possibly; namely, $x_0 = 0$.

Pre-step 2 Get the differential equation into the form

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0$$

where A, B, and C are polynomials.

To get the given differential equation (equation (13.1)) into the form desired, we multiply the equation by $4x^2$. That gives us the differential equation

$$4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + [4x^2 - 1]y = 0 \quad . \tag{13.2}$$

(Yes, we could have just multiplied by x^2, but getting rid of any fractions will simplify computation.)

Now for the basic method of Frobenius:

Step 1: *(a)* Start by assuming a solution of the form

$$y = y(x) = (x - x_0)^r \sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{13.3a}$$

where a_0 is an arbitrary constant. Since it is arbitrary, we can and will assume $a_0 \neq 0$ in the following computations.[1]

(b) Then simplify the formula for the following computations by bringing the $(x - x_0)^r$ factor into the summation,

$$y = y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+r} \quad . \tag{13.3b}$$

(c) And then compute the corresponding modified power series for y' and y'' from the assumed series for y by differentiating "term-by-term".

[1] Insisting that $a_0 \neq 0$ helps limit the possible values of r.

Since we've already decided $x_0 = 0$, we assume

$$y = y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r} \tag{13.4}$$

with $a_0 \neq 0$. Differentiating this term-by-term, we see that

$$\begin{aligned} y' &= \frac{d}{dx} \sum_{k=0}^{\infty} a_k x^{k+r} \\ &= \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^{k+r}\right] = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \quad , \end{aligned}$$

$$\begin{aligned} y'' &= \frac{d}{dx} \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \\ &= \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k (k+r) x^{k+r-1}\right] = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} \end{aligned}$$

Step 2: Plug these series for y, y', and y'' back into the differential equation, "multiply things out", and divide out the $(x - x_0)^r$ to get the left side of your equation in the form of the sum of a few power series.

Some Notes:

ii. Absorb any x's in A, B and C (of the differential equation) into the series.

iii. Dividing out the $(x - x_0)^r$ isn't necessary, but it simplifies the expressions slightly and reduces the chances of silly errors later.

iv. You may want to turn your paper sideways for more room!

Combining the above series formulas for y, y' and y'' with our differential equation (equation (13.2)), we get

$$\begin{aligned} 0 &= 4x^2 y'' + 4xy' + [4x^2 - 1]y \\ &= 4x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} + 4x \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \\ &\quad + [4x^2 - 1] \sum_{k=0}^{\infty} a_k x^{k+r} \\ &= 4x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} + 4x \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \\ &\quad + 4x^2 \sum_{k=0}^{\infty} a_k x^{k+r} - 1 \sum_{k=0}^{\infty} a_k x^{k+r} \end{aligned}$$

$$= \sum_{k=0}^{\infty} a_k 4(k+r)(k+r-1)x^{k+r} + \sum_{k=0}^{\infty} a_k 4(k+r)x^{k+r}$$
$$+ \sum_{k=0}^{\infty} a_k 4x^{k+2+r} + \sum_{k=0}^{\infty} a_k(-1)x^{k+r} \quad .$$

Dividing out the x^r from each term then yields

$$0 = \sum_{k=0}^{\infty} a_k 4(k+r)(k+r-1)x^{k} + \sum_{k=0}^{\infty} a_k 4(k+r)x^{k}$$
$$+ \sum_{k=0}^{\infty} a_k 4x^{k+2} + \sum_{k=0}^{\infty} a_k(-1)x^{k} \quad .$$

Step 3: For each series in your last equation, do a change of index so that each series looks like

$$\sum_{n=\text{something}}^{\infty} \big[\text{something not involving } x\big](x-x_0)^n \quad .$$

Be sure to appropriately adjust the lower limit in each series.

In all but the third series in the example, the "change of index" is trivial, $n = k$. In the third series, we will set $n = k+2$ (equivalently, $n-2 = k$). This means, in the third series, replacing k with $n-2$, and replacing $k = 0$ with $n = 0+2 = 2$:

$$0 = \underbrace{\sum_{k=0}^{\infty} a_k 4(k+r)(k+r-1)x^{k}}_{n=k} + \underbrace{\sum_{k=0}^{\infty} a_k 4(k+r)x^{k}}_{n=k}$$
$$+ \underbrace{\sum_{k=0}^{\infty} a_k 4x^{k+2}}_{n=k+2} + \underbrace{\sum_{k=0}^{\infty} a_k(-1)x^{k}}_{n=k}$$

$$= \sum_{n=0}^{\infty} a_n 4(n+r)(n+r-1)x^{n} + \sum_{n=0}^{\infty} a_n 4(n+r)x^{n}$$
$$+ \sum_{n=2}^{\infty} a_{n-2} 4x^{n} + \sum_{n=0}^{\infty} a_n(-1)x^{n} \quad .$$

Step 4: Convert the sum of series in your last equation into one big series. The first few terms will probably have to be written separately. Simplify what can be simplified.

Since one of the series in the last equation begins with $n = 2$, we need to separate out the terms corresponding to $n = 0$ and $n = 1$ in the other series *before combining series:*

$$
\begin{aligned}
0 \;=\; & \sum_{n=0}^{\infty} a_n 4(n+r)(n+r-1)x^n \;+\; \sum_{n=0}^{\infty} a_n 4(n+r)x^n \\
& +\; \sum_{n=2}^{\infty} a_{n-2} 4x^n \;+\; \sum_{n=0}^{\infty} a_n(-1)x^n \\
=\; & \Bigg[a_0 \underbrace{4(0+r)(0+r-1)}_{4r(r-1)} x^0 \;+\; a_1 \underbrace{4(1+r)(1+r-1)}_{4(1+r)r} x^1 \\
& \qquad +\; \sum_{n=2}^{\infty} a_n 4(n+r)(n+r-1)x^n \Bigg] \\
& +\; \Bigg[a_0 \underbrace{4(0+r)}_{4r} x^0 \;+\; a_1 \underbrace{4(1+r)}_{4(1+r)} x^1 \;+\; \sum_{n=2}^{\infty} a_n 4(n+r)x^n \Bigg] \\
& +\; \sum_{n=2}^{\infty} a_{n-2} 4x^n \;+\; \Bigg[-a_0 x^0 \;-\; a_1 x^1 \;+\; \sum_{n=2}^{\infty} a_n(-1)x^n \Bigg] \\
=\; & a_0 \big[\underbrace{4r(r-1) \;+\; 4r \;-\; 1}_{4r^2-4r+4r-1} \big] x^0 \;+\; a_1 \big[\underbrace{4(1+r)r \;+\; 4(1+r) \;-\; 1}_{4r+4r^2+4+4r-1} \big] x^1 \\
& +\; \sum_{n=2}^{\infty} \big[\underbrace{a_n 4(n+r)(n+r-1) + a_n 4(n+r) + a_{n-2}4 + a_n(-1)}_{a_n[4(n+r)(n+r-1)+4(n+r)-1]+4a_{n-2}} \big] x^n \\
=\; & a_0 \big[4r^2 - 1 \big] x^0 \;+\; a_1 \big[4r^2 + 8r + 3 \big] x^1 \\
& +\; \sum_{n=2}^{\infty} \Big[a_n \big[4(n+r)(n+r) - 1 \big] \;+\; 4a_{n-2} \Big] x^n \quad .
\end{aligned}
$$

So our differential equation reduces to

$$
a_0 \big[4r^2 - 1 \big] x^0 \;+\; a_1 \big[4r^2 + 8r + 3 \big] x^1 \;+\; \sum_{n=2}^{\infty} \Big[a_n \big[4(n+r)^2 - 1 \big] \;+\; 4a_{n-2} \Big] x^n \;=\; 0 \quad . \tag{13.5}
$$

Observe that the end result of this step will be an equation of the form

$$
\text{some big power series} \;=\; 0 \quad .
$$

This, in turn, tells us that each term that big power series must be 0.

Step 5: The first term in the last equation just derived will be of the form

$$
a_0 \big[\text{formula of } r \big] (x - x_0)^{\text{something}} \quad .
$$

But, remember, each term in that series must be 0. So we must have

$$a_0\big[\text{formula of } r\big] = 0 \quad .$$

Moreover, since $a_0 \neq 0$ (by assumption), the above must reduce to

$$\text{formula of } r = 0 \quad .$$

This is the *indicial equation* for r. It will always be a quadratic equation for r (i.e., of the form $\alpha r^2 + \beta r + \delta = 0$). Solve this equation for r. You will get two solutions (sometimes called either the *exponents* of the solution or the *exponents* of the singularity). Denote them by r_2 and r_1 with $r_2 \leq r_1$.[2]

In our example, the first term in the "big series" is the first term in equation (13.5),

$$a_0\big[4r^2 - 1\big]x^0 \quad .$$

Since this must be zero (and $a_0 \neq 0$ by assumption) the indicial equation is

$$4r^2 - 1 = 0 \quad .$$

Thus,

$$r = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2} \quad .$$

Following the convention given,

$$r_2 = -\frac{1}{2} \qquad \text{and} \qquad r_1 = \frac{1}{2} \quad .$$

Step 6: Using r_1, the *larger* r just found:

(a) Plug r_1 into the last series equation (and simplify, if possible). This will give you an equation of the form

$$\sum_{n=n_0}^{\infty} \big[n^{\text{th}} \text{ formula of } a_k\text{'s}\big](x - x_0)^n = 0 \quad .$$

Since each term must vanish, we have

$$n^{\text{th}} \text{ formula of } a_k\text{'s} = 0 \qquad \text{for} \quad n_0 \leq n \quad .$$

(b) Solve this for

$$a_{\text{highest index}} = \text{formula of } n \text{ and lower indexed } a_k\text{'s} \quad .$$

A few of these equations may need to be treated separately, but you will also obtain a relatively simple formula that holds for all indices above some fixed value. This formula is the *recursion formula* for computing each coefficient a_n from the previously computed coefficients.

[2] We are assuming r_1 and r_2 are real numbers. This is the standard situation. We'll briefly discuss the cases where r_1 and r_2 are complex numbers later.

(c) To simplify things just a little, do another change of indices so that the recursion formula just derived is rewritten as

$$a_k = \text{formula of } k \text{ and lower-indexed coefficients} \quad .$$

Letting $r = r_1 = 1/2$ *in equation (13.5) yields*

$$\begin{aligned} 0 &= a_0\big[4r^2 - 1\big]x^0 + a_1\big[4r^2 + 8r + 3\big]x^1 \\ &\quad + \sum_{n=2}^{\infty}\Big[a_n\big[4(n+r)^2 - 1\big] + 4a_{n-2}\Big]x^n \\ &= a_0\left[4\left(\frac{1}{2}\right)^2 - 1\right]x^0 + a_1\left[4\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right) + 3\right]x^1 \\ &\quad + \sum_{n=2}^{\infty}\left[a_n\left[4\left(n+\frac{1}{2}\right)^2 - 1\right] + 4a_{n-2}\right]x^n \\ &= a_0 0x^0 + a_1 8x^1 + \sum_{n=2}^{\infty}\left[a_n\left[4n^2 + 4n + 1 - 1\right] + 4a_{n-2}\right]x^n \quad . \end{aligned}$$

The first term vanishes (as it should since $r = 1/2$ *satisfies the indicial equation, which came from making the first term vanish). Doing a little more simple algebra, we see that, with* $r = 1/2$*, equation (13.5) reduces to*

$$0a_0x^0 + 8a_1x^1 + \sum_{n=2}^{\infty}4\left[n(n+1)a_n + a_{n-2}\right]x^n = 0 \quad . \tag{13.6}$$

From the above series, we must have

$$n(n+1)a_n + a_{n-2} = 0 \qquad \text{for} \quad n = 2, 3, 4 \ldots \quad .$$

Solving for a_n *leads to the recursion formula*

$$a_n = \frac{-1}{n(n+1)}a_{n-2} \qquad \text{for} \quad n = 2, 3, 4 \ldots \quad .$$

Using the trivial change of index, $k = n$*, this is*

$$a_k = \frac{-1}{k(k+1)}a_{k-2} \qquad \text{for} \quad k = 2, 3, 4 \ldots \quad . \tag{13.7}$$

Step 7: Use the recursion formula (and any corresponding formulas for the lower-order terms) to find all the a_k's in terms of a_0 and, possibly, one other a_m. Look for patterns!

From the first two terms in equation (13.6),

$$0a_0 = 0 \implies a_0 \text{ is arbitrary.}$$

$$8a_1 = 0 \implies a_1 = 0 \quad .$$

Using these values and the recursion formula (equation (13.7)) with $k = 2, 3, 4, \ldots$ *(and looking for patterns):*

$$a_2 = \frac{-1}{2(2+1)} a_{2-2} = \frac{-1}{2 \cdot 3} a_0 \quad ,$$

$$a_3 = \frac{-1}{3(3+1)} a_{3-2} = \frac{-1}{3 \cdot 4} a_1 = \frac{-1}{3 \cdot 4} \cdot 0 = 0 \quad ,$$

$$a_4 = \frac{-1}{4(4+1)} a_{4-2} = \frac{-1}{4 \cdot 5} a_2 = \frac{-1}{4 \cdot 5} \cdot \frac{-1}{2 \cdot 3} a_0 = \frac{(-1)^2}{5 \cdot 4 \cdot 3 \cdot 2} a_0 = \frac{(-1)^2}{5!} a_0 \quad ,$$

$$a_5 = \frac{-1}{5(5+1)} a_{5-2} = \frac{-1}{5 \cdot 6} \cdot 0 = 0 \quad ,$$

$$a_6 = \frac{-1}{6(6+1)} a_{6-2} = \frac{-1}{6 \cdot 7} a_4 = \frac{-1}{7 \cdot 6} \cdot \frac{(-1)^2}{5!} a_0 = \frac{(-1)^3}{7!} a_0 \quad ,$$

$$\vdots$$

The patterns should be obvious here:

$$a_k = 0 \qquad \text{for} \quad k = 1, 3, 5, 7, \ldots \quad ,$$

and

$$a_k = \frac{(-1)^{k/2}}{(k+1)!} a_0 \qquad \text{for} \quad k = 2, 4, 6, 8, \ldots \quad .$$

Using $k = 2m$ *, this be written more conveniently as*

$$a_{2m} = (-1)^m \frac{a_0}{(2m+1)!} \qquad \text{for} \quad m = 1, 2, 3, 4, \ldots \quad .$$

Moreover, this last equation reduces to the trivially true statement "$a_0 = a_0$*" if* $m = 0$ *. So, in fact, it gives all the even-indexed coefficients,*

$$a_{2m} = (-1)^m \frac{a_0}{(2m+1)!} \qquad \text{for} \quad m = 0, 1, 2, 3, 4, \ldots \quad .$$

Step 8: Using $r = r_1$ along with the formulas just derived for the coefficients, write out the resulting series for y . Try to simplify it and factor out the arbitrary constant(s).

Plugging $r = 1/2$ *and the formulas just derived for the* a_n*'s into the formula originally assumed for* y *(equation (13.4) on page 13–3), we get*

$$\begin{aligned}
y &= x^r \sum_{k=0}^{\infty} a_k x^k \\
&= x^r \left[\sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} a_k x^k + \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} a_k x^k \right] \\
&= x^{1/2} \left[\sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} 0 \cdot x^k + \sum_{m=0}^{\infty} (-1)^m \frac{a_0}{(2m+1)!} x^{2m} \right] \\
&= x^{1/2} \left[0 + a_0 \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!} x^{2m} \right] \quad .
\end{aligned}$$

So one solution to Bessel's equation of order $^1\!/_2$ *(equation (13.1) on page 13–1) is given by*

$$y = a_0 x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} \quad . \tag{13.8}$$

Step 9: If the indical equation had two distinct solutions, now repeat steps 6 through 8 with the smaller r, r_2. Sometimes (but not always) this will give you a second independent solution to the differential equation. Sometimes, also, the series formula derived in this mega-step will include the series formula already derived.

Letting $r = r_2 = -^1\!/_2$ *in equation (13.5) yields*

$$\begin{aligned}
0 &= a_0\big[4r^2 - 1\big]x^0 + a_1\big[4r^2 + 8r + 3\big]x^1 \\
&\quad + \sum_{n=2}^{\infty}\Big[a_n\big[4(n+r)^2 - 1\big] + 4a_{n-2}\Big]x^n \\
&= a_0\left[4\left(-\tfrac{1}{2}\right)^2 - 1\right]x^0 + a_1\left[4\left(-\tfrac{1}{2}\right)^2 + 8\left(-\tfrac{1}{2}\right) + 3\right]x^1 \\
&\quad + \sum_{n=2}^{\infty}\left[a_n\left[4\left(n - \tfrac{1}{2}\right)^2 - 1\right] + 4a_{n-2}\right]x^n \\
&= a_0 0 x^0 + a_1 0 x^1 + \sum_{n=2}^{\infty}\left[a_n\left[4n^2 - 4n + 1 - 1\right] + 4a_{n-2}\right]x^n
\end{aligned}$$

Then …

$\vdots$

yielding

$$y = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \quad . \tag{13.9}$$

Note that the second series term is the same series (slightly rewritten) as in equation (13.8) (since $x^{-1/2}x^{2m+1} = x^{1/2}x^{2m}$ *).*

?►Exercise 13.1: *"Fill in the dots" in the last statement. That is, do all the computations that were omitted.*

Step 10 If the last step yielded y as an arbitrary linear combination of two different series, then that is the general solution to the original differential equation. If the last step yielded y as just one arbitrary constant times a series, then the general solution to the original differential equation is the linear combination of the two series obtained at the end of steps 8 and 9. Either way, write down the general solution (using different symbols for the two different arbitrary constants!). If step 9 did not yield a new series solution, then at least write down the one solution previously derived, noting that a second solution is still needed for the general solution to the differential equation.

We are in luck. In the last step we obtained y as the linear combination of two different series (equation (13.9)). So

$$y = a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \quad ,$$

is the general solution to our original differential equation (equation (13.1) — Bessel's equation of order $1/2$).

Last Step See if you recognize the series as the series for some well-known function (you probably won't!).

Our luck continues! The two series are easily recognized as the series for the sine and the cosine functions:

$$\begin{aligned} y &= a_0 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} + a_1 x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \\ &= a_0 x^{-1/2} \cos(x) + a_1 x^{-1/2} \sin(x) \quad . \end{aligned}$$

So the general solution to Bessel's equation of order $1/2$ is

$$y = a_0 \frac{\cos(x)}{\sqrt{x}} + a_1 \frac{\sin(x)}{\sqrt{x}} \quad . \qquad (13.10)$$

13.2 Practical Advice and Possibly Helpful Comments

1. If you get something like

$$2a_1 = 0 \quad ,$$

then you know $a_1 = 0$. On the other hand, if you get something like

$$0a_1 = 0 \quad ,$$

then you have an equation that tells you nothing about a_1. This means that a_1 is an arbitrary constant (unless something else tells you otherwise).

2. If the recursion formula blows up at some point, then some of the coefficients must be zero. For example, if

$$a_n = \frac{3}{(n+2)(n-5)} a_{n-2} \quad ,$$

then, for $n = 5$,

$$a_5 = \frac{3}{(7)(0)} a_3 = \infty \cdot a_3 \quad ,$$

which can only make sense if $a_3 = 0$. Note also, that, unless otherwise indicated, a_5 here would be arbitrary. (Remember, the last equation is equivalent to $(7)(0)a_5 = 3a_3$.)

3. If you get a coefficient being zero, it is a good idea to check back using the recursion formula to see if any of the previous coefficients must also be zero, or if many of the following coefficients are zero. In some cases, you may find that an "infinite" series solution only contains a finite number of nonzero terms, in which case we have a "terminating series"; i.e., a solution which is simply a polynomial.

   On the other hand, obtaining $a_0 = 0$, contrary to our basic assumption that $a_0 \neq 0$, tells you that there is no series solution of the form assumed for the basic Frobenius method using that value of r.

4. It is possible to end up with a three term recursion formula, say,

$$a_n = \frac{1}{n^2+1}a_{n-1} + \frac{2}{3n(n+3)}a_{n-2} \quad .$$

   This, naturally, makes "finding patterns" rather difficult.

5. Keep in mind that, even if you find that "finding patterns" and describing them by "nice" formulas is beyond you, you can always use the recursion formulas to compute (or have a computer compute) as many terms as you wish of the series solutions.

6. The computations can become especially messy and confusing when $x_0 \neq 0$. In this case, simplify matters by using the substitutions

$$Y(X) = y(x) \qquad \text{with} \quad X = x - x_0 \quad .$$

   You can then easily verify that, under these substitutions,

$$Y'(X) = \frac{dY}{dX} = \frac{dy}{dx} = y'(x)$$

   and the differential equation

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0 \tag{13.11}$$

   becomes

$$A_1(X)\frac{d^2Y}{dX^2} + B_1(X)\frac{dY}{dX} + C_1(X)Y = 0 \tag{13.11$'$}$$

   with

$$A_1(X) = A(X+x_0) \quad , \quad B_1(X) = B(X+x_0) \quad \text{and} \quad C_1(X) = C(X+x_0) \quad .$$

   Use the method of Frobenious to find the modified power series solutions

$$Y(X) = X^r \sum_{k=0}^{\infty} a_k X^k$$

   for equation (13.11$'$). The corresponding solutions to the original differential equation, equation (13.11), are then given from this via the above substitution,

$$y(x) = Y(X) = X^r \sum_{k=0}^{\infty} a_k X^k = (x-x_0)^r \sum_{k=0}^{\infty} a_k (x-x_0)^k \quad .$$

13.3 Theory Behind the Method (Without Proofs)

Basic Theory and Ordinary Points

Let us consider any differential equation of the form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \tag{13.12}$$

where a, b and c are known functions. Remember that the general solution over some interval (α, β) to such a differential equation is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where y_1 and y_2 is a linearly independent pair of particular solutions (no arbitrary constants) over (α, β), and c_1 and c_2 are two arbitrary constants.

For describing theory (though rarely for simplifying computations), it helps to divide the differential equation by $a(x)$, obtaining the equivalent equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \tag{13.12$'$}$$

(with $P = {}^b/_a$ and $Q = {}^c/_a$). Now, for each point z_0 in the complex plane, we say that:

1. z_0 is an *ordinary point* for the differential equation if and only if both $P(z)$ and $Q(z)$ (viewed as functions of a complex variable) are analytic at z_0.[3]

2. z_0 is a *singular point* for the differential equation if and only if it is not an ordinary point for the equation.

Keep in mind that $P = {}^b/_a$ and $Q = {}^c/_a$. So if $a(z)$ vanishes at some point z_0 where either $b(z)$ or $c(z)$ is nonzero, then that point, z_0, will be a singular point for the differential equation.

The importance of whether a point is ordinary or singular is indicated in the following theorem.

Theorem 13.1 (Big Theorem on Solving DEs about Ordinary Points)
Let x_0 be an ordinary point (on the real line) for the differential equation

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \quad .$$

Then a general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

exists on some interval containing x_0, with y_1 and y_2 being analytic at x_0. These functions, y_1 and y_2, can be obtained by the basic method of Frobenius. Moreover:

1. *The indicial equation from the Frobenius method is $r(r-1) = 0$. (Hence $r_1 = 1$ and $r_2 = 0$.)*

[3] That is, $P(z)$ and $Q(z)$ can each be described as a power series about z_0 over some neighborhood of that point. (We are anticipating some stuff from our study of complex variables.)

and

2. *The series formulas for both* y_1 *and* y_2 *obtained by the method of Frobenius are valid at least on the interval* $(x_0 - R, x_0 + R)$ *where* R *is the distance from* x_0 *to the closest point on the complex plane at which either* P *or* Q *is not analytic.*

For the proof of this theorem, see a good intermediate or higher level text on differential equations.[4] We won't attempt it here. Our interest is simply that it assures us that someone has confirmed that the basic Frobenius method works when we attempt it using power series about ordinary points, and has even given us an idea as to the interval over which the series will converge. It even states that the series will be true power series.

Keep in mind that, while the theorem describes cases where the method of Frobenius works, it does not say that it is always the *best* method. For example, it assures you that you can find infinite series solutions to

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

when a, b and c are constants — but why bother? It would be much easier to simply use a method described in section A.3 of the appendix reviewing elementary ordinary differential equations.

In fact, whenever x_0 is an ordinary point for the differential equation, using the full Frobenius method is excessive. With a little thought, you'll realize that the above theorem says that you can skip the whole "indicial equation" part of the Frobenius method. Instead, just use the relevant parts of the method after assuming

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

with a_0 and a_1 being arbitrary constants.

More Advanced Theory

If you go back and check our example in the section 13.1, you will find that our series expansion was not done about an ordinary point for the differential equation. The point $x_0 = 0$ turns out to be a singular point for Bessel's equation of order $1/2$. Indeed, mathematical physics is full of situations where series expansions are attempted about singular points. Fortunately, most of these singular points are not "badly singular".

Time for more terminology:

Suppose z_0 is a singular point for

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \quad .$$

Divide through by $a(x)$ to get

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

(with $P = b/a$ and $Q = c/a$). We say that z_0 is a *regular singular point* for the differential equation if and only if

$$P(z) = \frac{p(z)}{z - z_0} \qquad \text{and} \qquad Q(z) = \frac{q(z)}{(z - z_0)^2}$$

[4] say, the text at `www.uah.edu/math/howell/DEtext` (chap. 31–35).

where $p(z)$ and $q(z)$ are analytic functions at z_0. In other words, a singular point z_0 is a regular singular point for the differential equation if and only if the differential equation can be rewritten as

$$\frac{d^2y}{dx^2} + \frac{p(x)}{x - z_0}\frac{dy}{dx} + \frac{q(x)}{(x - z_0)^2}y = 0$$

or, equivalently, as

$$(x - z_0)^2\frac{d^2y}{dx^2} + (x - z_0)p(x)\frac{dy}{dx} + q(x)y = 0$$

for some functions $p(x)$ and $q(x)$ analytic at x_0.

If you go back and check, you will find that $x_0 = 0$ is a regular singular point for Bessel's equation of order $^1\!/_2$. As the next theorem shows, regular singular points are not that badly singular.

Theorem 13.2 (The Big Theorem on the Frobenius Method)
Let x_0 be a regular singular point (on the real line) for

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \quad .$$

Then the indicial equation arising in the basic method of Frobenius exists and is a quadratic equation with two solutions r_1 and r_2 (which may be one solution, repeated).[5] *If r_2 and r_1 are real, assume $r_2 \leq r_1$. Then:*

1. *The basic method of Frobenius will yield at least one solution of the form*

$$y_1(x) = (x - x_0)^{r_1}\sum_{k=0}^{\infty} a_k(x - x_0)^k$$

   where a_0 is the one and only arbitrary constant.

2. *If $r_1 - r_2$ is* not *an integer, then the basic method of Frobenius will yield a second independent solution of the form*

$$y_2(x) = (x - x_0)^{r_2}\sum_{k=0}^{\infty} a_k(x - x_0)^k$$

   where a_0 is an arbitrary constant.

3. *If $r_1 - r_2 = N$ is* positive *integer, then the method of Frobenius might yield a second independent solution of the form*

$$y_2(x) = (x - x_0)^{r_2}\sum_{k=0}^{\infty} a_k(x - x_0)^k$$

   where a_0 is an arbitrary constant. If it doesn't, then a second independent solution exists of the form

$$y_2(x) = y_1(x)\ln|x - x_0| + (x - x_0)^{r_2}\sum_{k=0}^{\infty} b_k(x - x_0)^k$$

[5] See the first note after this theorem regarding the use of the basic method when the coefficients are not rational functions.

or, equivalently,

$$y_2(x) = y_1(x)\left[\ln|x - x_0| + (x - x_0)^{-N}\sum_{k=0}^{\infty} c_k(x - x_0)^k\right]$$

where b_0 and c_0 are nonzero constants.

4. *If $r_1 = r_2$, then there is a second solution of the form*

$$y_2(x) = y_1(x)\ln|x - x_0| + (x - x_0)^{1+r_1}\sum_{k=0}^{\infty} b_k(x - x_0)^k$$

or, equivalently,

$$y_2(x) = y_1(x)\left[\ln|x - x_0| + (x - x_0)^1\sum_{k=0}^{\infty} c_k(x - x_0)^k\right] \quad .$$

In this case, b_0 and c_0 might be zero.

Moreover, if we let R be the distance between x_0 and the nearest singular point (other than x_0) in the complex plane (with $R = \infty$ if x_0 is the only singular point), then the series solutions described above converge at least on the intervals $(x_0 - R, x_0)$ and $(x_0, x_0 + R)$.[6]

Again, for the proof, go to some decent intermediate differential equation text.[7]
Some notes regarding this theorem:

1. In first describing the method of Frobenious to solve

$$a(x)\frac{d^2y}{ds^2} + b(x)\frac{dy}{dx} + c(x)y = 0$$

we assumed a, b and c were rational functions so that we could get the differential equation into the form

$$A(x)\frac{d^2y}{ds^2} + B(x)\frac{dy}{dx} + C(x)y = 0$$

where $A(x)$, $B(x)$ and $C(x)$ are polynomials (Pre-step 2). More generally, you want to get the equation into the above form where $A(x)$, $B(x)$ and $C(x)$ are analytic functions about x_0, and then express these functions as power series about x_0. After all, a power series is just a really big polynomial, isn't it?

Fortunately, it seems safe to say that the use of the Frobenius method when $a(x)$, $b(x)$ and $c(x)$ are not rational functions is rare.

2. In case 3 (where $r_1 - r_2$ is a positive integer), if the method of Frobenius does yield a second solution, then you will end up rederiving the first one again as you derive the second.

3. The formulas for $y_2(x)$ involving $\ln|x - x_0|$ can be derived via the method of "reduction of order" for finding second solutions to second order differential equations (see your old DE text, not AW&H). This is of theoretical, not practical, significance. It would usually

[6] The power series parts will converge on the interval $(x_0 - R, x_0 + R)$, indeed, on the disk of radius R in the complex plane about x_0. However, the $(x - x_0)^r$ will behave badly at $x = x_0$ if r is not a positive real number.
[7] e.g., the one at `www.uah.edu/math/howell/DEtext` (chap. 31–35).

be a mistake to attempt to actually compute these series using reduction of order. Instead, just plug these series formulas into the differential equation and find recursion formulas for the coefficients. At some point, you will probably have to write out the series formula for $y_1(x)$ and multiply two series together. It will not be pretty.

4. In practice, one major application of many of these series to determine the general behavior of solutions near x_0. For example, the logarithmic singularity in some of the solutions may allow us to reject them in certain applications.

5. Again, keep in mind that simpler methods might be available. For instance, you can use the Frobenius method on Cauchy-Euler equations (see section A.3 of the appendix on ordinary differential equations), but that would be silly.

6. Obvious variations of the Frobenius method can be applied to first-order and third-order (and fourth-order, etc.) differential equations.

7. As long as we allow complex numbers, we will have only a little difficulty with expressions of the form $(x - x_0)^r$ when $x - x_0 < 0$ and, say, $r = 1/2$. When complex numbers are being avoided, it may be best to write your final answer using

$$|x - x_0|^r \qquad \text{instead of} \qquad (x - x_0)^r \quad .$$

   But *we* will be allowing complex numbers. In fact, x and x_0 may be complex in future work, in which case we will see that the above mentioned replacement of $(x - x_0)^r$ with $|x - x_0|^r$ would be BAD.

Behavior of Solutions Near Regular Singular Points

Power Series Near Their Centers

Assume f is given by a power series about some point x_0,

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 \neq 0 \quad .$$

Observe that

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + a_4(x - x_0)^4 \cdots \\ &= a_0 + (x - x_0)\left[a_1 + a_2(x - x_0) + a_3(x - x_0)^2 + a_4(x - x_0)^3 + \cdots\right] \\ &= a_0 + (x - x_0)\sum_{k=0}^{\infty} a_{k+1}(x - x_0)^k \quad . \end{aligned}$$

Now, if $x \approx x_0$, then $x - x_0$ is small, and

$$f(x) = a_0 + \text{something small} \approx a_0 \quad .$$

Behavior of the Solutions

Suppose we've used the method of Frobenius to solve

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0$$

about a regular singular point x_0. Let r_1 and r_2 be the solutions to the indicial equation (we will assume them to be real with $r_1 \geq r_2$), and let $y_1(x)$ and $y_2(x)$ be the solutions as described in our big theorem on the Frobenius method (theorem 13.2).

Let's look at those solutions when $x \approx x_0$.

No matter what, we know that

$$y_1(x) = (x - x_0)^{r_1} \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 \neq 0 \quad .$$

So, for $x \approx x_0$

$$y_1(x) \approx (x - x_0)^{r_1} a_0 \quad .$$

In particular, then,

$$\lim_{x \to x_0} |y_1(x)| = \lim_{x \to x_0} |x - x_0|^{r_1} |a_0| = \begin{cases} 0 & \text{if} \quad r_1 > 0 \\ |a_0| & \text{if} \quad r_1 = 0 \\ +\infty & \text{if} \quad r_1 < 0 \end{cases} \quad .$$

Now assume $r_2 \neq r_1$. "With luck", our second solution is similar to the first,

$$y_2(x) = (x - x_0)^{r_2} \sum_{k=0}^{\infty} a_k (x - x_0)^k \qquad \text{with} \quad a_0 \neq 0 \quad .$$

So, for $x \approx x_0$,

$$y_2(x) \approx a_0 (x - x_0)^{r_2} \quad .$$

Still, it is possible that our second solution is of the form

$$y_2(x) = y_1(x) \left[\ln|x - x_0| + (x - x_0)^{-N} \sum_{k=0}^{\infty} c_k (x - x_0)^k \right]$$

where $N = r_1 - r_2$ is a positive integer and $c_0 \neq 0$. Then

$$\begin{aligned} y_2(x) &\approx a_0 (x - x_0)^{r_1} \left[\ln|x - x_0| + (x - x_0)^{-N} c_0 \right] \\ &= a_0 (x - x_0)^{r_1 - N} \left[(x - x_0)^N \ln|x - x_0| + c_0 \right] \quad . \end{aligned}$$

But, $r_1 - N = r_2$ and, as you can easily verify for yourself,

$$(x - x_0)^N \ln|x - x_0| \approx 0 \qquad \text{when} \quad x \approx x_0 \quad .$$

So the last approximation above for $y_2(x)$ reduces to

$$y_2(x) \approx a_0 (x - x_0)^{r_2} [0 + c_0] \quad .$$

Thus, no matter what, if $r_1 > r_2$ and $x \approx x_0$, then

$$y_2(x) \approx A_0(x - x_0)^{r_2}$$

for some (nonzero) constant A_0. In particular, then,

$$\lim_{x \to x_0} |y_2(x)| \;=\; \lim_{x \to x_0} |x - x_0|^{r_2} |A_0| \;=\; \begin{cases} 0 & \text{if} \quad r_2 > 0 \\ |A_0| & \text{if} \quad r_2 = 0 \\ +\infty & \text{if} \quad r_2 < 0 \end{cases} \quad .$$

Finally, if $r_1 = r_2$, then the second solution can be written as

$$y_2(x) \;=\; y_1(x)\left[\ln|x - x_0| \;+\; (x - x_0)^1 \sum_{k=0}^{\infty} c_k (x - x_0)^k \right] \quad .$$

Since $\ln|x - x_0|$ "blows up" as $x \to 0$ and the other terms in the brackets go to zero as $x \to 0$, we have, for $x \approx x_0$,

$$y_2(x) \;\approx\; a_0(x - x_0)^{r_1} [\ln|x - x_0| \;+\; 0] \;=\; a_0(x - x_0)^{r_1} \ln|x - x_0| \quad .$$

Using L'Hôpital's rule, you can then easily show that

$$\lim_{x \to x_0} |y_2(x)| \;=\; \lim_{x \to x_0} |a_0|\, |x - x_0|^{r_1}\, |\ln|x - x_0|| \;=\; \begin{cases} 0 & \text{if} \quad r_1 > 0 \\ +\infty & \text{if} \quad r_1 \leq 0 \end{cases} \quad .$$

!► ***Example 13.1:*** *Suppose we are only interested in those solutions to Bessel's equation of order* $^1\!/_2$,

$$\frac{d^2y}{dx^2} \;+\; \frac{1}{x}\frac{dy}{dx} \;+\; \left[1 - \frac{1}{4x^2}\right] y \;=\; 0 \quad ,$$

that do not "blow up" at $x_0 = 0$*; that is, we only want those solutions satisfying*

$$\lim_{x \to 0} y(x) \;=\; \textit{a finite value} \quad .$$

This is the one equation we've solved by the Frobenius method, and we obtained the exponents

$$r_1 \;=\; \frac{1}{2} \qquad \textit{and} \qquad r_2 \;=\; -\frac{1}{2} \quad .$$

Hence, for some nonzero constants a_0 *and* A_0*, the two solutions* y_1 *and* y_2 *obtained by the Frobenius method satisfy*

$$\lim_{x \to 0} y_1(x) \;=\; \lim_{x \to 0} a_0 x^{1/2} \;=\; 0$$

and

$$\lim_{x \to 0} y_2(x) \;=\; \lim_{x \to 0} A_0 x^{-1/2} \;=\; \pm\infty \quad .$$

And, thus, if we only need those solutions that do not "blow up" at $x_0 = 0$*, then we only need to find the "first" solution given by the Frobenius method. There is no need to do all that work to obtain the solutions corresponding to* r_2*. What's more, we also know that all the solutions of interest actually go to zero at* $x_0 = 0$*.*

More on the Indicial Equation and Complex Solutions

Again, assume x_0 is a regular singular point for

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \quad ,$$

and let p and q be the analytic functions given by

$$\frac{b(x)}{a(x)} = \frac{p(x)}{x - x_0} \qquad \text{and} \qquad \frac{c(x)}{a(x)} = \frac{q(x)}{(x - x_0)^2} \quad ,$$

It's not hard to show that the indicial equation is actually given by

$$r(r-1) + p(x_0)r + q(x_0) = 0 \quad .$$

Knowing this, and solving for the r's early can save a little work in finding the series solutions, especially if you can make use of the big theorem on the Frobenius method (Theorem 13.2).

In practice, the functions a, b, and c in the differential equation are always real valued, and you expect to be able to obtain real-valued solutions to the differential equation. And in almost all (if not all) examples of the Frobenius method in textbooks, the roots of the indicial equation turn out to be real. In fact, few of the references I've had time to check even mention the possibility of the r's being complex, and the comments that I did find appear to be last minute changes because someone said "What about complex roots?"[8]

Still, even if we require a, b, and c to be real-valued functions, it is quite possible to construct a differential equation whose indicial equation yields nontrivial complex roots $r = \alpha \pm i\beta$. The result of using the Frobenius method on this differential equation will be that described in case 2 of Theorem 13.2 because the difference between the r's would not be an integer. In this case it would not matter which r you refer to as r_1. Moreover, if a, b, and c are real-valued functions, then you can easily show that, if y_1 is a complex-valued solution to the differential equation, then so is its complex conjugate $y_2 = y_1{}^*$ and the two form a linearly independent pair. Hence an arbitrary linear combination of y_1 and its conjugate will serve as a general solution to the differential equation. It then follows that the real and imaginary parts of y_1 are also a linearly independent pair of solutions. Hence, a general solution in terms of real-valued functions can also be given by

$$y(x) = c_1 \operatorname{Re}[y_1(x)] + c_2 \operatorname{Im}[y_2(x)] \quad .$$

What all this means is that, in theory, if r is complex and your first series solution obtained via Frobenius has complex terms, then you need not find the other series solution. Just use the real and imaginary parts of the series solution you already have. Just how practical this is depends on how easy it is to extract those real and imaginary parts.

If you want, you can play with an equation whose indical equation has complex roots by doing the next exercise.

?►*Exercise 13.2 (OPTIONAL):* *Consider the Cauchy-Euler equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 1 = 0 \quad .$$

[8] In one case the changes were rather poorly considered.

a: *Solve it using the methods described for Cauchy-Euler equations in the appendix on elementary ordinary differential equations. (The answer will be*

$$y(x) \;=\; c_1 \cos(\ln|x|) \;+\; c_2 \sin(\ln|x|) \quad .$$

To get this, you may have to use the fact that $x^\alpha = e^{\alpha \ln|x|}$ *for* $x > 0$ *.)*

b: *Solve it using the method of Frobenius. (Warning: This may get messy — I have not tried it yet.)*

Additional Exercises

These problems were all stolen from chapters 31–35 of the online text for MA238. That text can be found at `www.uah.edu/math/howell/DEtext`.

13.3. *Use the basic method of Frobenius to find modified power series solutions about $x_0 = 0$ to each of the following. In particular:*

i *Find and solve the corresponding indicial equation for the equation's exponents r_1 and r_2.*

ii *Find the recursion formula corresponding to each exponent.*

iii *Find and explicitly write out at least the first four nonzero terms of all series solutions about $x_0 = 0$ that can be found by the basic Frobenius method (if a series terminates, find all the nonzero terms).*

iv *Try to find a general formula for all the coefficients in each series.*

v *When a second particular solution cannot be found by the basic method, give a reason that second solution cannot be found.*

a. $(1+x^2)y'' - 2y = 0 \quad \text{with} \quad x_0 = 0$

b. $x^2y'' - 2xy' + \left(x^2+2\right)y = 0$

c. $4x^2y'' + (1-4x)y = 0$

d. $x^2y'' + xy' + (4x-4)y = 0$

e. $\left(x^2-9x^4\right)y'' - 6xy' + 10y = 0$

f. $x^2y'' - xy' + \dfrac{1}{1-x}y = 0$

g. $2x^2y'' + \left(5x-2x^3\right)y' + (1-x^2)y = 0$

h. $x^2y'' - \left(5x+2x^2\right)y' + (9+4x)y = 0$

i. $\left(3x^2-3x^3\right)y'' - \left(4x+5x^2\right)y' + 2y = 0$

j. $x^2y'' - \left(x+x^2\right)y' + 4xy = 0$

k. $4x^2y'' + 8x^2y' + y = 0$

l. $x^2y'' + \left(x-x^4\right)y' + 3x^3y = 0$

m. $\left(9x^2+9x^3\right)y'' + \left(9x+27x^2\right)y' + (8x-1)y = 0$

13.4. A Hermite equation *is any differential equation that can be written as*

$$y'' - 2xy' + \lambda y = 0$$

where λ is a constant.

a. *Derive the general recursion formula (in terms of λ) for the power series solutions about $x_0 = 0$ to the above Hermite equation.*

b. *Over what interval are these power series solutions guaranteed to be valid according to our theorems?*

c. *Using your recursion formula, show that when $\lambda = 2m$ for a nonnegative integer m , then the resulting series solution reduces to*

$$y_m(x) = a_0 y_{m,1}(x) + a_1 y_{m,2}(x)$$

where one of the $y_{m,j}$'s is a polynomial of degree m and the other is a power series (but not a polynomial). (The polynomials, multiplied by suitable constants, are called the Hermite polynomials.)

d. *Find those polynomial solutions when*

i. $m = 0$ **ii.** $m = 1$ **iii.** $m = 2$ **iv.** $m = 3$

v. $m = 4$ **vi.** $m = 5$

13.5. *For any constant λ , the* Legendre equation[9] with parameter λ *is*

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \quad .$$

Do the following:

a. *Derive the general recursion formula (in terms of λ) for the general power series solution $y_\lambda(x) = \sum_{k=0}^{\infty} a_k x^k$ to the above Legendre equation.*

b. *Using the recursion formula just found, derive the fact that the general power series solution y_λ can be written as*

$$y_\lambda(x) = a_0 y_{\lambda,E}(x) + a_1 y_{\lambda,O}(x)$$

where $y_{\lambda,E}$ and $y_{\lambda,O}$ are, respectively, even- and odd-termed series

$$y_{\lambda,E}(x) = \sum_{\substack{k=0 \\ k \text{ is even}}}^{\infty} c_k x^k \qquad \text{and} \qquad y_{\lambda,O}(x) = \sum_{\substack{k=0 \\ k \text{ is odd}}}^{\infty} c_k x^k$$

with $c_0 = 1$, $c_1 = 1$ and the other c_k's determined from c_0 or c_1 via the recursion formula.

c. *Now assume m is an nonnegative integer.*

i. *Find the one value λ_m for λ such that the above-found recursion formula yields $a_{m+2} = 0$.*

ii. *Using this, deduce that if and only if $\lambda = \lambda_m$, then exactly one of the two power series $y_{\lambda,E}(x)$ or $y_{\lambda,O}(x)$ reduces to an even or odd m^{th} degree polynomial p_m , with*

$$p_m(x) = \begin{cases} y_{\lambda,E}(x) & \text{if } m \text{ is even} \\ y_{\lambda,O}(x) & \text{if } m \text{ is odd} \end{cases} \quad .$$

[9] The Legendre equations arise in problems involving three-dimensional spherical objects (such as the Earth).

d. Based on the above, find the following:

i. λ_0 , $p_0(x)$, and $y_{0,O}(x)$ **ii.** λ_1 , $p_1(x)$, and $y_{1,E}(x)(x)$

iii. λ_2 and $p_2(x)$ **iv.** λ_3 and $p_3(x)$

v. λ_4 and $p_4(x)$ **vi.** λ_5 and $p_5(x)$

e. Now let λ be any constant (not necessarily λ_m).

i. What is the largest interval over which these power series solutions to the Legendre equation are guaranteed to be valid according to our theorems?

ii. Use the recursion formula along with the ratio test to confirm that the radius of convergence for both $y_{\lambda,E}(x)$ and for $y_{\lambda,O}(x)$ is 1 , provided the series does not terminate as polynomials.

f. Again, let λ be any constant (not necessarily λ_m) and let y_λ be any solution on $(-1, 1)$ to the Legendre equation with parameter λ . Using the series $y_{\lambda,E}(x)$ and $y_{\lambda,E}(x)$ from above, we know that there are constants a_0 and a_1 such that

$$\begin{aligned} y_\lambda(x) &= a_0 y_{\lambda,E}(x) + a_1 y_{\lambda,O}(x) \\ &= a_0 \sum_{\substack{k=0 \\ k \text{ is even}}}^{\infty} c_k x^k + a_1 \sum_{\substack{k=0 \\ k \text{ is odd}}}^{\infty} c_k x^k = \sum_{k=0}^{\infty} a_k x^k \end{aligned}$$

where

$$a_k = \begin{cases} a_0 c_k & \text{if } k \text{ is even} \\ a_1 c_k & \text{if } k \text{ is odd} \end{cases} .$$

Using this and the work already done in this exercise, verify each of the following:

i. If $\lambda = m(m+1)$ for some nonnegative integer m , then the Legendre equation with parameter λ has polynomial solutions, and they are all constant multiples of $p_m(x)$.

ii. If $\lambda \neq m(m+1)$ for every nonnegative integer m , then y_λ is not a polynomial.

iii. If y_λ is not a polynomial, then it is given by a power series about $x_0 = 0$ with a radius of convergence of exactly 1 .

13.6. The Chebyshev equation with parameter λ is given by

$$(1 - x^2)y'' - xy' + \lambda y = 0 .$$

The parameter λ may be any constant.

The solutions to this equation can be analyzed in a manner analogous to the way we analyzed solutions to the Legendre equation in the last exercise set. Do so, in particular:

a. Derive the general recursion formula (in terms of λ) for the general power series solution $y_\lambda(x) = \sum_{k=0}^{\infty} a_k x^k$ to the above Chebyshev equation.

b. Using the recursion formula just found, derive the fact that the general power series solution y_λ can be written as

$$y_\lambda(x) = a_0 y_{\lambda,E}(x) + a_1 y_{\lambda,O}(x)$$

where $y_{\lambda,E}$ and $y_{\lambda,O}$ are, respectively, even- and odd-termed series

$$y_{\lambda,E}(x) = \sum_{\substack{k=0 \\ k \text{ is even}}}^{\infty} c_k x^k \qquad \text{and} \qquad y_{\lambda,O}(x) = \sum_{\substack{k=0 \\ k \text{ is odd}}}^{\infty} c_k x^k$$

with $c_0 = 1$, $c_1 = 1$ and the other c_k's determined from c_0 or c_1 via the recursion formula.

c. *Now assume m is an nonnegative integer.*

i. *Find the one value λ_m for λ such that the above-found recursion formula yields $a_{m+2} = 0$.*

ii. *Using this, deduce that if and only if $\lambda = \lambda_m$, then exactly one of the two power series $y_{\lambda,E}(x)$ or $y_{\lambda,O}(x)$ reduces to an even or odd m^{th} degree polynomial p_m, with*

$$p_m(x) = \begin{cases} y_{\lambda,E}(x) & \text{if } m \text{ is even} \\ y_{\lambda,O}(x) & \text{if } m \text{ is odd} \end{cases} .$$

d. *Now, find the following:*

i. λ_0, *and* $p_0(x)$ **ii.** λ_1 *and* $p_1(x)$ **iii.** λ_2 *and* $p_2(x)$

iv. λ_3 *and* $p_3(x)$ **v.** λ_4 *and* $p_4(x)$ **vi.** λ_5 *and* $p_5(x)$

e. *Now let λ be any constant (not necessarily λ_m).*

i. *What is the largest interval over which these power series solutions to the Chebyshev equation are guaranteed to be valid according to our theorems?*

ii. *Use the recursion formula along with the ratio test to show that the radius of convergence for both $y_{\lambda,E}(x)$ and for $y_{\lambda,O}(x)$ is 1, provided the series does not terminate as polynomials.*

f. *Verify each of the following:*

i. *If $\lambda = m^2$ for some nonnegative integer m, then the Chebyshev equation with parameter λ has polynomial solutions, all of which are all constant multiples of $p_m(x)$.*

ii. *If $\lambda \neq m^2$ for every nonnegative integer m, then the Chebyshev equation with parameter λ has no polynomial solutions (other than $y = 0$).*

iii. *If y_λ is a nonpolynomial solution to a Chebyshev equation on $(-1, 1)$, then it is given by a power series about $x_0 = 0$ with a radius of convergence of exactly 1.*

13.7. *Identify all of the singular points for each of the following differential equations, and determine which are regular singular points, and which are irregular singular points. Also, find the Frobenius radius of convergence R for the given differential equation about the given x_0.*

a. $x^2 y'' + \dfrac{x}{x-2} y' + \dfrac{2}{x+2} y = 0 \quad , \quad x_0 = 0$

b. $x^3 y'' + x^2 y' + y = 0 \quad , \quad x_0 = 2$

c. $\left(x^3 - x^4\right) y'' + (3x - 1) y' + 827 y = 0 \quad , \quad x_0 = 1$

d. $y'' + \dfrac{1}{x-3}y' + \dfrac{1}{x-4}y = 0 \quad , \quad x_0 = 3$

e. $y'' + \dfrac{1}{(x-3)^2}y' + \dfrac{1}{(x-4)^2}y = 0 \quad , \quad x_0 = 4$

f. $y'' + \left(\dfrac{1}{x} - \dfrac{1}{3}\right)y' + \left(\dfrac{1}{x} - \dfrac{1}{4}\right)y = 0 \quad , \quad x_0 = 0$

g. $\left(4x^2 - 1\right)y'' + \left(4 - \dfrac{2}{x}\right)y' + \dfrac{1-x^2}{1+x^2}y = 0 \quad , \quad x_0 = 0$

h. $\left(4 + x^2\right)^2 y'' + y = 0 \quad , \quad x_0 = 0$

13.8. *For each of the following, verify that the given x_0 is a regular singular point for the given differential equation, and then use the basic method of Frobenius to find modified power series solutions to that differential equation. In particular:*

i *Find and solve the corresponding indicial equation for the equation's exponents r_1 and r_2.*

ii *Find the recursion formula corresponding to each exponent.*

iii *Find and explicitly write out at least the first four nonzero terms of all series solutions about $x_0 = 0$ that can be found by the basic Frobenius method (if a series terminates, find all the nonzero terms).*

iv *Try to find a general formula for all the coefficients in each series.*

v *When a second particular solution cannot be found by the basic method, give a reason that second solution cannot be found.*

a. $(x-3)y'' + (x-3)y' + y = 0 \quad , \quad x_0 = 3$

b. $y'' + \dfrac{2}{x+2}y' + y = 0 \quad , \quad x_0 = -2$

c. $4y'' + \dfrac{4x-3}{(x-1)^2}y = 0 \quad , \quad x_0 = 1$

d. $(x-3)^2 y'' + \left(x^2 - 3x\right)y' - 3y = 0 \quad , \quad x_0 = 3$