21

Special Functions

The fact that the radial equation in the "heat flow on a disk" problem leads to the Bessel equation suggests that we should spend some time getting to know the solutions to the Bessel equation more intimately. Surprisingly, these solutions are called Bessel functions, and they are members of a large class of functions known as "special functions". Typically, what makes these functions 'special' is that they are associated with solutions to Sturm-Liouville problems, and their study involves a lot of exciting analysis and weird formulas.[1] Because of time, we will limit most of our study of special functions to Bessel functions. However, the analysis we'll carry out is analogous to that done for other special functions (such as Hankle functions, Legendre and Hermite polynomials, and Mathieu functions).

We will return to actually solving the radial boundary-value problem we ended with in the last chapter. First, though, for no apparent reason, we will discuss something called the Gamma function.

21.1 The Gamma Function

Frankly, the "Gamma function" is not of much interest in itself (unless you like weird functions), but it does figure in the formulas of many functions that are of interest in applications.

Basic Definition and Identities

Let z be any complex number with positive real part, $\text{Re}[z] > 0$. The *Gamma function* Γ is the complex-valued function on the right half of the complex plane given by

$$\Gamma(z) = \int_{t=0}^{\infty} e^{-t} t^{z-1} \, dt \qquad \text{for} \quad z \quad \text{with} \quad \text{Re}[z] > 0 \quad .$$

Equivalently, of course,

$$\Gamma(x + iy) = \int_{t=0}^{\infty} e^{-t} t^{x+iy-1} \, dt \qquad \text{for} \quad x > 0 \quad .$$

[1] Technically, one could consider the sine and cosine functions as 'special functions' if we didn't already know everything about them.

The $x > 0$ requirement ensures that the integral is finite.[2] Note that, for any positive value x,

$$\Gamma(x) = \int_{t=0}^{\infty} \underbrace{e^{-t}t^{x-1}}_{>0}\, dt > 0 \quad ,$$

and that

$$\begin{aligned}\lim_{x\to 0}\Gamma(x) = \lim_{x\to 0}\int_{t=0}^{\infty} e^{-t}t^{x-1}\, dt &> \lim_{x\to 0}\int_{t=0}^{1} e^{-t}t^{x-1}\, dt \\ &> \lim_{x\to 0}\int_{t=0}^{1} e^{-1}t^{x-1}\, dt = \lim_{x\to 0} e^{-1}\frac{t^x}{x} = \infty \quad .\end{aligned}$$

(Oddly enough, though, we will see that $\lim_{x\to 0}\Gamma(x+iy) \neq \infty$ if $y \neq 0$. This is not immediately obvious from the basic formula.)

Unfortunately, the integral for the Gamma function can be computed exactly for only a few values of z. For most values of z, the integral for $\Gamma(z)$ can only be approximated. Fortunately,

1. Γ is considered to be a "well-known" function. Most modern mathematical packages (Maple, Mathematica, etc.) contain the Gamma functions as one of their "standard" functions. (And the computer-phobic can find tables for computing $\Gamma(z)$ in old texts.)

2. Those few values of z for which $\Gamma(z)$ can be exactly computed are the values most commonly of interest

For example, if $x = 1$, $\Gamma(x)$ is easily computed:

$$\Gamma(1) = \int_{t=0}^{\infty} e^{-t}t^{1-1}\, dt = \int_{t=0}^{\infty} e^{-t}\, dt = 1 \quad .$$

If $x = 1/2$, then $\Gamma(x)$ can be computed using the substitution $s = t^{1/2}$ and a "well-known" integral:

$$\begin{aligned}\Gamma\left(\tfrac{1}{2}\right) = \int_{t=0}^{\infty} e^{-t}t^{\frac{1}{2}-1}\, dt &= \int_{t=0}^{\infty} e^{-t}t^{-\frac{1}{2}}\, dt \\ &= 2\int_{s=0}^{\infty} e^{-s^2}\, ds = \int_{s=-\infty}^{\infty} e^{-s^2}\, ds = \sqrt{\pi} \quad .\end{aligned}$$

?►Exercise 21.1: *Fill in any details of the above computations.*

Thus

$$\Gamma(1) = 1 \qquad \text{and} \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \quad . \tag{21.1}$$

On the other hand,

$$\begin{aligned}\Gamma(1+i) = \int_{t=0}^{\infty} e^{-t}t^{1+i-1}\, dt &= \int_{t=0}^{\infty} e^{-t}t^{i}\, dt \\ &= \int_{t=0}^{\infty} e^{-t}e^{i\ln t}\, dt = \int_{t=0}^{\infty} e^{-t}\left[\cos(\ln t) + i\sin(\ln t)\right] dt \quad ,\end{aligned}$$

which, to the best of my knowledge, is not an integral anyone has managed to evaluate exactly.

[2] It also keeps the t^{z-1} single valued since we naturally take t^z to be its principle value, and that is well defined if we limit ourselves to the right half of the complex plane.

The Big Identity for $\Gamma(z)$, and Factorials

Let z be any complex number with positive real part. Using integration by parts:

$$\begin{aligned}
\Gamma(z+1) &= \int_{t=0}^{\infty} e^{-t} t^{z+1-1}\, dt \\
&= \int_{t=0}^{\infty} \underbrace{t^z}_{u} \underbrace{e^{-t}\, dt}_{dv} \\
&= -t^z e^{-t}\Big|_{t=0}^{\infty} - \int_{t=0}^{\infty} \left(zt^{z-1}\right)\left(-e^{-t}\right)\, dt \\
&= 0 - 0 + z \underbrace{\int_{t=0}^{\infty} e^{-t} t^{z-1}\, dt}_{\Gamma(z)}
\end{aligned}$$

This gives us the most significant identity for the Gamma function:

$$\Gamma(z+1) = z\Gamma(z) \qquad \text{for} \quad \text{Re}[z] > 0 \quad . \tag{21.2a}$$

Equivalently,

$$\Gamma(z) = (z-1)\Gamma(z-1) \qquad \text{for} \quad \text{Re}[z] > 1 \quad . \tag{21.2b}$$

This, along with the fact that $\Gamma(1) = 1$, yields

$$\begin{aligned}
\Gamma(2) &= 1 \cdot \Gamma(1) = 1 \quad , \\
\Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1 \quad , \\
\Gamma(4) &= 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1 \quad , \\
\Gamma(5) &= 4 \cdot \Gamma(4) = 4 \cdot 3 \cdot 2 \cdot 1 \quad , \\
&\vdots
\end{aligned}$$

Clearly,

$$\Gamma(k) = (k-1)\cdots 3 \cdot 2 \cdot 1 = (k-1)! \qquad \text{for} \quad k = 1,\ 2,\ 3,\ \ldots \quad . \tag{21.3}$$

Thus we can view the Gamma function as a generalization of the factorial. In fact, it is somewhat standard practice to "redefine" the factorial by

$$z! = \Gamma(z+1) \qquad \text{whenever} \quad \Gamma(z+1) \text{ is defined} \quad .$$

?►**Exercise 21.2** **a:** *Using identity (21.2) and the value for $\Gamma\left({}^1\!/\!{}_2\right)$ found earlier, evaluate*

$$\Gamma\left(\frac{3}{2}\right) \quad , \quad \Gamma\left(\frac{5}{2}\right) \quad \textit{and} \quad \Gamma\left(\frac{7}{2}\right) \quad .$$

b: *Using the above "redefined" notion of the factorial, evaluate*

$$\frac{3}{2}! \quad , \quad \frac{5}{2}! \quad \textit{and} \quad \frac{7}{2}! \quad .$$

The Extended Definition of the Gamma Function

Observe that the big identity for the Gamma function can be rewritten as

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1) \quad . \tag{21.4}$$

While this was derived from the integral formula for $\Gamma(z)$ assuming $\mathrm{Re}(z) > 0$, there is nothing in the identity itself that prevents it from being applied so long as $z \neq 0$ and $\Gamma(z+1)$ is defined. This allows us to recursively extend the definition of the Gamma function into (most of) the rest of the complex plane. Thus, the full definition of the Gamma function Γ is that it is the function on the complex plane satisfying

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1)$$

in general, and

$$\Gamma(z) = \int_{t=0}^{\infty} e^{-t} t^{z-1}\, dt \qquad \text{when} \quad \mathrm{Re}[z] > 0 \quad .$$

!►*Example 21.1:* *By the above*

$$\Gamma\left(-\tfrac{1}{2}\right) = \frac{1}{-1/2}\Gamma\left(-\tfrac{1}{2}+1\right) = -2\Gamma\left(\tfrac{1}{2}\right) = -2\sqrt{\pi} \quad ,$$

and (making use of an earlier computation)

$$\Gamma(i) = \frac{1}{i}\Gamma(i+1) = -i\int_{t=0}^{\infty} e^{-t}\left[\cos(\ln t) + i\sin(\ln t)\right] dt \quad .$$

It isn't hard to verify that, under this definition, $\Gamma(z)$ is defined for every complex value z *except*

$$z = 0,\ -1,\ -2,\ -3,\ -4,\ \dots \quad .$$

In fact, you can even show that Γ is an analytic function on the complex plane except for simple poles at zero and the negative integers.

?►*Exercise 21.3:* *Here are a bunch of problems about the Gamma function. In many of them, you will want to use formula (21.4)*

a: *Evaluate*

$$\Gamma\left(-\tfrac{3}{2}\right) \quad , \quad \Gamma\left(-\tfrac{5}{2}\right) \quad , \quad -\tfrac{1}{2}! \quad \text{and} \quad -\tfrac{11}{2}! \quad .$$

b: *What problems do we have with $\Gamma(x)$ when $x = 0, -1, -2, -3, \dots$?*

c: *Compute*

$$\lim_{x\to 0^+}\Gamma(x) \quad , \quad \lim_{x\to 0^-}\Gamma(x) \quad , \quad \lim_{x\to -1^+}\Gamma(x) \quad \text{and} \quad \lim_{x\to -1^-}\Gamma(x) \quad .$$

Then make a rough graph of $\Gamma(x)$ on the real line. Compare your graph with the graph in figure 13.1 of AW&H.

d: *(Optional) Verify that the Gamma function is analytic on complex plane except for simple poles at the nonpositive integers.*

e: *(Optional) What is the residue of Γ at $z = 0$? at $z = -1$? at $z = -2$? …*

Even More Formulas and Identities for the Gamma Function

There are many other formulas for the Gamma function. Using reasonably clever changes of variables, for example, you can show that, if $\operatorname{Re}[z] > 0$, then

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1}\,dt \qquad \text{and} \qquad \Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt \quad .$$

?►*Exercise 21.4:* *Derive the above two integral formulas from our original integral formula for* $\Gamma(z)$.

Some identities that we will soon find mildly useful start by observing that, if k is any positive integer and γ is any complex number such that the following Gamma function values exist, then

$$\begin{aligned}
(k+\gamma)! &= \Gamma(k+\gamma+1) \\
&= (k+\gamma)\Gamma(k+\gamma) \\
&= (k+\gamma)(k+\gamma-1)\Gamma(k+\gamma-1) \\
&= \cdots \\
&= (k+\gamma)(k+\gamma-1)(k+\gamma-2)\cdots(1+\gamma)\Gamma(1+\gamma) \\
&= (k+\gamma)(k+\gamma-1)(k+\gamma-2)\cdots\gamma\Gamma(\gamma) \quad .
\end{aligned}$$

Thus, for $k = 1, 2, 3, \ldots$,

$$(k+\gamma)! = (k+\gamma)(k+\gamma-1)(k+\gamma-2)\cdots\gamma\Gamma(\gamma) \tag{21.5}$$

and

$$(k+\gamma)(k+\gamma-1)(k+\gamma-2)\cdots(1+\gamma) = \frac{\Gamma(k+\gamma+1)}{\Gamma(1+\gamma)} = \frac{(k+\gamma)!}{\gamma!} \quad . \tag{21.6}$$

It should be noted that identity (21.5) also holds, almost by definition, for $k = 0$.

A famous identity that you've probably already seen (at least partially) is *Stirling's approximation* for $\Gamma(x)$ when $x > 0$,

$$\Gamma(x) \approx x^x e^{-x}\sqrt{\frac{2\pi}{x}} \qquad \text{for} \quad x > 0 \quad . \tag{21.7}$$

The error in this approximation is less than

$$e^{\frac{1}{12x}} - 1 \quad .$$

This approximation can be derived using an asymptotic expansion, one of the many neat things we've not had time to discuss.

Two other identities that may be amusing (or not)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{21.8}$$

and

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2z) \quad . \tag{21.9}$$

Both of these hold for each complex value of z for which the Gamma functions in the identity exist.

The first of the last two identities is probably the more interesting. For one thing, it shows that $\Gamma(z)$ is never zero. Another reason it is of some interest is that its derivation makes use of stuff we learned in our study of analytic functions. To see this, turn to appendix 21.5 starting on page 21–24 where we actually do verify this identity.

?► ***Exercise 21.5:*** *Quickly skim through §8.1 of Arfken, Weber and Harris (it does have a few things about the Gamma function this I haven't mentioned and we won't use) and do problems 6 and 7 on page 608.*

21.2 Bessel Functions of the First Kind

The Frobenius Solutions to Bessel's Equation

Deriving the Basic Formulas

Let us consider *Bessel's equation of order* ν

$$\frac{d}{dz}\left[z\frac{dy}{dz}\right] - \frac{\nu^2}{z}y = -zy$$

or, equivalently,

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = 0$$

where $\nu \geq 0$ is some constant. Because of the 'heat flow on a disk' problem, we are especially interested in the solutions to this equation when

$$\nu = m = 0,\ 1,\ 2,\ 3,\ \ldots \qquad \text{and} \qquad |y(0)| < \infty \quad ;$$

however, for a number of reasons we will generally assume ν is some nonnegative real constant. On occasions where we want to think of ν as a nonnegative integer, we may let $\nu = m$.

You solved this equation via the Frobenius method in *Homework Handout II*. You discovered that the indicial equation was

$$\gamma^2 = \nu^2 \quad ;$$

so $\gamma = \pm\nu$. Using $\gamma = +\nu$, you got something of the form

$$\begin{aligned} y_\gamma(z) &= c_\gamma z^\gamma \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}\,k!\,(\gamma+1)(\gamma+2)\cdots(\gamma+k)}\,z^{2k} \\ &= c_\gamma 2^\gamma \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,[(\gamma+k)\cdots(\gamma+2)(\gamma+1)]}\left(\frac{z}{2}\right)^{2k+\gamma} \end{aligned}$$

where c_γ is some arbitrary constant. The same formula was obtained using $\gamma = -\nu$, provided ν is not an integer. Now here is a terribly clever thing to do: Using identity (21.6),

$$(k+\gamma)(k+\gamma-1)(k+\gamma-2)\cdots(1+\gamma) = \frac{\Gamma(k+\gamma+1)}{\Gamma(1+\gamma)} \quad ,$$

we rewrite this formula for y_γ as

$$y_\gamma(z) \;=\; C_\gamma\, J_\gamma(z)$$

where

$$C_\gamma \;=\; c_\gamma 2^\gamma \Gamma(1+\gamma)$$

and

$$J_\gamma(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\gamma+1)} \left(\frac{z}{2}\right)^{2k+\gamma} \quad . \tag{21.10}$$

The C_γ, of course, is just an arbitrary constant. The function defined by formula (21.10) is formally known as the *Bessel function of order γ (of the first kind)*.[3] Remember, $\Gamma(\zeta)$ is an analytic function on $\mathbb{C}$ except for simple poles at $\zeta = 0, -1, -2, -3, \ldots$. This means that, as long as $\gamma + k$ is not a negative integer,

$$\frac{(-1)^k}{k!\,\Gamma(\gamma+k+1)}$$

is a well-defined number. And even if $k+\gamma$ is a negative integer $-N$, the fact that the Gamma function only has simple poles means that we can view this expression as

$$\frac{(-1)^k}{k!\,\Gamma(k+\gamma+1)} \;=\; \frac{(-1)^k}{k!\,\Gamma(1-N)} \;=\; \frac{(-1)^k}{k! \times (\pm\infty)} \;=\; 0 \qquad \text{for} \quad N \;=\; 1,\, 2,\, 3,\, \ldots \quad .$$

So we can take formula (21.10) as defining the Bessel function J_γ for any value γ.

In particular, if γ is a nonnegative integer, say,

$$\gamma \;=\; m \;=\; 0,\, 1,\, 2,\, 3,\, \ldots \quad ,$$

then

$$\Gamma(k+\gamma+1) \;=\; \Gamma(k+m+1) \;=\; (k+m)!$$

and the formula for the J_γ becomes

$$J_m(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+m)!} \left(\frac{z}{2}\right)^{2k+m} \quad . \tag{21.11}$$

Even more particularly,

$$J_0(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \;=\; 1 \;-\; \tfrac{1}{2}z^2 \;+\; \tfrac{1}{64}z^4 \;-\; \cdots \quad .$$

?►*Exercise 21.6* ***a:*** *Write out both the full series formula, as well as the first three nonzero terms of the series formula for each of the following:*

$$J_1(z) \qquad , \qquad J_2(z) \qquad , \qquad J_{\frac{1}{2}}(z) \qquad \text{and} \qquad J_{-\frac{1}{2}}(z) \quad .$$

b: *Using your answers to the above, verify that*

$$J_{\frac{1}{2}}(z) \;=\; \sqrt{\frac{2}{\pi z}}\,\sin(z) \qquad \textit{and} \qquad J_{-\frac{1}{2}}(z) \;=\; \sqrt{\frac{2}{\pi z}}\,\cos(z) \quad .$$

(Note: Identity (21.9) on page 21–5 may be helpful.)

[3] We'll discuss the second kind much later.

Analyticity of the Bessel Functions

It's worth rewriting the series formula for the Bessel function with $({}^{z}\!/_{2})^{\gamma}$ factored out:

$$J_{\gamma}(z) \;=\; \left(\frac{z}{2}\right)^{\gamma} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\gamma+1)} \left(\frac{z}{2}\right)^{2k} \quad . \tag{21.12}$$

For integral values of the order,

$$J_m(z) \;=\; \left(\frac{z}{2}\right)^{m} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+m)!} \left(\frac{z}{2}\right)^{2k} \quad . \tag{21.13}$$

So $J_{\gamma}(z)$ is simply z^{γ} multiplied by $2^{-\gamma}$ and a power series with just even powers of z. Using the ratio test (or the big theorem on the Frobenius method), you can easily verify that this series converges for all $z \in \mathbb{C}$, and, hence, defines an analytic function on the complex plane. Thus, if $\gamma = m$ is a nonnegative integer, then $J_m(z)$ is also analytic on $\mathbb{C}$.

However, if γ is not an integer, then z^{γ} is, technically, multivalued (with an essential singularity at 0). Naturally, to make our function single valued, we choose the branch with $-\pi < \operatorname{Arg}(z) < \pi$ (equivalently, we take the negative X–axis as the cut line. That is, when γ is not an integer, J_{γ} is an analytic function on the complex plane except for an essential singularity at 0 and a cut line taken to be the negative X–axis.

But what if γ is a negative integer, say, $\gamma = -2$? It's tempting to look at the formula (21.13) with $m = -2$,

$$J_{-2}(z) \;=\; \left(\frac{z}{2}\right)^{-2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k-2)!} \left(\frac{z}{2}\right)^{2k} \quad ,$$

and conclude that J_{-2} has a double pole at $z = 0$. But remember,

$$\frac{1}{(-2)!} \;=\; \frac{1}{\Gamma(-1)} \;=\; \frac{1}{\infty} \;=\; 0 \qquad \text{and} \qquad \frac{1}{(-1)!} \;=\; \frac{1}{\Gamma(0)} \;=\; \frac{1}{\infty} \;=\; 0 \quad .$$

This fact, along with the use of index substitution $n = k - 2$, means that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k-2)!} \left(\frac{z}{2}\right)^{2k} &= \frac{(-1)^0}{0!\,(0-2)!} \left(\frac{z}{2}\right)^{2\cdot 0} + \frac{(-1)^1}{1!\,(1-2)!} \left(\frac{z}{2}\right)^{2\cdot 1} + \sum_{k=2}^{\infty} \frac{(-1)^k}{k!\,(k-2)!} \left(\frac{z}{2}\right)^{2k} \\
&= \frac{1}{(-2)!} - \frac{1}{(-1)!} \left(\frac{z}{2}\right)^{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{(n+2)!\,n!} \left(\frac{z}{2}\right)^{2(n+2)} \\
&= 0 + 0 + (-1)^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!\,n!\,2^{-2}\,2^2} \left(\frac{z}{2}\right)^{2n} \left(\frac{z}{2}\right)^{4} \\
&= (-1)^2 \left(\frac{z}{2}\right)^{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+2)!} \left(\frac{z}{2}\right)^{2n}
\end{aligned}$$

So, in fact,

$$\begin{aligned}
J_{-2}(z) &= \left(\frac{z}{2}\right)^{-2} (-1)^2 \left(\frac{z}{2}\right)^{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+2)!} \left(\frac{z}{2}\right)^{2n} \\
&= (-1)^2 \left(\frac{z}{2}\right)^{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+2)!} \left(\frac{z}{2}\right)^{2n} \quad .
\end{aligned}$$

This tells us that $J_{-2}(z)$ does *not* have a pole at $z = 0$. It is analytic everywhere on $\mathbb{C}$. Moreover, if you compare the last formula above for $J_{-2}(z)$ with formula (21.13) for $J_m(z)$ with $m = 2$, you will realize that we have also derived

$$J_{-2}(z) = (-1)^2 J_2(z) \quad .$$

The only special feature about 2 that we really used in the last paragraph was that 2 is an integer. Redoing these calculations slightly more generally will yield

$$J_{-m}(z) = (-1)^m J_m(z) \qquad \text{for} \quad m = 0, 1, 2, 3, \ldots \quad . \tag{21.14}$$

Not only is this a nifty identity, it shows that these $J_m(z)$'s are all analytic on the entire complex plane.

Does an identity similar to equation (21.14) hold for $J_{-\gamma}$ when γ is not an integer? No. If $\gamma > 0$ is not an integer, then expanding formula (21.12) yields

$$J_\gamma(z) = z^\gamma \left[\underbrace{\frac{2^{-\gamma}}{\Gamma(1+\gamma)}}_{\neq 0} - \frac{2^{-\gamma}}{\Gamma(\gamma+2)} \left(\frac{z}{2}\right)^2 + \cdots \right] \quad .$$

Replacing γ with $-\gamma$, the same formula yields

$$J_{-\gamma}(z) = z^{-\gamma} \left[\underbrace{\frac{2^{\gamma}}{\Gamma(1-\gamma)}}_{\neq 0} - \frac{2^{\gamma}}{\Gamma(-\gamma+2)} \left(\frac{z}{2}\right)^2 + \cdots \right] \quad .$$

From this it should be clear that $J_{-\gamma}(z)$ cannot be any constant multiple of $J_\gamma(z)$ if γ is not an integer.

General Solutions to Bessel's Equation

Let's get back to Bessel's equation of order $\nu \geq 0$,

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) y = 0 \quad .$$

Remember, we derived $J_\gamma(z)$ with $\gamma = +\nu$ as a first solution to this differential equation obtained via the Frobenius method.

If ν is not an integer, then the basic Frobenius method yields a second solution. If you check back to your answers to the appropriate problem(s) in Homework Handout II, you will discover, amazingly, that the second solution can be written as a constant times J_γ with $\gamma = -\nu$. So J_ν and $J_{-\nu}$ is a pair of solutions. By the work just done a paragraph or two ago, we know neither can be written as a constant times the other (provided γ is not an integer). Thus, *if* ν *is not an integer*,

$$y(z) = c_1 J_\nu(z) + c_2 J_{-\nu}(z)$$

is a general solution to Bessel's equation of order ν.

On the other hand, if ν is an integer m, then the basic Frobenius method did not yield a second solution. That is reflected above by the discovery that $J_{-m} = (-1)^m J_m$. So J_{-m} cannot serve as a second (independent) solution. We have to go back to the big theorem on the Frobenius method (theorem 13.2 on page 13–14) to find that, in this case, the general solution to Bessel's equation of order m is

$$y(z) = c_1 J_m(z) + c_2 y_{m2}(z)$$

where

$$y_{m2}(z) = \begin{cases} J_0(z)\left[\ln z + \sum_{k=0}^{\infty} b_k z^k\right] & \text{if } \quad m = 0 \\ J_m(z)\left[\alpha \ln z + z^{-2m} \sum_{k=0}^{\infty} b_k z^k\right] & \text{if } \quad m = 1, 2, 3, \ldots \end{cases}$$

with, in either case, $b_0 \neq 0$ (by calculation if $m = 0$; by the Frobenius theory if $m \neq 0$).

Recursion Formulas and the Generating Function

Using the series formula for the Bessel functions, you can "easily" verify any of the following identities:

$$J_0{}'(z) = -J_1(z) \tag{21.15}$$

$$J_{\gamma-1}(z) - J_{\gamma+1}(z) = 2J_\gamma{}'(z) \qquad \text{for} \quad \gamma \neq 0, -1, -2, -3, \ldots \tag{21.16}$$

$$\frac{d}{dz}\left[z^\gamma J_\gamma(z)\right] = z^\gamma J_{\gamma-1}(z) \qquad \text{for} \quad \gamma \neq 0, -1, -2, -3, \ldots \tag{21.17}$$

$$\frac{d}{dz}\left[z^{-\gamma} J_\gamma(z)\right] = -z^{-\gamma} J_{\gamma+1}(z) \qquad \text{for} \quad \gamma \neq -1, -2, -3, \ldots \tag{21.18}$$

$$J_{\gamma-1}(z) + J_{\gamma+1}(z) = \frac{2\gamma}{z} J_\gamma(z) \qquad \text{for} \quad \gamma \neq 0, -1, -2, -3, \ldots \tag{21.19}$$

?► *Exercise 21.7:* *Using the series formula for the Bessel functions, verify/derive*

a: *identity (21.15)*

b: *and any one of the other identities (be able to verify/derive all of them).*

(Suggestion: Use x as the variable symbol, instead of z.)

Identities (21.15)—(21.19) are "recursion identities". Using them, we can express $J_\gamma(z)$ for any real γ in terms of one or two corresponding Bessel functions of lower order. This can be very useful in compiling tables.

?► *Exercise 21.8* **a:** *Verify that*

$$J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left[\frac{\sin(z)}{z} - \cos(z)\right] \quad .$$

What are the analogous formulas for $J_{-\frac{3}{2}}(z)$ and $J_{\frac{5}{2}}(z)$?

b: *Express each of the following in terms of $J_0(z)$ and $J_1(z)$:*

$$J_2(z) \quad , \quad J_3(z) \quad \text{and} \quad J_4(z) \quad .$$

Another thing you can show using the series formulas for the Bessel functions of integral order is that

$$\sum_{m=-\infty}^{\infty} J_m(z)t^m = \exp\left(\frac{z}{2}\left[t - \frac{1}{t}\right]\right) \quad .$$

The function on the right is said to be the *generating function* for the Bessel functions (of integral order). What's neat is that, if we treat t as the variable (and z as some arbitrary constant), then the above series is the Laurent series for the generating function about $t = 0$. So the coefficients — which just happen to be $J_k(z)$'s — can be determined from the generating function via the integrals described in our discussion of Laurent series (see theorem 16.1 on page 16–4). After applying some straighforward changes of variables, these integrals for the Laurent coefficients become the following integral formulas for the Bessel functions of integral order:

$$J_m(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin(\theta) - m\theta)\,d\theta \qquad \text{for} \quad m = 0, 1, 2, 3, \ldots \quad . \tag{21.20}$$

In particular,

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin(\theta))\,d\theta \quad .$$

These integral formulas give an alternative to the series formulas for computing Bessel functions of integral order.

If you check appropriate texts (including our text by Arfken, Weber & Harris), you will find many other identities involving the Bessel functions, as well as other integral formulas for Bessel functions of integral and non-integral order. You will also find different approaches to the development of "Bessel function theory". A few more formulas will be developed in the next few pages, but we won't come near to exhausting the subject. Feel free this summer to spend a few pleasant weeks in a really thorough self-study of Bessel function formulas/identities.

Qualitative Behavior of Bessel Functions (of the First Kind) on the Real Line

For the following, we will restrict ourselves to trying to figure out what the graphs of $J_\gamma(x)$ look like where x is restricted to the real line and γ is a real constant. Remember, if γ is not an integer, then the negative real axis is used as a cut line for J_γ. So if γ is not an integer, we will further restrict x to being positive.

Remember, also, that our main interest is in the cases where γ is an integer.

Upper Bounds and Symmetry (with Integral Orders)

For the following, we will limit our attention to $J_m(x)$ where m is an integer, and $x \in \mathbb{R}$.

From integral formula (21.20),

$$\begin{aligned} |J_m(x)| &= \left|\frac{1}{\pi}\int_0^{\pi} \cos(x\sin(\theta) - m\theta)\,d\theta\right| \\ &\leq \frac{1}{\pi}\int_0^{\pi} |\cos(x\sin(\theta) - m\theta)|\,d\theta \\ &\leq \frac{1}{\pi}\int_0^{\pi} 1\,d\theta = 1 \quad . \end{aligned}$$

So,

$$|J_m(x)| \leq 1 \qquad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad m = 0,\ 1,\ 2,\ 3,\ \dots \quad .$$

Of course, this bound also holds if m is a negative integer simply because $J_{-m}(x) = (-1)^m J_m(x)$.

Now take another look at series formula (21.13) for J_m, noting in particular that the powers of x in the series are all even integers:

$$J_m(x) = \left(\frac{x}{2}\right)^m \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+m)!} \left(\frac{x}{2}\right)^{2k}}_{\text{an even function}} \quad .$$

Since we basically have x^m multiplied by an even function, we clearly have that

$J_m(x)$ is an even function if m is an even integer.

and

$J_m(x)$ is an odd function if m is an odd integer.

Behavior Near Zero

For these results, we do not need γ to be an integer.

Let us write out the first few terms of formula (21.12) for the Bessel function:

$$\begin{aligned} J_\gamma(x) &= \left(\frac{x}{2}\right)^\gamma \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\gamma+1)} \left(\frac{x}{2}\right)^{2k} \\ &= x^\gamma \left[\frac{1}{\Gamma(\gamma+1)2^\gamma} - \frac{x^2}{\Gamma(\gamma+2)2^{\gamma+1}} + \frac{x^4}{\Gamma(\gamma+3)2^{\gamma+2}} - \cdots \right] \quad . \end{aligned}$$

Assuming $\gamma \neq 0$, the derivative of this is

$$J_\gamma{}'(x) = x^{\gamma-1} \left[\frac{\gamma}{\Gamma(\gamma+1)2^\gamma} - \frac{(2+\gamma)x^2}{\Gamma(\gamma+2)2^{\gamma+1}} + \frac{(4+\gamma)x^4}{\Gamma(\gamma+3)2^{\gamma+2}} - \cdots \right] \quad .$$

If x is very close to 0, then only the first few terms of either series is significant; the rest of these series will be much smaller, and vanish much more quickly as $x \to 0$. So we let us write these formulas as

$$J_\gamma(x) = x^\gamma \left[\frac{1}{\Gamma(\gamma+1)2^\gamma} - \frac{x^2}{\Gamma(\gamma+2)2^{\gamma+1}} + \varepsilon_{\gamma 0}(x) \right]$$

and

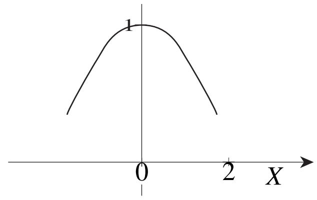
$$J_\gamma{}'(x) = x^{\gamma-1} \left[\frac{\gamma}{\Gamma(\gamma+1)2^\gamma} + \varepsilon_{\gamma 1}(x) \right] \quad ,$$

and look at what they tell us about the graph of $J_\gamma(x)$ when $x \approx 0$ (keeping in mind that $\varepsilon_{\gamma 0}(x)$ and $\varepsilon_{\gamma 1}(x)$, the rest of the series for these functions, are negligibly small for these values of x).

For $\gamma = 0$: We've already noted that $J_0(x)$ is an even function defined for $-\infty < x < \infty$. By the above,

$$J_0(x) = 1 - \frac{1}{4}x^2 + \varepsilon_{00}(x)$$

So the graph of $J_0(x)$ looks like the parabola $1 - {x^2}/{4}$ right around $x = 0$. That is, it looks something like



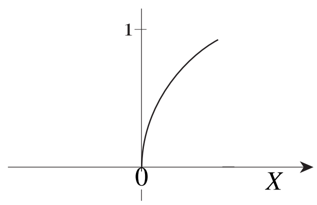
For $0 < \gamma < 1$: Remember that, in this case, the negative real axis is a cut line for J_γ. So we can only graph this J_γ on the positive X–axis. By the above.

$$J_\gamma(x) = x^\gamma \left[\frac{1}{\Gamma(\gamma+1)2^\gamma} - \frac{x^2}{\Gamma(\gamma+2)2^{\gamma+1}} + \varepsilon_{\gamma 0}(x) \right] \longrightarrow 0 \quad \text{as} \quad x \to 0^+$$

and

$$J_\gamma{}'(x) = x^{\gamma-1} \left[\frac{\gamma}{\Gamma(\gamma+1)2^\gamma} + \varepsilon_{\gamma 1}(x) \right] \longrightarrow +\infty \quad \text{as} \quad x \to 0^+ \quad ,$$

which looks something like



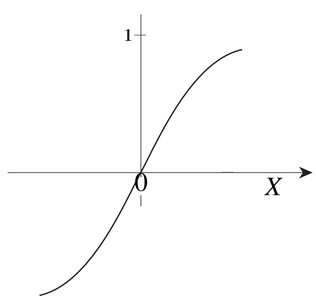
For $\gamma = 1$: $J_1(x)$ is an odd function on the X–axis. By the above.

$$\begin{aligned} J_1(x) &= x \left[\frac{1}{2} - \frac{1}{16}x^2 + \varepsilon_{10}(x) \right] \\ &= \frac{1}{2}x - \frac{1}{16}x^3 + x\varepsilon_{10}(x) \longrightarrow 0 \quad \text{as} \quad x \to 0 \end{aligned}$$

and

$$J_1{}'(x) = \frac{1}{2} - \frac{3}{16}x^2 + \varepsilon_{11}(x) \longrightarrow \frac{1}{2} \quad \text{as} \quad x \to 0 \quad ,$$

which looks something like



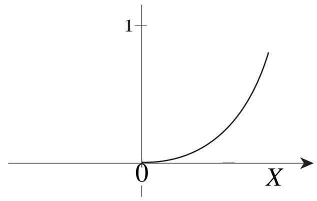
For $\gamma > 1$: Here we have

$$J_\gamma(x) = x^\gamma \left[\frac{1}{\Gamma(\gamma+1)2^\gamma} - \frac{x^2}{\Gamma(\gamma+2)2^{\gamma+1}} + \varepsilon_{\gamma 0}(x) \right] \longrightarrow 0 \quad \text{as} \quad x \to 0^+$$

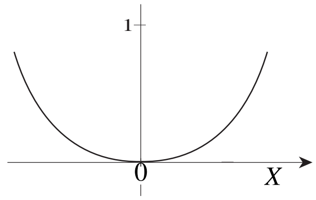
and

$$J_\gamma{}'(x) \;=\; x^{\gamma-1}\left[\frac{\gamma}{\Gamma(\gamma+1)2^\gamma} \;+\; \varepsilon_{\gamma 1}(x)\right] \quad\longrightarrow\quad 0 \qquad \text{as} \quad x \to 0^+ \quad .$$

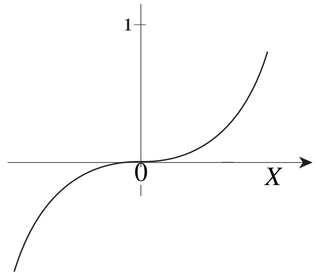
For $x \geq 0$ this looks like



This is all we can say if γ is not an integer. If $\gamma = m$ is a (positive) integer, then $J_m(x)$ is an even or odd function on $\mathbb{R}$, depending on whether m is an even or odd integer. Thus, for $m = 2, 4, 6, \ldots$, the graph of $J_m(x)$ around $x = 0$ looks something like



While the graph of $J_m(x)$ around $x = 0$ for $m = 3, 4, 5, \ldots$ looks something like



For $\gamma < 0$ and not an integer: For psychological purposes, let $\nu = -\gamma > 0$. Then the formula for J_γ gives

$$J_\gamma(x) \;=\; J_{-\nu}(x) \;=\; \frac{1}{x^\nu}\left[\frac{1}{\Gamma(1-\nu)2^{-\nu}} \;-\; \cdots\right] \quad\longrightarrow\quad \pm\infty \qquad \text{as} \quad x \to 0^+ \quad ,$$

with the sign on the ∞ depending on whether $\Gamma(2-\nu)$ is positive or negative. Thus, in this case, $J_\gamma(x)$ blows up near $x = 0$.

For $\gamma = -m = -1, -2, -3, \cdots$: In this case, we already know that $J_{-m}(x) \;=\; (-1)^m J_m(x)$, so the graph of $J_{-m}(x)$ is simply $\pm$ the graph of $J_m(x)$.

?►*Exercise 21.9:* *Let m be a nonnegative integer, and let y_{m2} be the "second" solution to Bessel's equation of order m. Using the formula described earlier for y_{m2} and the above, verify that*

$$\lim_{x\to 0^+} y_{m2}(x) \;=\; \pm\infty \quad .$$

It is worth noting that

$$\lim_{x\to 0} J_0(x) = 1 \quad ,$$

while

$$\lim_{x\to 0^+} J_\gamma(x) = 0 \qquad \text{if} \quad \gamma > 0 \quad .$$

It is also worth noting that our analysis shows that all other solutions to Bessel's equations (other than constant multiples of the above) "blow up" as $x \to 0^+$. This gives us the following corollary:

Corollary 21.1
Let $\nu \geq 0$. The only solutions to Bessel's equation of order ν,

$$z^2 \frac{d^2 y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = 0$$

which also satisfy

$$|y(0)| < \infty$$

are given by

$$y(z) = c_\nu J_\nu(z)$$

where c_ν is an arbitrary constant.

Behavior Far from Zero

Now let's consider $J_\gamma(x)$ when x is large.

Recall that the independent pair of solutions we've found to Bessel's equation of order $1/2$ are

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin(x) \qquad \text{and} \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos(x) \quad .$$

That is,

$$J_{\pm\frac{1}{2}} = x^p s_\pm(x)$$

where $p = -1/2$,

$$s_+(x) = \sqrt{\frac{2}{\pi}}\sin(x) \qquad \text{and} \qquad s_-(x) = \sqrt{\frac{2}{\pi}}\cos(x) \quad .$$

Could it be that, for any other γ,

$$J_\gamma(x) = x^p s(x)$$

where p is some constant and s is some reasonably nice (maybe periodic) function?

This turns out to be a question worth pursuing.

Accordingly, let us seek all solutions to the Bessel equation of order $\nu \geq 0$ which are of the form

$$y(x) = x^p s(x)$$

where p and $s(x)$ are, respectively, a constant and function to be determined. Plugging this into the Bessel's equation, we get

$$\begin{aligned} 0 &= x^2 \frac{d}{dx}\left[x^p s(x)\right] + x\frac{d}{dx}\left[x^p s(x)\right] + \left(x^2 - \nu^2\right) x^p s(x) \\ &= \cdots \\ &= x^{p+2}\left[\frac{d^2 s}{dx^2} + \frac{2p+1}{x}\frac{ds}{dx} + \left(1 + \frac{p^2 - \nu^2}{x^2}\right) s\right] \quad , \end{aligned}$$

which means that $s(x)$ must satisfy

$$\frac{d^2 s}{dx^2} + \frac{2p+1}{x}\frac{ds}{dx} + \left(1 + \frac{p^2 - \nu^2}{x^2}\right) s = 0 \quad .$$

If we now choose $p = -1/2$, then $2p + 1 = 0$ and the above differential equation reduces to

$$\frac{d^2 s}{dx^2} + \left(1 + \frac{1 - 4\nu^2}{4x^2}\right) s = 0 \quad .$$

Unless $\nu = 1/2$, this is not a differential equation that can be solved by elementary means (and we already know about the solutions when $\nu = 1/2$ — they are what inspired our search). So we aren't finding quite what we want. However, do observe that

$$\frac{1 - 4\nu^2}{4x^2} \to 0 \qquad \text{as} \quad x \to \infty \quad .$$

Thus, "for large values of x", the function $s(x)$ satisfies

$$\frac{d^2 s}{dx^2} + s \approx 0 \quad ,$$

which has general solution

$$s(x) \approx a\sin(x) + b\cos(x)$$

where a and b are arbitrary constants. Thus, for large values of x,

$$y(x) = x^p s(x) \approx x^{-1/2}\left[a\sin(x) + b\cos(x)\right] \quad .$$

That is, for large values of x, any solution to Bessel's equation — including any Bessel function — must look like a sinusoidal function whose amplitude is decreasing like $x^{-1/2}$ as $x \to \infty$. In particular, the full graph of $J_0(x)$ looks something like that drawn (by Maple) in figure 21.7.

?► *Exercise 21.10:* *Roughly sketch the graphs of J_1, J_2, and J_3.*

Now recall a little trigonometry: Given any two real values a and b, we can find two other pairs of real values (C, ϕ) and (D, ψ) such that

$$a\sin(x) + b\cos(x) = C\sin(x - \phi) = D\cos(x - \psi) \quad .$$

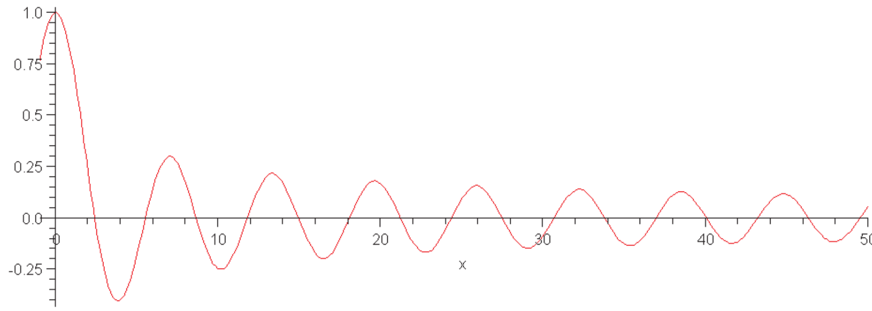


Figure 21.7: Graph of J_0 .

Thus, given any real value γ , there are corresponding constants a_γ , b_γ , C_γ , ϕ_γ , D_γ and ψ_γ such that, for large values of x ,

$$\begin{aligned} J_\gamma(x) &\approx x^{-1/2}\left[a_\gamma \sin(x) + b_\gamma \cos(x)\right] \\ &= C_\gamma \frac{\sin(x-\phi_\gamma)}{\sqrt{x}} \\ &= D_\gamma \frac{\cos(x-\psi_\gamma)}{\sqrt{x}} \quad . \end{aligned}$$

Deriving the precise values of these constants, as well as the error in this approximation, requires tools we haven't developed (asymptotic expansions, again – see AW&H chapter 12.6).[4] It can be shown that, at least for $\gamma \geq 0$,

$$D_\gamma = \sqrt{\frac{2}{\pi}} \qquad \text{and} \qquad \psi_\gamma = \left(\gamma + \frac{1}{2}\right)\frac{\pi}{2} \quad .$$

That is,

$$J_\gamma(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \left(\gamma + \frac{1}{2}\right)\frac{\pi}{2}\right) \qquad \text{for large } x \quad .$$

In particular,

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) \qquad \text{for large } x \quad .$$

To get an idea of just how good this approximation is, turn to page 694 of AW&H and look at figure 14.13 in which the graphs of $J_0(x)$ and the above approximation are drawn on the same coordinate system. The two graphs sketched become virtually identical when $x > 4$.

[4] And, frankly, isn't nearly as important as it was before programs like Maple and Mathematica made computing and graphing Bessel functions so easy.

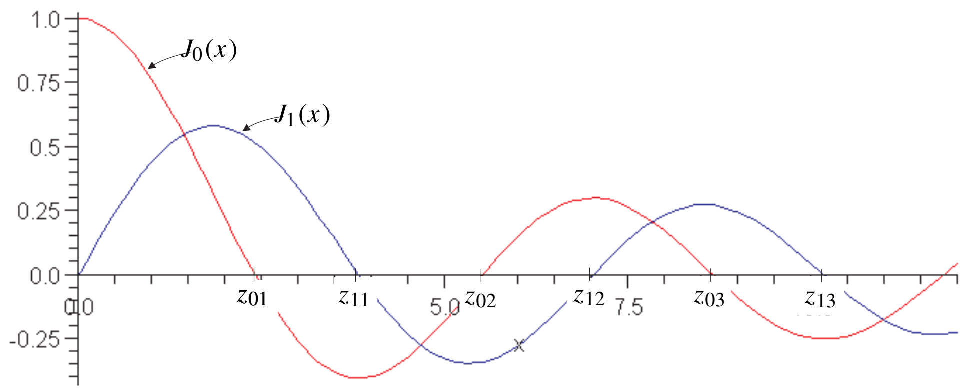


Figure 21.8: Graphs of J_0 and J_1 including the first three zeroes of each (drawn with the aid of Maple).

Zeroes of the Bessel Functions

Let's recall a little standard (but still poorly considered) terminology: Given a function f, any value z_0 such that $f(z_0) = 0$ is called a *zero* for f. For $f(z) = \sin(z)$,

$$\sin(z) = 0 \iff z = z_k = k\pi \qquad \text{for} \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots \quad .$$

So the zeroes for the sine function are the above z_k values. In particular, the "positive zeros" of the sine function are

$$\pi \quad , \quad 2\pi \quad , \quad 3\pi \quad , \quad 4\pi \quad , \quad \cdots \quad .$$

Knowing these zeroes was important in solving some of our partial differential equation problems on an interval $0 < x < L$. Remember? We often had to find all $\lambda > 0$ satisfying

$$\sin\left(\sqrt{\lambda}\, L\right) = 0 \quad .$$

So $\sqrt{\lambda}\, L$ must be one of the above zeroes for the sine function, leading to the "eigenvalue formula"

$$\lambda = \lambda_k = \left(\frac{k\pi}{L}\right)^2 \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

Because of the "heat flow on a disk" problem, we are currently most interested in the positive zeroes of the Bessel functions of the first kind and integral order. Following common convention, we will let

$$z_{mk} = k^{\text{th}} \text{ positive zero of } J_m \quad .$$

The first few positive zeroes of J_0 and J_1 are indicated figure 21.8.
It turns out that

$$z_{01} \approx 2.4 \qquad\qquad z_{11} \approx 3.83$$

$$z_{02} \approx 5.52 \qquad\qquad z_{12} \approx 7.02$$

and

$$z_{03} \approx 8.65 \qquad\qquad z_{12} \approx 10.17 \quad .$$

Solving $J_m(z) = 0$ for z is not as easy as solving $\sin(z) = 0$ for z. At best, we can only find approximations using numerical methods (such as Newton's method for finding roots). Of course, the approxiation

$$J_\gamma(z) \approx \sqrt{\frac{2}{\pi x}} \cos\left(z - \left(\gamma + \frac{1}{2}\right)\frac{\pi}{2}\right) \qquad \text{for large } x$$

tells us that the big zeroes are approximately given by solving

$$z - \left(\gamma + \frac{1}{2}\right)\frac{\pi}{2} = \frac{(2n+1)\pi}{2}$$

for z using large integral values for n. In practice, it's generally agreed that this means $n >> \gamma^2/\pi$.

Fortunately, we no longer need to compute the zeroes for Bessel functions from scratch. A lot of work has been done by tireless computational experts in compiling tables of these values. These tables can be found in many texts and compendiums of tables. One dauntless soul, in 1958, published a table of "the first 700 zeroes of Bessel functions". That had to have been a lot of work, then.

Today, however, you can do much better using Maple or Mathematica. In Maple

$$\texttt{BesselJZeros}(m, k) \quad ,$$

gives $z_{m,k}$ as quickly and accurately as anyone could reasonably wish. On my computer, Maple cranks out

$$z_{0,1000} \approx 3140.807295 \qquad \text{and} \qquad z_{1000,1} \approx 1018.660881$$

in less time than it takes to pour a beer.

Using the Bessel Functions

Our interest in Bessel equations came from the "heat flow on a disk of radius a" problem in the previous chapter of these notes. In separating that problem, we got the eigenproblems

$$\frac{d}{dr}\left[r\frac{d\phi}{dr}\right] - \frac{m^2}{r}\phi = -\lambda r\phi$$

$$|\phi(0)| < \infty \qquad \text{and} \qquad \phi(a) = 0$$

for $m = 0, 1, 2, 3, \ldots$. For each of these values of m we need to solve this eigen-problem for all possible eigenvalues and eigenfunctions,

$$(\lambda, \phi) = (\lambda_{mk}, \phi_{mk}) \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

From the Rayleigh quotient, we learned that λ had to be positive. Then, using the change of variables

$$z = \sqrt{\lambda}\, r$$

and letting "$\phi(r) = y(z)$", we found[5] that

$$\phi(r) = y\left(\sqrt{\lambda}\, r\right)$$

[5] well, you found (in homework)

where the function $y = y(z)$ satisfies Bessel's equation of order m, with y and λ, together, satisfying the boundary conditions

$$|y(0)| < \infty \qquad \text{and} \qquad y\big(\sqrt{\lambda}\, a\big) = 0 \quad .$$

Because of the boundedness of the solution at $z = 0$, we now know the solution y must be of the form

$$y(z) = \text{constant} \times J_m(z) \quad .$$

Using this for $y(z)$, the other boundary condition becomes

$$J_m\big(\sqrt{\lambda}\, a\big) = 0 \quad .$$

Thus, $\sqrt{\lambda}\, a$ must be one of the positive zeroes for J_m,

$$\sqrt{\lambda}\, a = z_{mk} \qquad \text{for} \quad k = 1, 2, 3, \dots \quad .$$

Solving for the λ's, we get all the possible eigenvalues (for each choice of m) as being

$$\lambda = \lambda_{mk} = \left(\frac{z_{mk}}{a}\right)^2 \qquad \text{for} \quad k = 1, 2, 3, \dots \quad .$$

The corresponding eigenfunctions (for each choice of m) are then given by

$$\begin{aligned} \phi_{mk}(r) &= \text{“} y\big(\sqrt{\lambda}\, r\big) \text{”} \\ &= c_{mk}\, J_m\big(\sqrt{\lambda_{mk}}\, r\big) \\ &= c_{mk}\, J_m\left(\frac{z_{mk}}{a} r\right) \qquad \text{for} \quad k = 1, 2, 3, \dots \end{aligned}$$

where the c_{mk}'s are arbitrary constants.

21.3 Other Bessel Functions

In all of the following, assume $\nu \geq 0$.

We will be concerned with "other" basic solutions to Bessel's equation of order ν. At this point, we have two. The first (and our current favorite) is J_ν, the Bessel function of the first kind of order ν. It is a favorite because it is bounded at 0. The other solution is y_ν where

$$y_\nu(z) = J_{-\nu}(z) \qquad \text{if} \quad \nu \neq \text{an integer}$$

and, for $\nu = m = 0, 1, 2, 3, \dots,$ $y_m(z)$ is of the form

$$y_m(z) = J_m(z)\left[\alpha \ln z + z^{-2m} \sum_{k=0}^{\infty} \beta_k z^k\right] \quad .$$

Remember, whether ν is or is not an integer, this second solution was not bounded at 0.

Bessel Functions of the Second Kind

As an alternative to the second solution y_ν given above, we can use any linear combination $A_\nu J_\nu(z) + B_\nu y_\nu(z)$ with $B_\nu \neq 0$. The Bessel function of the second kind of order ν, also called the Neumann function of order ν and denoted by $N_\nu(z)$, is just such a linear combination.[6] For $\nu \neq$ an integer, this function is given by

$$N_\nu(z) = A_\nu J_\nu(z) + B_\nu J_{-\nu}(z)$$

with

$$A_\nu = \frac{\cos(\nu\pi)}{\sin(\nu\pi)} \qquad \text{and} \qquad B_\nu = \frac{-1}{\sin(\nu\pi)} \quad .$$

That is,

$$N_\nu(z) = \frac{\cos(\nu\pi)\, J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)} \quad .$$

(It may be worth while to observe that, for $m = 0, 1, 2, 3, \ldots,$

$$N_{m+\frac{1}{2}} = \frac{\cos\Big(\big(m+\frac{1}{2}\big)\pi\Big)\, J_{m+\frac{1}{2}}(z) - J_{-m-\frac{1}{2}}(z)}{\sin\Big(\big(m+\frac{1}{2}\big)\pi\Big)} = (-1)^m J_{-m-\frac{1}{2}}(z) \quad .\Big)$$

The Neumann function of an integral order m is defined by

$$N_m(z) = \lim_{\nu\to m} N_\nu(z) = \lim_{\nu\to m} \frac{\cos(\nu\pi)\, J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)} \quad ,$$

which can be shown to exist and be a linear combination of J_m and y_m.

It can be shown that N_ν satisfies relations similar to those satisfied by J_ν. For example,

$$\frac{d}{dx}\left[x^{-m} N_m(x)\right] = -x^{-m} N_{m+1}(x)$$

$$N_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh(t))\, dt$$

and

$$N_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\sin\left(x - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right) \qquad \text{for large } x \quad .$$

The last relation helps explains why the Neumann functions are often considered the natural "second" solution to the Bessel equation, at least it does once you recall that

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right) \qquad \text{for large } x \quad .$$

In a sense, $J_\nu(x)$ and $N_\nu(x)$ can be viewed as a natural pair of solutions to Bessel's equation of order ν, just as the the pair $\cos(x)$ and $\sin(x)$ are considered to be a natural pair of solutions to $y'' + y = 0$.

[6] Many texts use $Y_\nu(z)$ instead of $N_\nu(z)$. I also vaguely recall the Bessel functions of the first and second kind, J_ν and N_ν, being represented in some texts by $J_\nu^{(1)}$ and $J_\nu^{(2)}$.

The Hankel Functions

The two *Hankel functions* of order ν can be defined as follows:

$$H_\nu^{(1)}(z) = \text{"the Hankel function of the first kind of order } \nu\text{"}$$
$$= J_\nu(z) + i N_\nu(z)$$

and

$$H_\nu^{(2)}(z) = \text{"the Hankel function of the second kind of order } \nu\text{"}$$
$$= J_\nu(z) - i N_\nu(z)$$

Again, it can be shown that the Hankel functions satisfy relations similar to those satisfied by the Bessel functions of the first and second kind.

Note that the general solution to Bessel's equation of order ν can be written as either

$$y(z) = c_1 J_\nu(z) + c_2 N_\nu(z)$$

or as

$$y(z) = c_1 H_\nu^{(1)}(z) + c_2 H_\nu^{(2)}(z)$$

Crudely speaking, J_ν and N_ν are the "trig-like" solutions, and $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the "complex-exponential-like" solutions.

?►*Exercise 21.11:* *Confirm that the Hankel functions are "complex-exponential-like" by deriving the approximations for $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ when x is a large positive value. (Use the corresponding approximations for the Bessel and Neumann functions.)*

One advantage of the Hankel functions is that it is a little easier (and more 'elegant') to develop the full theory of Bessel functions by starting with the Hankel functions, and then deriving results for the Bessel functions of the first kind from corresponding results for the Hankel functions. This is something one discovers in retrospect.

?►*Exercise 21.12:* *What are the formulas for J_ν and N_ν in terms of $H_\nu^{(1)}$ and $H_\nu^{(2)}$?*

21.4 Other Special Functions

Even More 'Bessel' Functions

We were led to the Bessel functions, and especially the Bessel functions of the first kind of integral order, by our attempts to solve a heat flow problem on a disk. (We would also have gotten them if the problem had involved a vibrating membrane attached to a circle — i.e., a vibrating drumhead.)

If, instead we had been interested in the heat flow in either a finitely long cylinder or a ball, then we would have been lead, respectively, to either the "modified Bessel functions" or the "spherical Bessel functions". It turns out that all of these 'Bessel' functions are related.

The *modified Bessel functions* of order ν, usually denoted by I_ν and K_ν, are basically the corresponding Bessel and Hankel functions computed along the imaginary axis. In particular,

$$I_\nu(x) = e^{-i\nu\pi/2} J_\nu(ix) \qquad \text{and} \qquad K_\nu(x) = i\frac{\pi}{2} e^{i\nu\pi/2} H_\nu^{(1)}(ix) \qquad \text{when} \quad x > 0 \quad .$$

More careful authors refer to I_ν as the *modified Bessel function of the first kind* of order ν, and call K_ν either the *modified Bessel function of the third kind* or the *MacDonald's function* of order ν. When treated as a function of a complex variable $z = re^{i\theta}$, one must use slightly different formulas that take into account the value of θ.)

The collection of "spherical Bessel functions of order ν" include the *spherical Bessel function (of the first kind)* j_ν, the *spherical Neumann function* n_ν, and the *spherical Hankel functions* $h_\nu^{(1)}$ and $h_\nu^{(2)}$. They are related to the ordinary Bessel/Neumann/Hankel functions via

$$j_\nu(z) = \sqrt{\frac{\pi}{2z}}\, J_{\nu+\frac{1}{2}}(z)$$

$$n_\nu(z) = \sqrt{\frac{\pi}{2z}}\, N_{\nu+\frac{1}{2}}(z)$$

$$h_\nu^{(1)}(z) = \sqrt{\frac{\pi}{2z}}\, H_{\nu+\frac{1}{2}}^{(1)}(z)$$

and

$$h_\nu^{(2)}(z) = \sqrt{\frac{\pi}{2z}}\, H_{\nu+\frac{1}{2}}^{(2)}(z) \quad .$$

In practice, ν is usually a nonnegative integer. If you recall the definitions and that

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin(z) \qquad \text{and} \qquad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos(z) \quad ,$$

then you will realize that

$$j_0(z) = \frac{\sin(z)}{z}$$

$$n_0(z) = -\frac{\cos(z)}{z}$$

$$h_0^{(1)}(z) = -\frac{i}{z}\, e^{iz}$$

and

$$h_0^{(2)}(z) = \frac{i}{z}\, e^{-iz} \quad .$$

Similar relatively simple formulas for higher integral order spherical Bessel functions can be derived using various recursion formulas. (See AW&H, §14.7.)

And Others

There are many other collections of special functions that arise in solving various classes of partial differential equation problems. Legendre polynomials (and the related spherical harmonics) would probably be the next collection worth studying because of their importance in problems in which spherical coordinates are used (as in the classical hydrogen atom problem of quantum mechanics). Other applications lead to the collections associated with Laguerre, Chebyshev, Hermite, Mathieu, and Professor Hypergeometric. My suggestion: learn about them as the need arises.

21.5 Appendix: Verifying Identity (21.8)

Our goal is to verify that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin([1-z]\pi)}$$

for every z at which either side is defined.

For convenience, let

$$f(z) = \Gamma(z)\Gamma(1-z) \qquad \text{and} \qquad g(z) = \frac{\pi}{\sin([1-z]\pi)} \quad ,$$

so that we can rephrase our goal as showing that

$$f(z) = g(z)$$

whenever $f(z)$ or $g(z)$ is defined.

By what we know about the Gamma and sine functions, it immediately follows that both f and g are defined and analytic on all the complex plane except for simple poles at the integers. Also, if $0 < \text{Re}[z] < 1$, we can use the integral formula for the Gamma function to get

$$\begin{aligned} f(z) = \Gamma(z)\Gamma(1-z) &= \left(\int_{t=0}^{\infty} e^{-t} t^{z-1}\, dt\right)\left(\int_{s=0}^{\infty} e^{-s} s^{[1-z]-1}\, ds\right) \\ &= \int_{t=0}^{\infty}\int_{s=0}^{\infty} e^{-[t+s]} t^{z-1} s^{-z}\, ds\, dt \\ &= \int_{t=0}^{\infty}\int_{s=0}^{\infty} e^{-[t+s]} \left(\frac{t}{s}\right)^{z} t^{-1}\, ds\, dt \end{aligned}$$

The s and t in the above integrals can be viewed as a set of coordinates in a Euclidean system. Let us now introduce a (u, v) coordinate system on the upper right quarter of the ST–plane related to the (s, t) coordinate system by

$$u = s + t \qquad \text{and} \qquad v = \frac{t}{s} \tag{21.21}$$

(see figure 21.9a). Note that

$$s > 0 \quad \text{and} \quad t > 0 \qquad \Longleftrightarrow \qquad u > 0 \quad \text{and} \quad v > 0$$

Solving for s and t in terms of v and u, we get

$$s = \frac{u}{1+v} \qquad \text{and} \qquad t = \frac{vu}{1+v} \quad .$$

The corresponding Jacobian is then easily computed,

$$\frac{\partial(s,t)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial s}{\partial u} & \dfrac{\partial t}{\partial u} \\ \dfrac{\partial s}{\partial v} & \dfrac{\partial t}{\partial v} \end{vmatrix} = \cdots = \frac{u}{(1+v)^2} \quad .$$

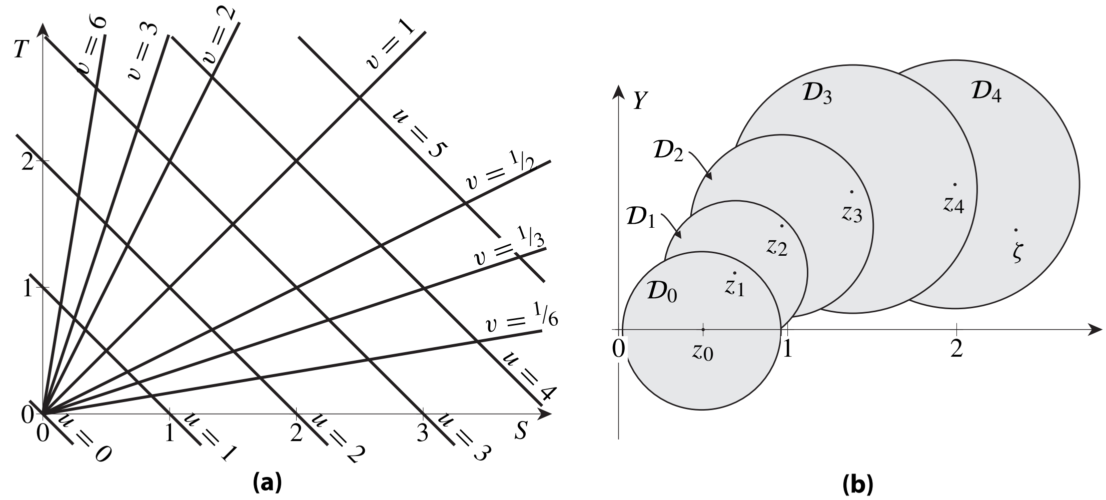


Figure 21.9: (a) The (s, t) and (u, v) coordinate systems corresponding to the change of coordinates formula (21.21). (b) A chain of disks linking z_0 to ζ .

So (see section 10.3 of the lecture notes),

$$\begin{aligned} f(z) &= \int_{t=0}^{\infty}\int_{s=0}^{\infty} e^{-[t+s]}\left(\frac{t}{s}\right)^{z} t^{-1}\, ds\, dt \\ &= \int_{v=0}^{\infty}\int_{u=0}^{\infty} e^{-u} v^{z}\left(\frac{uv}{1+v}\right)^{-1} \frac{\partial(s,t)}{\partial(u,v)}\, du\, dv \\ &= \int_{v=0}^{\infty}\int_{u=0}^{\infty} e^{-u} v^{z}\frac{1+v}{uv}\frac{u}{(1+v)^2}\, du\, dv \;=\; \int_{v=0}^{\infty}\int_{u=0}^{\infty} e^{-u}\frac{v^{z-1}}{1+v}\, du\, dv \quad . \end{aligned}$$

The integral with respect to u can be easily integrated, leaving us with

$$f(z) = \int_{v=0}^{\infty} \frac{v^{z-1}}{1+v}\, dv \qquad \text{when} \quad 0 < \mathrm{Re}[z] < 1 \quad .$$

Now compare this integral to the one you computed using residues for problem **M** in *Homework Handout V* (which refers to exercise 11.8.24 in AW&H). From that problem, we know

$$f(z) = \int_{v=0}^{\infty} \frac{v^{z-1}}{1+v}\, dv = \frac{\pi}{\sin([1-z]\pi)} \qquad \text{when} \quad 0 < z < 1 \quad ,$$

Verifying that $f(z) = g(z)$ when z is restricted to the real axis between 0 and 1 .

To finish verifying our claim, we now need only show that

$$f(\zeta) = g(\zeta)$$

for any ζ in the complex plane other than a point on the real axis between 0 and 1 , and other than an integral point on the real axis. So let ζ be any such point, and, as illustrated in figure 21.9b, form a "chain" of overlapping disks

$$\mathcal{D}_0\ ,\ \mathcal{D}_1\ ,\ \mathcal{D}_2\ ,\ \mathcal{D}_3\ ,\ \dots\ ,\ \mathcal{D}_M$$

with nonzero radii and corresponding centers

$$z_0\ ,\ z_1\ ,\ z_2\ ,\ z_3\ ,\ \dots\ ,\ z_M$$

satisfying all the following:

1. $z_0 = {}^1\!/_2$.
2. For $k = 1, 2, \ldots, M$, z_k is inside disk $\mathcal{D}_{k-1}$.
3. ζ is inside disk $\mathcal{D}_M$.
4. For $k = 0, 1, 2, \ldots, M$, disk $\mathcal{D}_k$ contains no integral point on the real axis. (This ensures that f and g are analytic on $\mathcal{D}_k$, and are given by their power series about z_k .)

Now, since f and g are equal and analytic on the real axis between 0 and 1 , their derivatives at z_0 are equal. Hence, their power series about z_0 (i.e., their Taylor series about z_0) are the same. And since f and g are given by the same power series throughout $\mathcal{D}_0$, f must equal g throughout $\mathcal{D}_0$.

But z_1 is inside $\mathcal{D}_0$ (where $f = g$). So all the derivatives of f at z_1 must equal the corresponding derivatives of g at z_1 . Consequently, the power series about z_1 for f is the same as the power series about z_1 for g . Hence, f and g are given by the same power series throughout $\mathcal{D}_1$, and, hence, $f = g$ throughout $\mathcal{D}_1$.

Continuing this way, we can ultimately verify that $f = g$ throughout each of the disks, and since ζ is in the last disk, $\mathcal{D}_M$, we must have, in particular, that

$$f(\zeta) = g(\zeta) \quad ,$$

which, as noted a few paragraphs ago, finishes our verification of identity the identity given at the beginning of this appendix.

(By the way, the process we just used to extend

$$f(z) = g(z) \qquad \text{when} \quad 0 < z < 1$$

to an equation holding on a much larger region of the complex plane is call *analytic continuation*. It can, in theory, be used to extend the domain of any function given by a power series with a finite radius of convergence to include regions outside the initial disk on which the power series is convergent. You just have to be careful that none of the overlapping disks being used contain a singular point of the function.)