22

Nonhomogeneous PDE Problems

22.1 Eigenfunction Expansions of Solutions

Let us complicate our problems a little bit by replacing the homogeneous partial differential equation,

$$\sum_{jk} a_{jk} \frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{l} b_l \frac{\partial u}{\partial x_l} + cu = 0 \quad ,$$

with a corresponding nonhomogeneous partial differential equation,

$$\sum_{jk} a_{jk} \frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{l} b_l \frac{\partial u}{\partial x_l} + cu = f$$

where f is some nonzero function. To keep our discussion reasonably brief, let us limit ourselves to problems involving one spatial variable x and one temporal variable t.

For example, we might take our original heat flow problem on a rod of length L,

Find the solution $u = u(x,t)$ to the heat equation

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t \tag{22.1a}$$

that also satisfies the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0 \qquad \text{for} \quad 0 < t \tag{22.1b}$$

and the initial conditions

$$u(x,0) = u_0(x) \qquad \text{for} \quad 0 < x < L \tag{22.1c}$$

where u_0 is some known function (precisely what u_0 is will not be important in these discussions).

and, letting f denote any reasonable function of x and y, consider

Find the solution $u = u(x,t)$ to the heat equation

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t \tag{22.2a}$$

that also satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \quad \text{for} \quad 0 < t \tag{22.2b}$$

and the initial condition

$$u(x,0) = u_0(x) \quad \text{for} \quad 0 < x < L \tag{22.2c}$$

where u_0 is some known function (precisely what u_0 is will not be important in these discussions).

The only change we are making is the addition of the f in the partial differential equation. This $f(x,t)$ describes the generation of heat or cold at position x and time t due to, say, radioactivity of the material making up the rod, laser heating of spots of the rod, tiny furnaces or refrigerators installed in the rod, or magic.

One thing we are not changing (in this section, at least) are the boundary conditions. We want the same Sturm-Liouville appropriate boundary conditions as in the corresponding homogeneous problem. The reason is that we will be using the set of eigenfunctions

$$\{\phi_1(x),\ \phi_2(x),\ \phi_3(x),\ \ldots\}$$

obtained in solving the boundary-value/eigen-problem arising in the corresponding homogeneous problem.

Now here are the basic ideas: For each value of t, we can treat $f(x,t)$ and $u(x,t)$ as functions of just x. Since our set of eigenfunctions $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$ is a complete orthogonal set (with respect to some weight function $w(x)$ on our interval of interest (a,b)), we can express $f(x,t)$ and $u(x,t)$ in terms of these eigenfunctions,

$$f(x,t) = \sum_k f_k(t)\,\phi_k(x) \tag{22.3}$$

and

$$u(x,t) = \sum_k g_k(t)\,\phi_k(x) \quad . \tag{22.4}$$

Remember that the use of these eigenfunctions should ensure that $u(x,t)$ satisfies the given (Sturm-Liouville appropriate) boundary conditions.

The $f_k(t)$'s are the (generalized) Fourier coefficients of $f(x,t)$ with respect to the ϕ_k's at time t. For each t, they can be computed using the standard formula for computing these coefficients,

$$f_k(t) = \frac{\langle\, \phi_k(x) \mid f(x,t)\, \rangle}{\|\phi_k\|^2} = \frac{\int_a^b \phi^*(x) f(x,t) w(x)\, dx}{\int_a^b |\phi(x)|^2\, w(x)\, dx} \quad .$$

Of course, the easiest case will be when f doen't really depend on t. Then the f_k's will simply be constants.

Since $u(x,t)$ is the unknown function we are trying to find, we cannot compute the $g_k(t)$'s as we computed the $f_k(t)$'s. Often, though, you will find that plugging $u_k(x,t) = g_k(t)\phi_k(x)$ for u into the left side of the partial differential equation will yield

$$[\text{formula of } g_k(t) \text{ and its derivatives}] \times \phi_k(x) \quad .$$

This is because the ϕ_k's are eigenfunctions from the corresponding boundary-value problem. Consequently, replacing $u(x,t)$ and $f(x,t)$ in the given partial differential equation with their eigenfunction expansions will yield something like

$$\sum_k [\text{formula of } g_k(t) \text{ and its derivatives}] \times \phi_k(x) \;=\; \sum_k f_k(t)\,\phi_k(x) \quad .$$

The uniqueness of the generalized Fourier coefficients then assures us that, for each k, we must have

$$\text{formula of } g_k(t) \text{ and its derivatives} \;=\; f_k(t) \quad .$$

That is, we have a differential equation for each g_k. Solving this differential equation will give the formula for each $g_k(t)$, a formula which will involve an arbitrary constant or two. These constants can then be found as they were found in the homogeneous case, by setting

$$u_0(x) \;=\; u(x,0) \;=\; \sum_k g_k(0)\phi_k(x)$$

and using the fact that we must then have

$$g_k(0) \;=\; \frac{\langle\, \phi_k \mid u_0 \,\rangle}{\|\phi_k\|^2} \quad .$$

(This assumes $u(x,0) = u_0(x)$ is *the* initial condition. Obvious adjustments must be made if there are other initial conditions.)

!►*Example 22.1:* *Let us solve the sample problem given above assuming $f(x,t)$ depends only on x. That is, assume $f(x)$ and $u_0(x)$ are any two known 'reasonable' functions of x, and let's find the solution $u = u(x,t)$ to the heat equation*

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} \;=\; f(x,t) \qquad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t \tag{22.5}$$

that also satisfies the boundary conditions

$$u(0,t) \;=\; 0 \qquad \text{and} \qquad u(L,t) \;=\; 0 \qquad \text{for} \quad 0 < t \tag{22.6}$$

and the initial condition

$$u(x,0) \;=\; u_0(x) \qquad \text{for} \quad 0 < x < L \quad . \tag{22.7}$$

In solving the corresponding homogeneous problem (with $f \equiv 0$) we got the complete set of eigenfunctions

$$\{\, \phi_1(x),\ \phi_2(x),\ \phi_3(x),\ \dots \}$$

given by

$$\phi_k(x) \;=\; \sin\!\left(\frac{k\pi}{L}x\right) \qquad \text{for} \quad k = 1, 2, 3, \dots \quad .$$

The corresponding eigenfunction expansion of the supposedly known function f is

$$f(x,t) \;=\; \sum_{k=1}^{\infty} f_k(t)\sin\!\left(\frac{k\pi}{L}x\right)$$

where

$$f_k(t) = \frac{\left\langle \sin\left(\frac{k\pi}{L}x\right) \;\middle|\; f(x,t) \right\rangle}{\left\|\sin\left(\frac{k\pi}{L}x\right)\right\|^2} = \cdots = \frac{2}{L}\int_0^L f(x,t)\sin\left(\frac{k\pi}{L}x\right)\,dx \quad .$$

The corresponding eigenfunction expansion for $u(x,t)$ *for each* $t > 0$ *is*

$$u(x,t) = \sum_{k=1}^{\infty} g_k(t)\sin\left(\frac{k\pi}{L}x\right)$$

where the g_k*'s are functions to be determined. Now observe what happens if we replace* u *with*

$$u_k(x,t) = g_k(t)\phi_k(x) = g_k(t)\sin\left(\frac{k\pi}{L}x\right)$$

in the the left side of our partial differential equation:

$$\begin{aligned}
\frac{\partial u_k}{\partial t} - 6\frac{\partial^2 u_k}{\partial x^2} &= \frac{\partial}{\partial t}\left[g_k(t)\sin\left(\frac{k\pi}{L}x\right)\right] - 6\frac{\partial^2}{\partial x^2}\left[g_k(t)\sin\left(\frac{k\pi}{L}x\right)\right] \\
&= \frac{dg_k}{dt}\sin\left(\frac{k\pi}{L}x\right) - 6g_k(t)\left[-\left(\frac{k\pi}{L}\right)^2\sin\left(\frac{k\pi}{L}x\right)\right] \\
&= \left[\frac{dg_k}{dt} + 6\left(\frac{k\pi}{L}\right)^2 g_k(t)\right]\sin\left(\frac{k\pi}{L}x\right) \quad .
\end{aligned}$$

So we do get that

"*some formula of* $g_k(t) \times \phi_k(x)$ "

as desired. And thus, using the eigenfunction expansions for u *and* f, *we have*

$$\begin{aligned}
& \frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = f(x,t) \\
\Longrightarrow\quad & \frac{\partial}{\partial t}\left[\sum_{k=1}^{\infty} u_k(x,t)\right] - 6\frac{\partial^2}{\partial x^2}\left[\sum_{k=1}^{\infty} u_k(x,t)\right] = \sum_{k=1}^{\infty} f_k(t)\sin\left(\frac{k\pi}{L}x\right) \\
\Longrightarrow\quad & \sum_{k=1}^{\infty}\left[\frac{\partial u_k}{\partial t} - 6\frac{\partial^2 u_k}{\partial x^2}\right] = \sum_{k=1}^{\infty} f_k(t)\sin\left(\frac{k\pi}{L}x\right) \\
\Longrightarrow\quad & \sum_{k=1}^{\infty}\left[\frac{dg_k}{dt} + 6\left(\frac{k\pi}{L}\right)^2 g_k(t)\right]\sin\left(\frac{k\pi}{L}x\right) = \sum_{k=1}^{\infty} f_k(t)\sin\left(\frac{k\pi}{L}x\right) \quad .
\end{aligned}$$

Hence, for each k, *we must have*

$$\frac{dg_k}{dt} + 6\left(\frac{k\pi}{L}\right)^2 g_k(t) = f_k(t) \quad .$$

Recalling what the corresponding eigenvalues were, we see that we can also write this as

$$\frac{dg_k}{dt} + 6\lambda_k g_k(t) = f_k(t) \qquad \text{where} \quad \lambda_k = \left(\frac{k\pi}{L}\right)^2 \quad . \tag{22.8}$$

The above is a first-order linear ordinary differential equation that can be solved using the integrating factor $e^{6\lambda_k t}$ *.*[1] *Its solution is*

$$g_k(t) \ = \ e^{6\lambda_k t} \int f(t) e^{-6\lambda_k t} \, dt$$

which we can rewrite using a definite integral as

$$g_k(t) \ = \ e^{-6\lambda_k t} \int_{s=0}^{t} f(s) e^{6\lambda_k s} \, ds \ + \ c_k e^{-6\lambda_k t} \tag{22.9}$$

where c_k *is an arbitrary constant. For brevity, let's further rewrite this as*

$$g_k(t) \ = \ g_{0,k}(t) \ + \ c_k e^{-6\lambda_k t} \qquad \text{with} \quad g_{0,k}(t) \ = \ e^{-6\lambda_k t} \int_{s=0}^{t} f(s) e^{6\lambda_k s} \, ds \quad ,$$

and observe that

$$g_{0,k}(0) \ = \ e^{-6\lambda_k \cdot 0} \int_{s=0}^{0} f(s) e^{6\lambda_k s} \, ds \ = \ 0 \quad ,$$

which, in turn, gives us

$$g_k(0) \ = \ g_{0,k}(0) \ + \ c_k e^{-6\lambda_k \cdot 0} \ = \ c_k \quad .$$

Thus,

$$\begin{aligned} u(x,t) \ &= \ \sum_{k=1}^{\infty} g_k(t) \sin\left(\frac{k\pi}{L} x\right) \\ &= \ \sum_{k=1}^{\infty} \left[g_{0,k}(t) \ + \ c_k e^{-6\lambda_k t} \right] \sin\left(\frac{k\pi}{L} x\right) \quad . \end{aligned} \tag{22.10a}$$

with

$$g_{0,k}(t) \ = \ e^{-6\lambda_k t} \int_{s=0}^{t} f(s) e^{6\lambda_k s} \, ds \tag{22.10b}$$

Finally, we apply the initial condition:

$$\begin{aligned} u_0(x) \ = \ u(x,0) \ &= \ \sum_{k=1}^{\infty} \left[g_{0,k}(0) \ + \ c_k e^{-6\lambda_k \cdot 0} \right] \sin\left(\frac{k\pi}{L} x\right) \\ &= \ \sum_{k=1}^{\infty} [0 + c_k] \sin\left(\frac{k\pi}{L} x\right) \\ &= \ \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{L} x\right) \quad . \end{aligned}$$

Clearly, the c_k*'s here are the Fourier sine series coefficents for* u_0 *,*

$$c_k \ = \ \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{k\pi}{L} x\right) \, dx \quad .$$

[1] You may want to dash back to *Appendix A* (*A Brief Review of Elementary Ordinary Differential Equations*) to review how to solve these equations.

Combining this with formula (22.10) for $u(x,t)$ *then yields our final solution:*

$$u(x,t) = \sum_{k=1}^{\infty} \left[g_{0,k}(t) + c_k e^{-6\lambda_k t}\right] \sin\left(\frac{k\pi}{L}x\right) \tag{22.11a}$$

where

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2 \quad , \tag{22.11b}$$

$$g_{0,k}(t) = e^{-6\lambda_k t} \int_{s=0}^{t} f(s) e^{6\lambda_k s}\, ds \tag{22.11c}$$

and

$$c_k = \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{k\pi}{L}x\right) dx \quad . \tag{22.11d}$$

The final set of formulas for the generic solution given above used a particular form for the solution to the ordinary differential equations given in (22.8). Rather than memorize the above, just remember the general process, and don't worry about using any particular form for solving those ordinary differential equations.

!►Example 22.2: *Let's find the solution* $u = u(x,t)$ *to the heat equation*

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 1 \qquad \text{for} \quad 0 < x < 3 \quad \text{and} \quad 0 < t$$

that also satisfies the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(3,t) = 0 \qquad \text{for} \quad 0 < t$$

and the initial condition

$$u(x,0) = 0 \qquad \text{for} \quad 0 < x < 3 \quad .$$

This is just what we considered in the last example, but with $L = 3$ *,* $f = f(x) = 1$ *and* $u_0(x) = 0$ *. The corresponding eigenfunction expansions of this* f *and* u_0 *,*

$$1 = f(x) = \sum_{k=1}^{\infty} f_k \sin\left(\frac{k\pi}{3}x\right)$$

and

$$0 = u_0(x) = \sum_{k=1}^{\infty} u_{0,k} \sin\left(\frac{k\pi}{3}x\right) \quad ,$$

are easily found:

$$f_k = \frac{2}{3}\int_0^3 1 \cdot \sin\left(\frac{k\pi}{3}x\right) dx = \cdots = \frac{2\left[1-(-1)^k\right]}{k\pi}$$

and

$$u_{0,k} = 0 \quad .$$

The corresponding eigenfunction expansion for $u(x,t)$ for each $t > 0$ is

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) \qquad \text{with} \quad u_k(x,t) = g_k(t)\sin\left(\frac{k\pi}{3}x\right)$$

where the g_k's are functions to be determined. As before, we have

$$\begin{aligned}
\frac{\partial u_k}{\partial t} - 6\frac{\partial^2 u_k}{\partial x^2} &= \frac{\partial}{\partial t}\left[g_k(t)\sin\left(\frac{k\pi}{3}x\right)\right] - 6\frac{\partial^2}{\partial x^2}\left[g_k(t)\sin\left(\frac{k\pi}{3}x\right)\right] \\
&= \frac{dg_k}{dt}\sin\left(\frac{k\pi}{3}x\right) - 6g_k(t)\left[-\left(\frac{k\pi}{3}\right)^2\sin\left(\frac{k\pi}{3}x\right)\right] \\
&= \left[\frac{dg_k}{dt} + 6\left(\frac{k\pi}{3}\right)^2 g_k(t)\right]\sin\left(\frac{k\pi}{3}x\right) \quad .
\end{aligned}$$

From this, the fact that $\lambda_k = \left(\frac{k\pi}{3}\right)^2$ and our sine series for f, we then get

$$\begin{aligned}
& \frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 1 \\
\Longrightarrow \quad & \frac{\partial}{\partial t}\left[\sum_{k=1}^{\infty} u_k(x,t)\right] - 6\frac{\partial^2}{\partial x^2}\left[\sum_{k=1}^{\infty} u_k(x,t)\right] = \sum_{k=1}^{\infty} f_k \sin\left(\frac{k\pi}{3}x\right) \\
\Longrightarrow \quad & \sum_{k=1}^{\infty}\left[\frac{\partial u_k}{\partial t} - 6\frac{\partial^2 u_k}{\partial x^2}\right] = \sum_{k=1}^{\infty} f_k \sin\left(\frac{k\pi}{3}x\right) \\
\Longrightarrow \quad & \sum_{k=1}^{\infty}\left[\frac{dg_k}{dt} + 6\lambda_k g_k(t)\right]\sin\left(\frac{k\pi}{3}x\right) = \sum_{k=1}^{\infty} f_k \sin\left(\frac{k\pi}{3}x\right) \quad .
\end{aligned}$$

where

$$f_k = \frac{2\left[1-(-1)^k\right]}{k\pi} \quad .$$

Consequently, each $g_k(t)$ must satisfy

$$\frac{dg_k}{dt} + 6\lambda_k g_k(t) = f_k \quad .$$

Accidentally forgetting that we had already solved a more general version of this, we re-solve this simple, first-order linear differential equation, obtaining (this time)

$$\begin{aligned}
g_k(t) &= e^{-6\lambda_k t}\int f_k e^{6\lambda_k t}\,dt \\
&= e^{-6\lambda_k t}\left[\frac{f_k}{6\lambda_k}e^{6\lambda_k t} + c_k\right] = \frac{f_k}{6\lambda_k} + c_k e^{-6\lambda_k t}
\end{aligned}$$

Replacing f_k and λ_k in the first term with their values, this reduces to

$$g_k(t) = \frac{3\left[1-(-1)^k\right]}{(k\pi)^3} + c_k e^{-6\lambda_k t} \quad .$$

Thus,

$$u(x,t) = \sum_{k=1}^{\infty} g_k(t)\sin\left(\frac{k\pi}{L}x\right) = \sum_{k=1}^{\infty}\left[\frac{3\left[1-(-1)^k\right]}{(k\pi)^3} + c_k e^{-6\lambda_k t}\right]\sin\left(\frac{k\pi}{3}x\right) \quad .$$

Finally, we apply our initial condition:

$$0 = u(x,0) = \sum_{k=1}^{\infty}\left[\frac{3\left[1-(-1)^k\right]}{(k\pi)^3} + c_k e^{-6\lambda_k 0}\right]\sin\left(\frac{k\pi}{3}x\right)$$

$$= \sum_{k=1}^{\infty}\left[\frac{3\left[1-(-1)^k\right]}{(k\pi)^3} + c_k\right]\sin\left(\frac{k\pi}{3}x\right) \quad .$$

Since each term must be zero, we must have

$$c_k = -\frac{3\left[1-(-1)^k\right]}{(k\pi)^3} \qquad \text{for} \quad k = 1,\ 2,\ 3,\ \dots \quad .$$

Plugging this back into our last formula for $u(x,t)$ and simplifying slightly, we get

$$u(x,t) = \sum_{k=1}^{\infty}\frac{3\left[1-(-1)^k\right]}{\pi^3 k^3}\left[1-e^{-6\lambda_k t}\right]\sin\left(\frac{k\pi}{3}x\right) \qquad \text{with} \quad \lambda_k = \left(\frac{k\pi}{3}\right)^2 \quad . \tag{22.12}$$

It's may be worth noting that the exponential terms shrink to zero rapidly as t increases, and that the k^{-3} terms also shrink quickly to zero as k increases. So, for 'large t',

$$u(x,t) \approx \sum_{k=1}^{\infty}\frac{3\left[1-(-1)^k\right]}{\pi^3 k^3}[1-0]\sin\left(\frac{k\pi}{3}x\right)$$

$$= \frac{3\left[1-(-1)^1\right]}{\pi^3\cdot 1^3}\sin\left(\frac{1\pi}{3}x\right) + \frac{3\left[1-(-1)^2\right]}{\pi^3 2^3}\sin\left(\frac{2\pi}{3}x\right) + \frac{3\left[1-(-1)^3\right]}{\pi^3 3^3}\sin\left(\frac{3\pi}{3}x\right) + \cdots$$

$$\approx \frac{6}{\pi^3}\sin\left(\frac{\pi}{3}x\right) \quad .$$

?► ***Exercise 22.1:*** *Consider the problem in example 22.1.*

a: *Solve it assuming $L = 3$, $u_0 \equiv 0$ and*

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 3/2 \\ 0 & \text{if} \quad 3/2 < x \le 3 \end{cases} \quad .$$

b: *As t gets large, what does $u(x,t)$ become? (Try graphing it, using a computer if necessary.)*

c: *Is the initial condition u_0 really relevant to the solution of the problem when t is large?*

?► ***Exercise 22.2:*** *Let* α *denote some real value, and consider the problem of finding the solution* $u = u(x,t)$ *to the heat equation*

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = e^{\alpha t} f(x) \qquad \text{for} \quad 0 < x < 3 \quad \text{and} \quad 0 < t$$

that also satisfies the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(3,t) = 0 \qquad \text{for} \quad 0 < t$$

and the initial condition

$$u(x,0) = u_0(x) \qquad \text{for} \quad 0 < x < 3 \quad .$$

Assume $f(x)$ *and* $u_0(x)$ *are 'reasonable' known functions on* $(0, L)$ *.*

a: *Find the formula (analogous to formula (22.11)) for the solution* $u(x,t)$ *.*

b: *What happens to* $u(x,t)$ *as* $t \to \infty$ *when* $\alpha > 0$ *?*

c: *What happens to* $u(x,t)$ *as* $t \to \infty$ *when* $\alpha < 0$ *?*

?► ***Exercise 22.3:*** *Consider the problem of finding the solution* $u = u(x,t)$ *to the heat equation*

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = f(x) \qquad \text{for} \quad 0 < x < 3 \quad \text{and} \quad 0 < t$$

that also satisfies the boundary conditions

$$u_x(0,t) = 0 \qquad \text{and} \qquad u_x(3,t) = 0 \qquad \text{for} \quad 0 < t$$

and the initial condition

$$u(x,0) = u_0(x) \qquad \text{for} \quad 0 < x < 3 \quad .$$

Assume $f(x)$ *and* $u_0(x)$ *are 'reasonable' known functions on* $(0, L)$ *.*

a: *Find the formula (analogous to formula (22.11)) for the solution* $u(x,t)$ *.*

b: *What happens to* $u(x,t)$ *as* $t \to \infty$ *?*