9

Multidimensional Calculus: Mainly Differential Theory

In the following, we will attempt to quickly develop the basic differential theory of calculus in multidimensional spaces. You've probably already seen much of this theory. Hopefully, we will develop a better understanding of the material than is usually imparted in the more elementary treatments, and see how to extend it to more general spaces and coordinate systems.

By the way, in keeping with the common practice in physics of denoting the position of a moving object by $\boldsymbol{r}$, I will relent and often use this notation instead of $\boldsymbol{x}$.

9.1 Motion, Curves, Arclength, Acceleration and the Christoffel Symbols

For all the following, assume we are considering motion in some space of positions $\mathcal{S}$, and that

$$\left\{\left(x^1, x^2, \ldots, x^N\right)\right\}$$

is any coordinate system for this space. As before,

$$\{h_1, h_2, \ldots, h_N\} \quad , \quad \{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \ldots, \boldsymbol{\varepsilon}_N\} \quad \text{and} \quad \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$$

are the associated scaling factors, tangent vectors and normalized tangent vectors.

We also assume that we have some position-valued function $\boldsymbol{r}(t)$ that traces out a curve C as t varies over some interval (t_0, t_1). Since this is a math/physics course, we naturally view $\boldsymbol{r}(t)$ as being the position of an object, say, George the Gerbil, at time t. In terms of our coordinate system, we have some coordinate formula for $\boldsymbol{r}$

$$\boldsymbol{r}(t) \sim \left(x^1(t),\ x^2(t),\ \ldots,\ x^N(t)\right) \quad \text{for} \quad t_0 < t < t_1 \quad .$$

Velocity and Speed

The formula for velocity $\mathbf{v}$ at any given time is easily computed using the chain rule:

$$\mathbf{v} = \frac{d\boldsymbol{r}}{dt} = \sum_{i=1}^{N} \frac{\partial \boldsymbol{r}}{\partial x^i}\frac{dx^i}{dt} = \sum_{i=1}^{N} h_i \mathbf{e}_i \frac{dx^i}{dt} \quad ,$$

which we may prefer to rewrite as

$$\mathbf{v} = \frac{d\boldsymbol{r}}{dt} = \sum_{i=1}^{N} h_i \frac{dx^i}{dt}\mathbf{e}_i \qquad \text{or} \qquad \mathbf{v} = \frac{d\boldsymbol{r}}{dt} = \sum_{i=1}^{N} \frac{dx^i}{dt}\boldsymbol{\varepsilon}_i \quad .$$

The corresponding speed, then, is

$$\frac{ds}{dt} = \left\| \frac{d\boldsymbol{r}}{dt} \right\| = \sqrt{\left[\sum_{i=1}^{N} h_i \frac{dx^i}{dt}\mathbf{e}_i\right] \cdot \left[\sum_{j=1}^{N} h_j \frac{dx^j}{dt}\mathbf{e}_j\right]} \quad .$$

That is,

$$\frac{ds}{dt} = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} h_i h_j \frac{dx^i}{dt}\frac{dx^j}{dt}\,\mathbf{e}_i \cdot \mathbf{e}_j} = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{dx^i}{dt}\frac{dx^j}{dt}\,g_{ij}} \quad . \tag{9.1}$$

If the coordinate system is orthogonal, this reduces to

$$\frac{ds}{dt} = \sqrt{\sum_{i=1}^{N} \left(h_i \frac{dx^i}{dt}\right)^2} \quad . \tag{9.2}$$

!►*Example 9.1:* *Assume that, using polar coordinates in the plane, the position of an object at time t is given by*

$$\boldsymbol{r}(t) \sim \big(\rho(t),\, \phi(t)\big) = \big(t^2,\, 2\pi t\big) \quad .$$

Then the velocity at time t is

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= h_\rho \frac{d\rho}{dt}\,\mathbf{e}_\rho + h_\phi \frac{d\phi}{dt}\,\mathbf{e}_\phi \\ &= 1\left(\frac{d}{dt}\left[t^2\right]\right)\mathbf{e}_\rho + \rho\left(\frac{d}{dt}\left[2\pi t\right]\right)\mathbf{e}_\phi \\ &= 1(2t)\,\mathbf{e}_\rho + t^2(2\pi)\,\mathbf{e}_\phi = 2t\,\mathbf{e}_\rho + 2\pi t^2\,\mathbf{e}_\phi\end{aligned}$$

and the corresponding speed is

$$\frac{ds}{dt} = \left\|\frac{d\boldsymbol{r}}{dt}\right\| = \sqrt{(2t)^2 + \left(2\pi t^2\right)^2} = 2t\sqrt{1 + \pi^2 t^2} \quad .$$

?►*Exercise 9.1:* *In the above example and exercises, I've completely forgotten about explicitly stating the general polar coordinate formulas for velocity $\mathbf{v}$ and speed for a given*

$$\boldsymbol{r}(t) = (\rho(t),\, \phi(t)) \quad .$$

So you should derive and state them. That is, show that

$$\mathbf{v} = \frac{d\rho}{dt}\mathbf{e}_\rho + \rho\frac{d\phi}{dt}\mathbf{e}_\phi \qquad \textit{and} \qquad \frac{ds}{dt} = \sqrt{\left[\frac{d\rho}{dt}\right]^2 + \rho^2\left[\frac{d\phi}{dt}\right]^2} \quad .$$

?►*Exercise 9.2:* *Let $\boldsymbol{r}(t)$ be the position at time t of George the Gerbil in a three-dimensional Euclidean space, and derive the spherical coordinate formulas for velocity and speed. (Compare your results to the formulas given in problem 3.10.27 on page 200 of Arfken, Weber & Harris. Ignore their hint for deriving these things!)*

Distance Traveled / Arclength

The distance traveled by our object (George) as t goes from t_0 to t_1 (i.e., the arclength of the curve traced out by the object) is simply the integral of the speed,

$$\text{arclength} = \int_{t_0}^{t_1} \frac{ds}{dt}\,dt = \int_{t_0}^{t_1} \|\mathbf{v}(t)\|\,dt \quad .$$

In general, the coordinate formula for this is

$$\text{arclength} = \int_{t=t_0}^{t_1} \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} h_i h_j \frac{dx^i}{dt}\frac{dx^j}{dt}\,\mathbf{e}_i \cdot \mathbf{e}_j}\,dt = \int_{t=t_0}^{t_1} \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{dx^i}{dt}\frac{dx^j}{dt}\,g_{ij}}\,dt \quad ,$$

which reduces to

$$\text{arclength} = \int_{t=t_0}^{t_1} \sqrt{\sum_{i=1}^{N}\left(h_i \frac{dx^i}{dt}\right)^2}\,dt$$

when the coordinate system is orthogonal, and further reduces to

$$\text{arclength} = \int_{t=t_0}^{t_1} \sqrt{\sum_{i=1}^{N}\left(\frac{dx^i}{dt}\right)^2}\,dt$$

if the space is Euclidean and the coordinate system is Cartesian.

!►*Example 9.2:* *Consider the curve C parameterized in polar coordinates by*

$$\boldsymbol{r}(t) \sim \big(\rho(t),\, \phi(t)\big) = \big(t^2,\, 2\pi t\big) \qquad \text{for} \quad 0 < t < 2 \quad ,$$

as in example 9.1 on page 9–2. Its arclength is

$$\begin{aligned}
\text{arclength} &= \int_{t=t_0}^{t_1} \sqrt{\sum_{i=1}^{N}\left(h_i \frac{dx^i}{dt}\right)^2}\,dt \\
&= \int_0^2 \sqrt{\left(h_\rho \frac{d\rho}{dt}\right)^2 + \left(h_\phi \frac{d\phi}{dt}\right)^2}\,dt \\
&= \int_0^2 \sqrt{\left(1\,\frac{d}{dt}\left[t^2\right]\right)^2 + \left(\rho\,\frac{d}{dt}\left[2\pi t\right]\right)^2}\,dt \\
&= \int_0^2 2t\sqrt{1+\pi^2 t^2}\,dt \\
&= \frac{2}{3\pi^2}\left(1+\pi^2 t^2\right)^{3/2}\Big|_0^2 = \frac{2}{3\pi^2}\left[\left(1+4\pi^2\right)^{3/2} - 1\right] \quad .
\end{aligned}$$

Acceleration and the Christoffel Symbols

Computing Acceleration

The acceleration $\mathbf{a}$ of George at each instant of t is, of course,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\sum_{i=1}^{N} h_i \mathbf{e}_i \frac{dx^i}{dt} = \sum_{i=1}^{N} \frac{d}{dt}\left[\frac{dx^i}{dt} h_i \mathbf{e}_i\right] \quad .$$

If the coordinate system is basis based, then the scaling factors and local basis vectors do not vary, and the above simplifies to

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \sum_{i=1}^{N} \frac{d^2x^i}{dt^2} h_i \mathbf{e}_i \quad .$$

This is certainly the case when our coordinate system is Cartesian. If, however, the scaling factor and/or local basis vectors vary from point to point (as with polar coordinates), then we must use the product rule

$$\frac{d}{dt}\left[\frac{dx^i}{dt} h_i \mathbf{e}_i\right] = \frac{d^2x^i}{dt^2} h_i \mathbf{e}_i + \frac{dx^i}{dt}\frac{d[h_i \mathbf{e}_i]}{dt} \quad .$$

yielding

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \sum_{i=1}^{N}\left[\frac{d^2x^i}{dt^2} h_i \mathbf{e}_i + \frac{dx^i}{dt}\frac{d[h_i \mathbf{e}_i]}{dt}\right] \quad .$$

Recalling that

$$\boldsymbol{\varepsilon}_i = h_i \mathbf{e}_i \quad ,$$

we can rewrite the above in the slightly more convenient form

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \sum_{i=1}^{N}\left[\frac{d^2x^i}{dt^2} \boldsymbol{\varepsilon}_i + \frac{dx^i}{dt}\frac{d\boldsymbol{\varepsilon}_i}{dt}\right] \quad .$$

Applying the chain rule to the derivatives of the $\boldsymbol{\varepsilon}_i$'s, we get[1]

$$\frac{dx^i}{dt}\frac{d\boldsymbol{\varepsilon}_i}{dt} = \frac{dx^i}{dt}\sum_{k=1}^{N}\frac{dx^k}{dt}\frac{\partial \boldsymbol{\varepsilon}_i}{\partial x^k} = \sum_{k=1}^{N}\frac{dx^i}{dt}\frac{dx^k}{dt}\frac{\partial \boldsymbol{\varepsilon}_i}{\partial x^k} \quad .$$

So,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \sum_{i=1}^{N}\left[\frac{d^2x^i}{dt^2} \boldsymbol{\varepsilon}_i + \sum_{k=1}^{N}\frac{dx^i}{dt}\frac{dx^k}{dt}\frac{\partial \boldsymbol{\varepsilon}_i}{\partial x^k}\right] \quad . \tag{9.3}$$

The last acceleration formula is probably the most important "new" formula of this section. Rather than memorizing it, you can always easily rederive it using the product and chain rules from calculus.

However, to use formula (9.3), we still need to know each $\partial \boldsymbol{\varepsilon}_i / \partial x^k$. Since this a partial derivative of the vector-valued function $\boldsymbol{\varepsilon}_i$, it is, itself, a vector-valued function. That is, for

[1] I'm cheating. We've not verified the chain rule for the differentiation of vector-valued functions, only for scalar- and position-valued functions. The chain rule used here with a vector-valued function can be rigorously justified in a Euclidean space or in a non-Euclidean space contained in a Euclidean space (trust me). Justifying it in a more general space is more involved and requires clever definitions.

each position $\boldsymbol{p}$ in $\mathcal{S}$, each ${}^{\partial \boldsymbol{\varepsilon}_i}/{}_{\partial x^k}$ is a vector in the tangent vector space at that point $\boldsymbol{p}$. So each ${}^{\partial \boldsymbol{\varepsilon}_i}/{}_{\partial x^k}$ can be expressed in terms of the $\{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \ldots, \boldsymbol{\varepsilon}_N\}$ basis at each point in space. This means that, for each point of the space $\mathcal{S}$, there must be corresponding scalars Γ^1_{ik}, Γ^2_{ik}, $\ldots$ and Γ^N_{ik} such that

$$\frac{\partial \boldsymbol{\varepsilon}_i}{\partial x^k} = \sum_{m=1}^{N} \Gamma^m_{ik} \, \boldsymbol{\varepsilon}_m \quad . \tag{9.4}$$

These Γ^m_{ik}'s are called the *Christoffel symbols* (for the coordinate system).[2] They are just the components of each ${}^{\partial \boldsymbol{\varepsilon}_i}/{}_{\partial x^k}$ with respect to the $\{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \ldots, \boldsymbol{\varepsilon}_N\}$ basis at each point.

Using the Christoffel symbols, formula (9.3) becomes

$$\begin{aligned}\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \sum_{i=1}^{N} \left[\frac{d^2 x^i}{dt^2} \boldsymbol{\varepsilon}_i + \sum_{k=1}^{N} \sum_{m=1}^{N} \frac{dx^i}{dt} \frac{dx^k}{dt} \Gamma^m_{ik} \boldsymbol{\varepsilon}_m \right] \\ &= \sum_{i=1}^{N} \frac{d^2 x^i}{dt^2} \boldsymbol{\varepsilon}_i + \sum_{i=1}^{N} \sum_{k=1}^{N} \sum_{m=1}^{N} \frac{dx^i}{dt} \frac{dx^k}{dt} \Gamma^m_{ik} \boldsymbol{\varepsilon}_m \quad .\end{aligned}$$

With a little relabeling of the indices, we have

$$\sum_{i=1}^{N} \sum_{k=1}^{N} \sum_{m=1}^{N} \frac{dx^i}{dt} \frac{dx^k}{dt} \Gamma^m_{ik} \boldsymbol{\varepsilon}_m = \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{i=1}^{N} \frac{dx^j}{dt} \frac{dx^k}{dt} \Gamma^i_{jk} \boldsymbol{\varepsilon}_i \quad ,$$

which allows us to rewrite the last formula for acceleration as

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \sum_{i=1}^{N} \left[\frac{d^2 x^i}{dt^2} + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{dx^j}{dt} \frac{dx^k}{dt} \Gamma^i_{jk} \right] \boldsymbol{\varepsilon}_i \quad . \tag{9.5}$$

Equivalently,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \sum_{i=1}^{N} \left[\frac{d^2 x^i}{dt^2} + \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{dx^j}{dt} \frac{dx^k}{dt} \Gamma^i_{jk} \right] h_i \mathbf{e}_i \quad . \tag{9.5$'$}$$

Of course, for the above to be useful, we need to find the values of the Christoffel symbols.

Computing ${}^{\partial \boldsymbol{\varepsilon}_i}/{}_{\partial x^k}$ and the Christoffel Symbols

Since the Christoffel symbols are, at each point in $\mathcal{S}$, components of vectors with respect to the basis $\{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \ldots, \boldsymbol{\varepsilon}_N\}$, we can use what we learn earlier this term about “finding components” to find the Γ^i_{jk}'s, provided we can, somehow, compute each

$$\frac{\partial \boldsymbol{\varepsilon}_j}{\partial x^k} \quad .$$

The simplest case is where the coordinate system is basis based. Then the scaling factors and local basis vectors do not vary. Thus,

$$\frac{\partial \boldsymbol{\varepsilon}_j}{\partial x^k} = 0 \quad ,$$

[2] More precisely, they are Christoffel symbols of the second kind.

and equation (9.4) becomes

$$0 = \frac{\partial \boldsymbol{\varepsilon}_j}{\partial x^k} = \sum_{i=1}^{N} \Gamma^i_{jk}\, \boldsymbol{\varepsilon}_i \quad ,$$

which means that, for every j, k and i,

$$\Gamma^i_{jk} = 0 \quad .$$

Remember, this is only if the coordinate system is basis based.

More generally, if the coordinate system is orthogonal, then (as you showed in 2.19 on page 2–15[3]), the Γ^i_{jk}'s in

$$\frac{\partial \boldsymbol{\varepsilon}_j}{\partial x^k} = \sum_{i=1}^{N} \Gamma^i_{jk}\, \boldsymbol{\varepsilon}_i$$

are given by

$$\Gamma^i_{jk} = \frac{\dfrac{\partial \boldsymbol{\varepsilon}_j}{\partial x^k} \cdot \boldsymbol{\varepsilon}_i}{\|\boldsymbol{\varepsilon}_i\|^2} \quad .$$

By the definitions of the scaling factors and normalized tangent vectors, this can also be written as

$$\Gamma^i_{jk} = \frac{1}{{h_i}^2} \frac{\partial \boldsymbol{\varepsilon}_j}{\partial x^k} \cdot \boldsymbol{\varepsilon}_i = \frac{1}{h_i} \frac{\partial [h_j \mathbf{e}_j]}{\partial x^k} \cdot \mathbf{e}_i \quad . \tag{9.6}$$

We should also note that we do have some symmetry in these computations. This is because

$$\frac{\partial \boldsymbol{\varepsilon}_j}{\partial x^k} = \frac{\partial}{\partial x^k} \frac{\partial \boldsymbol{r}}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial \boldsymbol{r}}{\partial x^k} = \frac{\partial \boldsymbol{\varepsilon}_k}{\partial x^j} \quad .$$

Hence,

$$\Gamma^i_{jk} = \Gamma^i_{kj} \quad ,$$

a fact that can reduce the number of computations we may have to carry out.

Of course, computing formula (9.6) requires that we can compute this dot product. If the space is Euclidean and you've determined how to express the $\boldsymbol{\varepsilon}_i$'s and $\mathbf{e}_i$'s in terms of the orthonormal basis corresponding to a Cartesian system, then computing the Γ^i_{jk}'s by the above equation is relatively straightforward.

!► *Example 9.3:* *Consider the polar coordinate system* $\{(\rho, \phi)\}$. *In example 8.9 on page 8–33 we saw that*

$$h_\rho = 1 \qquad \text{and} \qquad h_\phi = \rho \quad ,$$

and, in terms of the standard basis associated with the Cartesian system $\{(x, y)\}$,

$$\mathbf{e}_\rho = \cos(\phi)\,\mathbf{i} + \sin(\phi)\,\mathbf{j} \qquad \text{and} \qquad \mathbf{e}_\phi = -\sin(\phi)\,\mathbf{i} + \cos(\phi)\,\mathbf{j} \quad .$$

So,

$$\frac{\partial \boldsymbol{\varepsilon}_\rho}{\partial \rho} = \frac{\partial [h_\rho \mathbf{e}_\rho]}{\partial \rho} = \frac{\partial}{\partial \rho}\left[\cos(\phi)\,\mathbf{i} + \sin(\phi)\,\mathbf{j}\right] = \mathbf{0} \tag{9.7a}$$

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[3] I.e., that if a basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N\}$ is orthogonal, then $\mathbf{v} = \sum_{i=1}^{N} v_i \mathbf{b}_k \implies v_i = \dfrac{\mathbf{v} \cdot \mathbf{b}_i}{\|\mathbf{b}_i\|^2}$.

and

$$\frac{\partial \boldsymbol{\varepsilon}_\rho}{\partial \phi} = \frac{\partial [h_\rho \mathbf{e}_\rho]}{\partial \phi} = \frac{\partial}{\partial \phi}\big[\cos(\phi)\,\mathbf{i} + \sin(\phi)\,\mathbf{j}\big] = -\sin(\phi)\,\mathbf{i} + \cos(\phi)\,\mathbf{j} \quad . \tag{9.7b}$$

Just plugging into formula (9.6), above, we get

$$\Gamma^{\rho}_{\rho\rho} = \frac{1}{h_\rho}\frac{\partial [h_\rho \mathbf{e}_\rho]}{\partial \rho} \cdot \mathbf{e}_\rho = \frac{1}{1}\mathbf{0} \cdot \mathbf{e}_\rho = 0 \quad ,$$

$$\Gamma^{\phi}_{\rho\rho} = \frac{1}{h_\phi}\frac{\partial [h_\rho \mathbf{e}_\rho]}{\partial \rho} \cdot \mathbf{e}_\phi = \frac{1}{\rho}\mathbf{0} \cdot \mathbf{e}_\phi = 0 \quad ,$$

$$\Gamma^{\rho}_{\rho\phi} = \frac{1}{h_\rho}\frac{\partial [h_\rho \mathbf{e}_\rho]}{\partial \phi} \cdot \mathbf{e}_\rho = \frac{1}{1}\big[-\sin(\phi)\,\mathbf{i} + \cos(\phi)\,\mathbf{j}\big] \cdot \big[\cos(\phi)\,\mathbf{i} + \sin(\phi)\,\mathbf{j}\big] = 0$$

and

$$\Gamma^{\phi}_{\rho\phi} = \frac{1}{h_\phi}\frac{\partial [h_\rho \mathbf{e}_\rho]}{\partial \phi} \cdot \mathbf{e}_\phi = \frac{1}{\rho}\big[-\sin(\phi)\,\mathbf{i} + \cos(\phi)\,\mathbf{j}\big] \cdot \big[-\sin(\phi)\,\mathbf{i} + \cos(\phi)\,\mathbf{j}\big] = \frac{1}{\rho} \quad .$$

Thus (using (9.4)),

$$\frac{\partial \boldsymbol{\varepsilon}_\rho}{\partial \rho} = \Gamma^{\rho}_{\rho\rho}\boldsymbol{\varepsilon}_\rho + \Gamma^{\phi}_{\rho\rho}\boldsymbol{\varepsilon}_\phi = 0\boldsymbol{\varepsilon}_\rho + 0\boldsymbol{\varepsilon}_\phi = \mathbf{0}$$

and

$$\frac{\partial \boldsymbol{\varepsilon}_\rho}{\partial \phi} = \Gamma^{\rho}_{\rho\phi}\boldsymbol{\varepsilon}_\rho + \Gamma^{\phi}_{\rho\phi}\boldsymbol{\varepsilon}_\phi = 0\boldsymbol{\varepsilon}_\rho + \frac{1}{\rho}\boldsymbol{\varepsilon}_\phi = \frac{1}{\rho}\boldsymbol{\varepsilon}_\phi \quad .$$

In terms of the normalized tangent vectors,

$$\frac{\partial \boldsymbol{\varepsilon}_\rho}{\partial \rho} = \mathbf{0} \qquad \textit{and} \qquad \frac{\partial \boldsymbol{\varepsilon}_\rho}{\partial \phi} = \mathbf{e}_\phi \quad .$$

Keep in mind that the important quantities are the partial derivatives of the $\boldsymbol{\varepsilon}_j$'s. If we can compute these, as we did in equation set (9.7), above, then the only value of the Christoffel symbols is to help us express those partial derivatives in terms of the coordinate system's tangent vectors $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, ... (instead of $\mathbf{i}$, $\mathbf{j}$, ...). With some coordinate systems, though, you can just look at your initial formulas for these partial derivatives and, without using Christoffel symbols, write out the formulas in terms of the $\boldsymbol{\varepsilon}_j$'s. Certainly, for example, we did not really need to find the Christoffel symbols to derive

$$\frac{\partial \boldsymbol{\varepsilon}_\rho}{\partial \rho} = \mathbf{0} \qquad \text{and} \qquad \frac{\partial \boldsymbol{\varepsilon}_\rho}{\partial \phi} = \frac{1}{\rho}\boldsymbol{\varepsilon}_\phi = \mathbf{e}_\phi$$

from formulas (9.7).

?►*Exercise 9.3:* *Continue the work in the last example. In particular:*

a: *Verify that*

$$\Gamma^{\rho}_{\phi\rho} = 0 \quad , \quad \Gamma^{\phi}_{\phi\rho} = \frac{1}{\rho} \quad , \quad \Gamma^{\rho}_{\phi\phi} = -\rho \quad \textit{and} \quad \Gamma^{\phi}_{\phi\phi} = 0 \quad .$$

b: *Verify that*

$$\frac{\partial \boldsymbol{\varepsilon}_\phi}{\partial \rho} = \frac{1}{\rho}\boldsymbol{\varepsilon}_\phi = \mathbf{e}_\phi \qquad \text{and} \qquad \frac{\partial \boldsymbol{\varepsilon}_\phi}{\partial \phi} = -\rho\boldsymbol{\varepsilon}_\rho = -\rho\mathbf{e}_\rho \quad .$$

two ways: first, by inspecting these partial derivatives computed in terms of $\mathbf{i}$ *and* $\mathbf{j}$*, then using the Christoffel symbols.*

c: *Write out the full formula for acceleration in polar coordinates. Corresponding to a function of position* $\boldsymbol{r}(t) \sim (\rho(t), \phi(t))$*, you should get something like*

$$\mathbf{a} = \left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2\right]\mathbf{e}_\rho + \left[\rho\frac{d^2\phi}{dt^2} + 2\frac{d\rho}{dt}\frac{d\phi}{dt}\right]\mathbf{e}_\phi \quad .$$

d: *Compute the acceleration at time* t *when, in polar coordinates,*

$$\boldsymbol{r}(t) \sim \big(\rho(t), \phi(t)\big) = \big(t^2, 2\pi t\big) \quad ,$$

as in example 9.1.

?►Exercise 9.4 (spherical coordinates): *Let* $\{(r, \theta, \phi)\}$ *be the spherical coordinate system described at the start of section 8.3, page 8–9. Recall that you found the associated scaling factors and tangent vectors (in terms of* $\mathbf{i}$*,* $\mathbf{j}$ *and* $\mathbf{k}$*) in exercise 8.21 on page 8–34.*

a: *Find the nine partial derivatives*

$$\frac{\partial \boldsymbol{\varepsilon}_r}{\partial r} \;,\; \frac{\partial \boldsymbol{\varepsilon}_r}{\partial \theta} \;,\; \frac{\partial \boldsymbol{\varepsilon}_r}{\partial \phi} \;,\; \frac{\partial \boldsymbol{\varepsilon}_\theta}{\partial r} \;,\; \cdots \;\text{ and }\; \frac{\partial \boldsymbol{\varepsilon}_\phi}{\partial \phi} \quad .$$

(You might not need the Christoffel symbols for this.)

b: *How many Christoffel symbols are there? Find a few.*

c: *Let* $\boldsymbol{r}(t) \sim (r(t), \theta(t), \phi(t))$ *be the position at time* t *of George the Gerbil in a three-dimensional Euclidean space, and derive the spherical coordinate formulas for his acceleration. (Compare your results to the formula given in problem 3.10.27 on page 200 of Arfken, Weber & Harris. Again, ignore their hint for deriving this!)*

?►Exercise 9.5: *Once again, consider the circular paraboloid* $\mathcal{S}$ *we discussed in examples 8.3, 8.5, 8.8 and 8.10 (pages 8–16, 8–22 and 8–27 and 8–34). Recall that the coordinate system corresponding to polar coordinates,* $\{(\rho, \phi)\}$ *is orthogonal and that*

$$\boldsymbol{\varepsilon}_\rho = \cos(\phi)\,\mathbf{i} + \sin(\phi)\,\mathbf{j} + 2\rho\mathbf{k} \quad ,$$

$$\boldsymbol{\varepsilon}_\phi = -\rho\sin(\phi)\,\mathbf{i} + \rho\cos(\phi)\,\mathbf{j} \quad ,$$

$$h_\rho = \sqrt{1 + 4\rho^2} \quad ,$$

and

$$h_\phi = \rho \quad .$$

Now, compute the Christoffel symbols, and write out the full formula for acceleration for an object at position $\mathbf{r}(t) \sim (\rho(t), \phi(t))$*.*

Final Comments on Christoffel Symbols and Acceleration

Using the above, we can find formulas for acceleration (and Christoffel symbols) in Euclidean spaces and in nonEuclidean subpaces contained in Euclidean spaces (e.g., curves and spheres in Euclidean three-space), at least when the coordinates on the subspace are a subset of the coordinates of the coordinates of the larger Euclidean space. To compute the Christoffel symbols when we do not have a convenient Cartesian system lying around, we will need to further develop the "metric tensor". We will do that in the next chapter. (If you want, you can glance at the resulting formula in theorem 11.2 on page 11–16.)

9.2 Scalar and Vector Fields

Basic Definitions and Concepts

In the following, all points/positions refer to points in some N-dimensional space $\mathcal{S}$.

A *scalar field* Ψ (on $\mathcal{S}$) is a scalar-valued function of position,

$$\Psi(\boldsymbol{x}) \;=\; \text{some scalar value corresponding to position } \boldsymbol{x} \text{ in } \mathcal{S} \quad .$$

Some examples:

$$T(\boldsymbol{x}) \;=\; \text{temperature at position } \boldsymbol{x} \quad ,$$

$$\Phi(\boldsymbol{x}) \;=\; \text{the gravitational potential at } \boldsymbol{x} \text{ due to the surrounding masses} \quad ,$$

$$x^2(\boldsymbol{x}) \;=\; \text{the second coordinate of } \boldsymbol{x} \text{ with respect to some given coordinate system}$$

and

$$r(\boldsymbol{x}) \;=\; \text{“distance between } \boldsymbol{x} \text{ and some fixed point } \boldsymbol{O}\text{”} \;=\; \text{dist}(\boldsymbol{x}, \boldsymbol{O}) \quad .$$

The *coordinate formula* for a scalar field Ψ with respect to a given coordinate system $\{(x^1, x^2, \ldots, x^N)\}$ is just the formula $\psi(x^1, x^2, \ldots, x^N)$ for computing the value of $\Psi(\boldsymbol{x})$ from the coordinates for $\boldsymbol{x}$. So

$$\Psi(\boldsymbol{x}) \;=\; \psi(x^1, x^2, \ldots, x^N) \qquad \text{with} \quad \boldsymbol{x} \sim (x^1, x^2, \ldots, x^N) \quad .$$

For example, if our space is a plane and we are using Cartesian coordinates $\{(x, y)\}$ with $\boldsymbol{O}$ as the origin, then

$$\text{the coordinate formula for } r(\boldsymbol{x}) \;=\; \text{dist}(\boldsymbol{x}, \boldsymbol{O}) \quad \text{is} \quad \sqrt{x^2 + y^2} \quad .$$

If, instead, we are using polar coordinates $\{(\rho, \phi)\}$ with $\boldsymbol{O}$ as the origin, then

$$\text{the coordinate formula for } r(\boldsymbol{x}) \;=\; \text{dist}(\boldsymbol{x}, \boldsymbol{O}) \quad \text{is} \quad \rho \quad .$$

A *vector field* (on $\mathcal{S}$) is a vector-valued function of position,

$$\mathbf{V}(\boldsymbol{x}) \;=\; \text{some vector corresponding to position } \boldsymbol{x} \text{ in } \mathcal{S} \quad .$$

Some examples:

$$\mathbf{V}(\boldsymbol{x}) \;=\; \text{wind velocity at } \boldsymbol{x} \quad ,$$

$$\mathbf{F}(\boldsymbol{x}) \;=\; \text{force of gravity on some object at position } \boldsymbol{x}$$

and

$$\boldsymbol{\varepsilon}_k \;=\; \frac{\partial \boldsymbol{x}}{\partial x^k} \;=\; \text{`rate of change' in position as the } k^{\text{th}} \text{ coordinate varies} \quad .$$

At each point $\boldsymbol{x}$, $\mathbf{V}(\boldsymbol{x})$ will be a vector in the tangent space at that point. Often, though, we will be dealing with a subspace (say, a curve or a sphere in a three-dimensional space), in which case $\mathbf{V}(\boldsymbol{x})$ may or may not be in the tangent space of that subspace, depending on how $\mathbf{V}$ is generated. For example, $\mathbf{V}$ may be the velocity of some object moving on a curve, in which case $\mathbf{V}(\boldsymbol{x})$ is tangent to the curve at each point $\boldsymbol{x}$. On the other hand, $\mathbf{V}(\boldsymbol{x})$ may be a vector field perpendicular to a sphere in Euclidean three-space, in which case $\mathbf{V}(\boldsymbol{x})$ will not be in the tangent space of the sphere at any $\boldsymbol{x}$ in the sphere.

Because $\mathbf{V}(\boldsymbol{x})$ is in the tangent space of our overall space at $\boldsymbol{x}$, it will have components with respect to both of our associated bases. It can be expressed in terms of the $\boldsymbol{\varepsilon}_i$'s,

$$\mathbf{V}(\boldsymbol{x}) \;=\; \sum_{i=1}^{N} V^i(x^1, x^2, \dots, x^N)\,\boldsymbol{\varepsilon}_i \qquad \text{where} \quad \boldsymbol{x} \;\sim\; (x^1, x^2, \dots, x^N) \quad ,$$

and it can be expressed in terms of the $\mathbf{e}_i$'s,

$$\mathbf{V}(\boldsymbol{x}) \;=\; \sum_{i=1}^{N} v^i(x^1, x^2, \dots, x^N)\,\mathbf{e}_i \qquad \text{where} \quad \boldsymbol{x} \;\sim\; (x^1, x^2, \dots, x^N) \quad .$$

These formulas for $\mathbf{V}(\boldsymbol{x})$ should be refered to as *coordinate/component formulas* for $\mathbf{V}$. In practice, it is more common for a vector field to be expressed in terms of the normalized tangent vectors (the $\mathbf{e}_k$'s) than the $\boldsymbol{\varepsilon}_k$'s. Still, there are occasions where using the other representation is convenient.

Since $\boldsymbol{\varepsilon}_i = h_i \mathbf{e}_i$, these components are related by

$$v^i \;=\; h_i V^i \quad .$$

Note that these component functions are denoted with superscripts. That is standard when dealing with the slightly more general theory we are developing here. In practice, though, expect a lot of people to use subscripts.

On the other hand, there is no real convention on how to notationally distinguish components with respect to the $\boldsymbol{\varepsilon}_i$'s from components with respect to the $\mathbf{e}_i$'s. My use here of V^i and v^i will not be consistently followed — not even by me.

If the coordinate system is orthogonal, then $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ is orthonormal at each point, and, so, the corresponding components of $\mathbf{V}$ are given by

$$v^i \;=\; \mathbf{V} \cdot \mathbf{e}_i \quad .$$

?►*Exercise 9.6:* *Show that, if the coordinate system is orthogonal, then the components of vector field* $\mathbf{V}$ *with respect to the* $\boldsymbol{\varepsilon}_k$*'s are given by*

$$V^i \;=\; \frac{\mathbf{V} \cdot \boldsymbol{\varepsilon}_i}{(h_i)^2} \quad .$$

?►Exercise 9.7: *Using the product rule, show that*

$$\frac{\partial \mathbf{V}}{\partial x^j} = \sum_{k=1}^{N} \left[\frac{\partial V^k}{\partial x^j} + \sum_{i=1}^{N} V^i \Gamma_{ij}^k \right] \boldsymbol{\varepsilon}_k \quad .$$

What would this be in terms of the unit tangents $\mathbf{e}_1$ *, …,* $\mathbf{e}_N$ *? (By the way, the expression inside the brackets in the above equation is known as a "covariant derivative". of* $\mathbf{V}$ *)*

Some Warnings

1. Many authors use the terms "scalar" and "vector" when they mean "scalar field" and "vector field".
2. People often use the same symbols to denote both a scalar/vector field and its coordinate formula. This is not particularly bad if there is only one coordinate system being used (and everyone knows which system that is), but it can be quite confusing when we have multiple coordinate systems.

Change of Coordinates and the Coordinate Formulas

For us, this should be simple:

Suppose

$$\psi\left(x^1, x^2, \ldots, x^N\right)$$

is the coordinate formula for some scalar field Ψ using some coordinate system $\{(x^1, x^2, \ldots, x^N)\}$. To obtain the coordinate formula for Ψ using a different coordinate system $\{(x^{1'}, x^{2'}, \ldots, x^{N'})\}$, we simply replace each x^k in the formula for

$$\psi\left(x^1, x^2, \ldots, x^N\right)$$

with the k^{th} change of coordinates formula,

$$x^k = x^k\left(x^{1'}, x^{2'}, \ldots, x^{N'}\right) \quad .$$

If we have a coordinate formula for vector field $\mathbf{V}$,

$$\mathbf{V}(\boldsymbol{x}) = \sum_{i=1}^{N} V^i(x^1, x^2, \ldots, x^N)\, \boldsymbol{\varepsilon}_i \qquad \text{or} \qquad \mathbf{V}(\boldsymbol{x}) = \sum_{i=1}^{N} v^i(x^1, x^2, \ldots, x^N)\, \mathbf{e}_i$$

where

$$\boldsymbol{x} \sim (x^1, x^2, \ldots, x^N) \quad ,$$

then, in addition to converting the formulas to corresponding formulas in term of the $x^{k'}$, we need to express $\mathbf{V}$ in terms of the local bases associated with $\{(x^{1'}, x^{2'}, \ldots, x^{N'})\}$,

$$\mathbf{V}(\boldsymbol{x}) = \sum_{i=1}^{N} V^{i'}(x^{1'}, x^{2'}, \ldots, x^{N'})\, \boldsymbol{\varepsilon}_i{}' \qquad \text{or} \qquad \mathbf{V}(\boldsymbol{x}) = \sum_{i=1}^{N} v^{i'}(x^{1'}, x^{2'}, \ldots, x^{N'})\, \mathbf{e}_i{}'$$

where

$$\boldsymbol{x} \sim (x^{1'}, x^{2'}, \dots, x^{N'}) \quad .$$

For this, we can use what we learned earlier about "change of basis". In particular, if our coordinate systems are orthogonal (so that the sets of associated normalized tangent vectors are orthonormal at every point), then

$$\begin{bmatrix} v^{1'} \\ v^{2'} \\ \vdots \\ v^{N'} \end{bmatrix} = |\mathbf{V}\rangle_{\mathcal{B}'} = \mathbf{M}_{\mathcal{B}'\mathcal{B}} |\mathbf{V}\rangle_{\mathcal{B}} = \begin{bmatrix} v^{1} \\ v^{2} \\ \vdots \\ v^{N} \end{bmatrix}$$

where

$$\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\} \quad , \quad \mathcal{B}' = \left\{\mathbf{e}_1{}', \mathbf{e}_2{}', \dots, \mathbf{e}_N{}'\right\}$$

and

$$[\mathbf{M}_{\mathcal{B}'\mathcal{B}}]_{ij} = \mathbf{e}_i{}' \cdot \mathbf{e}_j \quad .$$

Further details will probably be assigned as exercises.

A Few Words About Continuity and Differentiability

Because our goal is to develop a substantial amount of "applicable multidimensional calculus" in a relatively short time, we are glossing over some of the mathematical fine points. Still, a few words on continuity and differentiability are in order, since there will be some issues involving the continuity and differentiability of scalar and vector fields.

Continuity is easy to rigorously define: We say that a scalar field Ψ and a vector field $\mathbf{V}$ are *continuous at a point* $\boldsymbol{p}$ if and only if

$$\lim_{\boldsymbol{x}\to\boldsymbol{p}} \Psi(\boldsymbol{x}) = \Psi(\boldsymbol{p}) \quad \text{and} \quad \lim_{\boldsymbol{x}\to\boldsymbol{p}} \mathbf{V}(\boldsymbol{x}) = \mathbf{V}(\boldsymbol{p}) \quad .$$

We then say that Ψ and $\mathbf{V}$ are continuous in some region if they are continuous at each point in the region.

Differentiability is a bit more difficult to rigorously define without going into more mathematical details than appropriate for this course. Here is a working definition for us:

> Ψ is *k-times differentiable* (at a point or in a region) if and only if, using any reasonable coordinate system $\{(x^1, x^2, \dots, x^N)\}$, the coordinate formula $\psi(x^1, x^2, \dots, x^N)$ for Ψ, as well as all partial derivatives of ψ up to order k, exist and are continuous (at the point or in the region).[4]

A similar working definition works for vector fields.

If the order of differentiability k is not mentioned, you should normally assume $k = 1$. That is, "Ψ is differentiable" means "Ψ is 1-time differentiable". If, however, we say "Ψ is *suitably* differentiable", then assume k is the order of the highest order derivative relevant to whatever we are discussing.

[4] Actually, mathematicians call this "k-times continuous differentiability" or "smoothness".

9.3 The Classic Gradient, Divergence and Curl

Basic Definitions in Euclidean Space

For expediency, we will first define the classical differential operators for scalar and vector fields in a Euclidean space $\mathcal{E}$ using the "del operator". Moreover, we will assume that our coordinate system $\{(x^1, x^2, \ldots, x^N)\}$ is *Cartesian*.

The *del operator* is the "vector differential operator" given in our Cartesian system by

$$\vec{\nabla} = \boldsymbol{\nabla} = \sum_{k=1}^{N} \frac{\partial}{\partial x^k} \mathbf{e}_k = \sum_{k=1}^{N} \mathbf{e}_k \frac{\partial}{\partial x^k} = \sum_{k=1}^{N} \frac{\partial \boldsymbol{x}}{\partial x^k} \frac{\partial}{\partial x^k} \quad .$$

For any sufficiently differentiable scalar field Ψ and vector field $\mathbf{F}$ with corresponding coordinate formulas

$$\Psi(\boldsymbol{x}) = \psi\left(x^1, x^2, \ldots, x^N\right) \qquad \text{and} \qquad \mathbf{F}(\boldsymbol{x}) = \sum_{k=1}^{N} F^k\left(x^1, x^2, \ldots, x^N\right) \mathbf{e}_k \quad ,$$

we define

$$\text{the gradient of } \Psi = \mathbf{grad}(\Psi) = \boldsymbol{\nabla}\Psi \quad ,$$

$$\text{the divergence of } \mathbf{F} = \operatorname{div}(\mathbf{F}) = \boldsymbol{\nabla} \cdot \mathbf{F} \quad ,$$

and

$$\text{the curl of } \mathbf{F} = \mathbf{curl}(\mathbf{F}) = \boldsymbol{\nabla} \times \mathbf{F}$$

by the coordinate formulas

$$\mathbf{grad}(\Psi) = \boldsymbol{\nabla}\Psi = \sum_{k=1}^{N} \mathbf{e}_k \frac{\partial}{\partial x^k} \psi\left(x^1, x^2, \ldots, x^N\right) = \sum_{k=1}^{N} \mathbf{e}_k \frac{\partial \psi}{\partial x^k} = \sum_{k=1}^{N} \frac{\partial \psi}{\partial x^k} \mathbf{e}_k \quad ,$$

$$\operatorname{div}(\mathbf{F}) = \boldsymbol{\nabla} \cdot \mathbf{F} = \left[\sum_{k=1}^{N} \frac{\partial}{\partial x^k} \mathbf{e}_k\right] \cdot \left[\sum_{j=1}^{N} F^j\left(x^1, x^2, \ldots, x^N\right) \mathbf{e}_j\right] = \sum_{k=1}^{N} \frac{\partial F^k}{\partial x^k} \quad ,$$

and

$$\begin{aligned}
\mathbf{curl}(\mathbf{F}) = \boldsymbol{\nabla} \times \mathbf{F} &= \left[\sum_{k=1}^{N} \frac{\partial}{\partial x^k} \mathbf{e}_k\right] \times \left[\sum_{j=1}^{N} F^j\left(x^1, x^2, \ldots, x^N\right) \mathbf{e}_j\right] \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x^1} & \dfrac{\partial}{\partial x^2} & \dfrac{\partial}{\partial x^3} \\ F^1 & F^2 & F^3 \end{vmatrix} \\
&= \left[\frac{\partial F^3}{\partial x^2} - \frac{\partial F^2}{\partial x^3}\right] \mathbf{e}_1 - \left[\frac{\partial F^3}{\partial x^1} - \frac{\partial F^1}{\partial x^3}\right] \mathbf{e}_2 + \left[\frac{\partial F^2}{\partial x^1} - \frac{\partial F^1}{\partial x^2}\right] \mathbf{e}_3 \quad .
\end{aligned}$$

Both $\mathbf{grad}(\Psi)$ and $\operatorname{div}(\mathbf{F})$ can be defined assuming our space is of any dimension. However, $\mathbf{curl}(\mathbf{F})$ requires that our space be three-dimensional (in which case, we usually use the traditional $\{(x, y, z)\}$ and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ notation).

Observe that $\mathbf{grad}(\Psi)$ and $\mathbf{curl}(\mathbf{F})$ are vector fields, while $\mathrm{div}(\mathbf{F})$ is a scalar field.

One thing not obvious from the above definitions is why anyone would be interested in these things. There are good reasons arising from both basic mathematics and from physics. The "geometric/physical significance" of $\mathbf{grad}(\Psi)$ will be discussed in a little bit. The importance of the divergence and curl of a vector field will be discussed later, in conjunction with some classic integral theorems. In anticipation of that discussion, let me introduce some terminology that you may encounter in homework:

$$\text{“}\mathbf{F}\text{ is solenoidal”} \iff \text{“}\mathbf{F}\text{ is divergence free”} \iff \nabla\cdot\mathbf{F} = 0$$

and

$$\text{“}\mathbf{F}\text{ is irrotational”} \iff \text{“}\mathbf{F}\text{ is curl free”} \iff \nabla\times\mathbf{F} = 0 \quad .$$

Invariance Under Change of Cartesian Coordinates

Because these formulas are given in terms of one Cartesian coordinate system, there is the issue of what these formulas become when we switch to any other Cartesian coordinate system. (Hopefully, these formulas are "Cartesian coordinate system independent".) To examine this issue, let $\{(x^{1'}, x^{2'}, \ldots, x^{N'})\}$ be another Cartesian coordinate system corresponding to an orthonormal basis $\{\mathbf{e}_1{}', \mathbf{e}_2{}', \ldots, \mathbf{e}_N{}'\}$. Remember, in problem *J 1* of *Homework Handout VII* you showed (among other things) that

$$\mathbf{e}_j{}' = \sum_{k=1}^{N} \frac{\partial x^{j'}}{\partial x^k}\,\mathbf{e}_k \quad .$$

Using the chain rule from elementary calculus and the above, we get

$$\nabla = \sum_{k=1}^{N} \frac{\partial}{\partial x^k}\,\mathbf{e}_k = \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{\partial x^{j'}}{\partial x^k}\frac{\partial}{\partial x^{j'}}\,\mathbf{e}_k = \sum_{j=1}^{N} \frac{\partial}{\partial x^{j'}}\sum_{k=1}^{N} \frac{\partial x^{j'}}{\partial x^k}\,\mathbf{e}_k = \sum_{j=1}^{N} \frac{\partial}{\partial x^{j'}}\,\mathbf{e}_j{}' \quad .$$

This shows that the basic formula

$$\nabla = \sum_{k=1}^{N} \frac{\partial}{\partial x^k}\,\mathbf{e}_k$$

is invariant under any change of Cartesian coordinates. Consequently, the formulas given for $\nabla\Psi$ and $\nabla\cdot\mathbf{F}$ and $\nabla\times\mathbf{F}$ are valid using any Cartesian coordinate system, and yield the same vector or scalar fields.

Product and Chain Rules

Using the Cartesian formulas and elementary calculus, we can derive or verify a number of "product rules" and "chain rules" that might simplify calculations later on. In particular, if Ψ and Φ are suitably differentiable scalar fields, and $\mathbf{F}$ and $\mathbf{G}$ are suitably differentiable vector fields, then

$$\nabla(\Phi\Psi) = \Psi\nabla\Phi + \Phi\nabla\Psi \quad , \tag{9.8a}$$

$$\nabla\cdot(\Phi\mathbf{F}) = (\nabla\Phi)\cdot\mathbf{F} + \Phi\nabla\cdot\mathbf{F} \quad , \tag{9.8b}$$

$$\nabla\times(\Phi\mathbf{F}) = (\nabla\Phi)\times\mathbf{F} + \Phi(\nabla\times\mathbf{F}) \tag{9.8c}$$

and

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) \;=\; (\nabla \times \mathbf{F}) \cdot \mathbf{G} \;-\; \mathbf{F} \cdot (\nabla \times \mathbf{G}) \quad . \tag{9.8d}$$

These are multidimensional versions of the product rule (note the "$-$" in (9.8d) due to the antisymmetry in the cross product). You can easily derive or verify any of them, yourself.

?►*Exercise 9.8* **a:** *Verify equation (9.8a).*

b: *Pick any one of the other equations in set (9.8), and verify it.*

Now let f be a real-valued function on $\mathbb{R}$ (so $f(\text{a real number}) = \text{a real number}$). Then, for each position $\boldsymbol{x}$, we can plug the real number $\Phi(\boldsymbol{x})$ into f, getting a 'new' scalar field

$$f(\Phi(\boldsymbol{x})) \quad .$$

You can then easily verify the chain rule

$$\nabla \left[f(\Phi(\boldsymbol{x})) \right] \;=\; \left[f'(\Phi(\boldsymbol{x})) \right] \nabla \Phi(\boldsymbol{x}) \quad . \tag{9.9}$$

?►*Exercise 9.9:* *Verify equation (9.9).*

Another chain rule involving the gradient will be derived in the next section.

9.4 The General Gradient Operator

Geometric Significance of $\nabla \Psi$

Consider a function f of the form

$$f(t) \;=\; \Psi(\boldsymbol{x}(t))$$

where $\Psi(\boldsymbol{x})$ is some scalar field on a Euclidean space $\mathcal{E}$ and $\boldsymbol{x}(t)$ is any (sufficiently smooth) position-valued function (with t being in some interval (α, β)). Both Ψ and $\boldsymbol{x}(t)$ will have coordinate formulas

$$\boldsymbol{x}(t) \;\sim\; \left(x^1(t),\, x^2(t),\, \ldots,\, x^N(t) \right) \qquad \text{for} \quad \alpha < t < \beta$$

and

$$\Psi(\boldsymbol{x}) \;=\; \psi\left(x^1, x^2, \ldots, x^N \right) \qquad \text{where} \quad \boldsymbol{x} \sim \left(x^1, x^2, \ldots, x^N \right) \quad .$$

Taking the composition of the two,

$$f(t) \;=\; \Psi(\boldsymbol{x}(t)) \;=\; \psi\left(x^1(t),\, x^2(t),\, \ldots,\, x^N(t) \right) \qquad \text{for} \quad \alpha < t < \beta \quad ,$$

gives us an ordinary real-valued function f such as you studied in "Calc. III". Its derivative can be found by the classical chain rule developed in that course. Computing this derivative, we see that, at least if the coordinate system is Cartesian,

$$\begin{aligned}
\frac{df}{dt} &= \frac{d}{dt}\left[\Psi(\boldsymbol{x}(t))\right] \\
&= \frac{d}{dt}\left[\psi\left(x^1(t),\, x^2(t),\, \ldots,\, x^N(t)\right)\right] \\
&= \frac{\partial\psi}{\partial x^1}\frac{dx^1}{dt} + \frac{\partial\psi}{\partial x^2}\frac{dx^2}{dt} + \cdots + \frac{\partial\psi}{\partial x^N}\frac{dx^N}{dt} \\
&= \left[\frac{\partial\psi}{\partial x^1}\mathbf{e}_1 + \frac{\partial\psi}{\partial x^2}\mathbf{e}_2 + \cdots + \frac{\partial\psi}{\partial x^N}\mathbf{e}_N\right]\cdot\left[\frac{dx^1}{dt}\mathbf{e}_1 + \frac{dx^2}{dt}\mathbf{e}_2 + \cdots + \frac{dx^N}{dt}\mathbf{e}_N\right] \\
&= \left[\sum_{k=1}^{N}\frac{\partial\psi}{\partial x^k}\mathbf{e}_k\right]\cdot\left[\sum_{k=1}^{N}\frac{dx^k}{dt}\mathbf{e}_k\right] \quad .
\end{aligned}$$

But remember

$$\nabla\Psi = \sum_{k=1}^{N}\frac{\partial\psi}{\partial x^k}\mathbf{e}_k \qquad \text{and} \qquad \frac{d\boldsymbol{x}}{dt} = \sum_{k=1}^{N}\frac{dx^k}{dt}\mathbf{e}_k \quad .$$

So the above boils down to

$$\frac{d}{dt}\left[\Psi(\boldsymbol{x}(t))\right] = \nabla\Psi\cdot\frac{d\boldsymbol{x}}{dt} \quad . \tag{9.10}$$

This is another multidimensional "chain rule". Keep in mind that the $\nabla\Psi$ in this equation is being evaluated at the position corresponding to whatever value of t is relevant. To be a little more precise, we should write this last equation as

$$\left.\frac{d}{dt}\left[\Psi(\boldsymbol{x}(t))\right]\right|_{t_0} = \left[\nabla\Psi|_{\boldsymbol{x}(t_0)}\right]\cdot\left[\left.\frac{d\boldsymbol{x}}{dt}\right|_{t_0}\right] \quad ,$$

but most of us are too lazy.

Some consequences of equation (9.10) can be seen if we rewrite this dot product as

$$\frac{d}{dt}\left[\Psi(\boldsymbol{x}(t))\right] = \|\nabla\Psi\|\left\|\frac{d\boldsymbol{x}}{dt}\right\|\cos(\theta) \qquad \text{where} \quad \theta = \text{angle between } \nabla\Psi \text{ and } \frac{d\boldsymbol{x}}{dt} \quad .$$

"Clearly", then:

1. As a function of θ, the above is maximum when $\theta = 0$. That is, $\frac{d}{dt}\Psi(\boldsymbol{x}(t))$ is largest when $\nabla\Psi$ and $d\boldsymbol{x}/dt$ are lined up. With a little thought, you'll realize this means that, at each point $\boldsymbol{x}_0$ in space, $\nabla\Psi$ evaluated at $\boldsymbol{x}_0$ points in the direction in which $\Psi(\boldsymbol{x})$ is increasing most rapidly as $\boldsymbol{x}$ moves through position $\boldsymbol{x}_0$.

2. On the other hand, if $\mathcal{S}$ is a curve or surface[5] on which Ψ is constant, and if $\boldsymbol{x}(t)$ traces out a curve in $\mathcal{S}$ (with $\boldsymbol{x}'(t)$ being nonzero), then $\Psi(\boldsymbol{x}(t))$ is a constant function of t. Hence

$$\|\nabla\Psi\|\left\|\frac{d\boldsymbol{x}}{dt}\right\|\cos(\theta) = \frac{d}{dt}\Psi(\boldsymbol{x}(t)) = 0 \quad ,$$

which means that $\theta = \pi/2$ (or $\nabla\Psi = 0$). In other words, at each point on this surface, $\nabla\Psi$ must be perpendicular to this curve or surface $\mathcal{S}$.

[5] or "hyper-surface" if $\mathcal{E}$ has dimension greater than three.

General Definition and Formula for the Gradient

Look, again, at equation (9.10). It gives a coordinate-free description of the gradient; namely, that $\nabla\Psi$ is the vector field making that equation true. Being coordinate-free, this is actually a better description for the gradient than the Cartesian formula given in the previous section. It can even apply when our space is not Euclidean.

So let us drop our old definition of the gradient, and, instead, define the *gradient of a scalar field* Ψ to be the vector field — denoted by either $\mathbf{grad}(\Psi)$ or $\nabla\Psi$ — such that

$$\frac{d}{dt}[\Psi(\boldsymbol{r}(t))] = [\nabla\Psi(\boldsymbol{r}(t))] \cdot \frac{d\boldsymbol{r}}{dt} \tag{9.11}$$

whenever $\boldsymbol{r}$ is a differentiable position-valued function. Note that this definition is free of coordinates, and does not require the space to be Euclidean.

Since $\nabla\Psi$ is a vector field, it has a coordinate/component formula in terms of the local basis at each point,

$$\nabla\Psi = \sum_{i=1}^{N} (\nabla\Psi)^i \mathbf{e}_i \quad . \tag{9.12}$$

For convenience, we've used $(\nabla\Psi)^i$ to denote the i^{th} component of $\nabla\Psi$ with respect to the local basis. Keep in mind that each $(\nabla\Psi)^i$ is a scalar field, and may vary with position just as $\nabla\Psi$ and each $\mathbf{e}_k$ varies with position. To find the coordinate formulas for these components, let us see what the left and right sides of equation (9.11) (defining the gradient) become when using an arbitrary position-valued function with its coordinate formula,

$$\boldsymbol{r}(t) \sim \left(x^1(t),\, x^2(t),\, \dots,\, x^N(t)\right) \quad .$$

First look at what we get when using this with the coordinate formula for Ψ in the left side of equation (9.11). Applying the chain rule from elementary calculus,

$$\frac{d}{dt}[\Psi(\boldsymbol{r}(t))] = \frac{d}{dt}\left[\psi\left(x^1(t),\, x^2(t),\, \dots,\, x^N(t)\right)\right] = \sum_{i=1}^{N} \frac{\partial\psi}{\partial x^i}\frac{dx^i}{dt} \quad . \tag{9.13}$$

On the other hand, using the coordinate/component formulas for $\nabla\Psi$ and $d\boldsymbol{r}/dt$ in the right side of (9.11) yields:

$$\begin{aligned}
[\nabla\Psi(\boldsymbol{r}(t))] \cdot \frac{d\boldsymbol{r}}{dt} &= \left[\sum_{i=1}^{N} (\nabla\Psi)^i \mathbf{e}_i\right] \cdot \left[\sum_{j=1}^{N} \frac{\partial\boldsymbol{r}}{\partial x^j}\frac{dx^j}{dt}\right] \\
&= \left[\sum_{i=1}^{N} (\nabla\Psi)^i \mathbf{e}_i\right] \cdot \left[\sum_{j=1}^{N} h_j \frac{dx^j}{dt}\,\mathbf{e}_j\right]
\end{aligned}$$

Assuming the coordinate system is orthogonal, the local basis $\{\mathbf{e}_1, \dots, \mathbf{e}_2\}$ is orthonormal, the last set of equations reduces to

$$[\nabla\Psi(\boldsymbol{r}(t))] \cdot \frac{d\boldsymbol{r}}{dt} = \sum_{i=1}^{N} (\nabla\Psi)^i h_j \frac{dx^j}{dt} \quad . \tag{9.14}$$

Thus, using equations (9.13) and (9.14) and any choice of

$$\boldsymbol{r}(t) \sim \left(x^1(t),\, x^2(t),\, \dots,\, x^N(t)\right) \quad ,$$

we can now rewrite

$$\frac{d}{dt}\left[\Psi(\boldsymbol{r}(t))\right] \;=\; \left[\boldsymbol{\nabla}\Psi(\boldsymbol{r}(t))\right]\cdot\frac{d\boldsymbol{r}}{dt}$$

as

$$\sum_{i=1}^{N}\frac{\partial\psi}{\partial x^i}\frac{dx^i}{dt} \;=\; \sum_{i=1}^{N}(\boldsymbol{\nabla}\Psi)^i h_i\frac{dx^i}{dt} \quad .$$

This “clearly” means that[6]

$$\frac{\partial\psi}{\partial x^i} \;=\; (\boldsymbol{\nabla}\Psi)^i h_i \qquad \text{for} \quad i = 1, 2, \ldots, N \quad .$$

Hence

$$(\boldsymbol{\nabla}\Psi)^i \;=\; \frac{1}{h_i}\frac{\partial\psi}{\partial x^i} \qquad \text{for} \quad i = 1, 2, \ldots, N \quad ,$$

and the coordinate/component formula for the gradient when using any orthogonal coordinate system is

$$\boldsymbol{\nabla}\Psi \;=\; \sum_{i=1}^{N}\frac{1}{h_i}\frac{\partial\psi}{\partial x^i}\,\mathbf{e}_i \quad . \tag{9.15}$$

!►*Example 9.4:* *In polar coordinates for the Euclidean plane,* $\{(\rho,\phi)\}$ *,*

$$\boldsymbol{\nabla}\Psi \;=\; \sum_{i=1}^{N}\frac{1}{h_i}\frac{\partial\psi}{\partial x^i}\mathbf{e}_i \;=\; \frac{1}{h_\rho}\frac{\partial\psi}{\partial\rho}\,\mathbf{e}_\rho \;+\; \frac{1}{h_\phi}\frac{\partial\psi}{\partial\phi}\,\mathbf{e}_\phi \quad .$$

Recalling that $h_\rho = 1$ *and* $h_\phi = \rho$ *, we see that*

$$\boldsymbol{\nabla}\Psi \;=\; \frac{\partial\psi}{\partial\rho}\,\mathbf{e}_\rho \;+\; \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\,\mathbf{e}_\phi \quad .$$

In particular, if the polar coordinate formula for Ψ *is*

$$\psi(\rho,\phi) \;=\; \rho^2\sin(\phi)$$

then, for $\boldsymbol{x}\sim(\rho,\phi)$ *,*

$$\begin{aligned}\boldsymbol{\nabla}\Psi(\boldsymbol{x}) \;&=\; \frac{\partial\psi}{\partial\rho}\,\mathbf{e}_\rho \;+\; \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\,\mathbf{e}_\phi \\ &=\; 2\rho\sin(\phi)\,\mathbf{e}_\rho \;+\; \frac{1}{\rho}\left[\rho^2\cos(\phi)\right]\mathbf{e}_\phi \;=\; 2\rho\sin(\phi)\,\mathbf{e}_\rho \;+\; \rho\cos(\phi)\,\mathbf{e}_\phi \quad .\end{aligned}$$

?►*Exercise 9.10:* *Find the formula for the gradient in spherical coordinates.*

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[6] If this isn’t clear, first fix your position $\boldsymbol{x}_0\sim\left(x_0^1, x_0^2, \ldots, x_0^N\right)$ and choose

$$\boldsymbol{r}(t) \;\sim\; \left(x^1(t),\, x^2(t),\, \ldots,\, x^N(t)\right) \;=\; \left(t,\, x_0^2,\, \ldots,\, x_0^N\right) \quad .$$

Then

$$\frac{dx^i}{dt} \;=\; \begin{cases} 1 & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i \neq 1 \end{cases}$$

and equation (9.14) reduces to

$$\frac{\partial\psi}{\partial x^1} \;=\; (\boldsymbol{\nabla}\Psi)^1 h_1 \quad .$$

9.5 General Formulas for the Divergence and Curl

Later, we will discover the geometric significance of the divergence and the curl of a vector field. Then, using those, we can both redefine divergence and curl in a coordinate-free manner, and obtain more general formulas for these. This will come from our development of the divergence (or Gauss's) theorem and the Stokes' theorem. In the meantime, I will simply tell you what those formulas are, and we will use them, as needed, with the understanding that, eventually, we will see how to derive them.

As usual, let

$$\left\{\left(x^1, x^2, \ldots, x^N\right)\right\}$$

be an *orthogonal* coordinate system for our space $\mathcal{S}$ with

$$\{h_1, h_2, \ldots, h_N\} \quad , \quad \{\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \ldots, \boldsymbol{\varepsilon}_N\} \quad \text{and} \quad \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$$

being the associated scaling factors, tangent vectors and normalized tangent vectors at each point. Assume $\mathbf{F}$ is a vector field with component/coordinate formula

$$\mathbf{F}(\boldsymbol{x}) = \sum_{i=1}^{N} F^i\left(x^1, x^2, \ldots, x^N\right)\mathbf{e}_i \qquad \text{where} \quad \boldsymbol{x} \sim \left(x^1, x^2, \ldots, x^N\right) \quad .$$

The corresponding formulas for the divergence of $\mathbf{F}$ depend somewhat on the dimension N of the space, though the pattern will become obvious. If $N = 2$, then

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial x^1}\left[F^1 h_2\right] + \frac{\partial}{\partial x^2}\left[F^2 h_1\right]\right\} \quad . \tag{9.16a}$$

If $N = 3$, then

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial x^1}\left[F^1 h_2 h_3\right] + \frac{\partial}{\partial x^2}\left[F^2 h_1 h_3\right] + \frac{\partial}{\partial x^3}\left[F^3 h_1 h_2\right]\right\} \quad . \tag{9.16b}$$

If $N = 4$, then

$$\begin{aligned}\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3 h_4}\bigg\{&\frac{\partial}{\partial x^1}\left[F^1 h_2 h_3 h_4\right] + \frac{\partial}{\partial x^2}\left[F^2 h_1 h_3 h_4\right] \\ &+ \frac{\partial}{\partial x^3}\left[F^3 h_1 h_2 h_4\right] + \frac{\partial}{\partial x^4}\left[F^4 h_1 h_2 h_3\right]\bigg\} \quad .\end{aligned} \tag{9.16c}$$

And so on.

The curl still only makes sense if the dimension N is three. Using Stokes' theorem, we will be able to show that

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial x^1} & \dfrac{\partial}{\partial x^2} & \dfrac{\partial}{\partial x^3} \\ h_1 F^1 & h_2 F^2 & h_3 F^3 \end{vmatrix}$$

$$\begin{aligned} &= \frac{1}{h_2 h_3}\left(\frac{\partial}{\partial x^2}\left[h_3 F^3\right] - \frac{\partial}{\partial x^3}\left[h_2 F^2\right]\right)\mathbf{e}_1 \\ &\quad - \frac{1}{h_1 h_3}\left(\frac{\partial}{\partial x^1}\left[h_3 F^3\right] - \frac{\partial}{\partial x^3}\left[h_1 F^1\right]\right)\mathbf{e}_2 \\ &\quad + \frac{1}{h_1 h_2}\left(\frac{\partial}{\partial x^1}\left[h_2 F^2\right] - \frac{\partial}{\partial x^2}\left[h_1 F^1\right]\right)\mathbf{e}_3 \quad . \end{aligned} \tag{9.17}$$

?►*Exercise 9.11:* *Verify that the above formulas reduce to the Cartesian formulas originally given when the coordinate system is Cartesian.*

?►*Exercise 9.12:* *Using the above, verify that the two-dimensional formula for divergence using polar coordinates* $\{(r,\phi)\}$ *is*

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}\left[\rho F^\rho\right] + \frac{\partial F^\phi}{\partial \phi}\right\} \quad ,$$

?►*Exercise 9.13:* *Verify that the the three-dimensional formulas for divergence and curl are, using cylindrical coordinates* $\{(\rho,\phi,z)\}$ *,*

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}\left[\rho F^\rho\right] + \frac{\partial F^\phi}{\partial \phi}\right\} + \frac{\partial F^z}{\partial z} \quad ,$$

and

$$\nabla \times \mathbf{F} = \left[\frac{1}{\rho}\frac{\partial F^z}{\partial \phi} - \frac{\partial F^\phi}{\partial z}\right]\mathbf{e}_\rho + \left[\frac{\partial F^\rho}{\partial z} - \frac{\partial F^z}{\partial \rho}\right]\mathbf{e}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho F^\phi\right) - \frac{\partial F^\rho}{\partial \phi}\right]\mathbf{e}_z \quad .$$

?►*Exercise 9.14:* *Find the formulas for the divergence and curl in spherical coordinates.*

?►*Exercise 9.15:* *Let* S_3 *be a sphere of radius* 3 *with the two-dimensional spherical coordinates system* $\{(\phi,\theta)\}$ *described in exercise 8.15 on page 8–27. Find the corresponding coordinate formula for the divergence of a vector field on* S_3 *. (Use the above along with the results derived in exercise 8.15.)*

It should be noted that you can also derive the formulas in the above exercises by taking the Cartesian coordinate formulas for these entities and *carefully* converting these formulas to polar/cylindrical/spherical coordinates.

9.6 Repeated Applications of the Del Operator

General Observations

For any sufficiently differentiable scalar field ψ and vector field $\mathbf{F}$, we can compute

$$\nabla \cdot (\nabla \psi) \quad , \quad \nabla \times (\nabla \psi) \quad ,$$
$$\nabla (\nabla \cdot \mathbf{F}) \quad , \quad \nabla \cdot (\nabla \times \mathbf{F}) \quad \text{and} \quad \nabla \times (\nabla \times \mathbf{F}) \quad .$$

I've included parentheses to emphasize that we are taking a "del operation" on an object resulting from a previous "del operation". However, except for the last expression, the parentheses are not really necessary and are not usually written.

?► ***Exercise 9.16*** ***a:*** *Which of the above ends up being*

i: *a scalar field?*

ii: *a vector field?*

b: *Give an example of an expression involving repeated use of* ∇ *that is "nonsense".*

Some of these formulas simplify greatly, with the verifications being rather straightforward (at least when our space is Euclidean). Consider, for example, $\nabla \times \nabla \psi$. If we are in a Euclidean space, then we can easily compute this using a standard Cartesian coordinate system:

$$\begin{aligned}
\nabla \times \nabla \psi &= \nabla \times \left[\frac{\partial \psi}{\partial x}\mathbf{i} + \frac{\partial \psi}{\partial y}\mathbf{j} + \frac{\partial \psi}{\partial z}\mathbf{k} \right] \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \psi}{\partial x} & \dfrac{\partial \psi}{\partial y} & \dfrac{\partial \psi}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 \psi}{\partial y \, \partial z} - \frac{\partial^2 \psi}{\partial z \, \partial y} \right) \mathbf{i} - \left(\frac{\partial^2 \psi}{\partial x \, \partial z} - \frac{\partial^2 \psi}{\partial z \, \partial x} \right) \mathbf{j} - \left(\frac{\partial^2 \psi}{\partial x \, \partial y} - \frac{\partial^2 \psi}{\partial y \, \partial x} \right) \mathbf{k}
\end{aligned}$$

But remember, the order in which we compute two partial derivatives of a sufficiently differentiable function is irrelevant. So

$$\frac{\partial^2 \psi}{\partial y \, \partial z} = \frac{\partial^2 \psi}{\partial z \, \partial y} \quad , \quad \frac{\partial^2 \psi}{\partial x \, \partial z} = \frac{\partial^2 \psi}{\partial z \, \partial x} \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x \, \partial y} = \frac{\partial^2 \psi}{\partial y \, \partial x} \quad ,$$

and the above computations reduce to

$$\nabla \times \nabla \psi = \mathbf{0} \quad .$$

In fact, this equation holds more generally.

?►Exercise 9.17: *Use the more general formulas for the gradient and curl to verify that*

$$\nabla \times \nabla \psi = \mathbf{0}$$

even when the space is nonEuclidean (but still three-dimensional).

?►Exercise 9.18: *Verify that* $\nabla \cdot \nabla \times \mathbf{F} = 0$

a: *when the space is Euclidean.*

b: *when the space is arbitrary.*

One other identity that can be "easily verified" by simply computing things out in Cartesian coordinates is

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla \cdot \nabla \mathbf{F}$$

where $\nabla \cdot \nabla \mathbf{F}$ actually denotes the "Laplacian of $\mathbf{F}$", which is discussed (briefly) at the end of the next subsection.

The Laplacian

Because many problems in physics involve "conservative, divergence-free vector fields" (to be discussed later), the expression $\nabla \cdot \nabla \psi$ arises fairly often in applications. Because of this, a shorthand has been universally adopted of using either ∇^2 or Δ for $\nabla \cdot \nabla$,

$$\Delta \psi = \nabla^2 \psi = \nabla \cdot \nabla \psi \quad .$$

This operator, whether denoted Δ or ∇^2, is called the *Laplacian*. In Cartesian coordinates,

$$\nabla^2 \psi = \nabla \cdot \nabla \psi = \nabla \cdot \left[\sum_{k=1}^{N} \frac{\partial \psi}{\partial x^k} \mathbf{e}_k \right] = \sum_{k=1}^{N} \frac{\partial}{\partial x^k} \left(\frac{\partial \psi}{\partial x^k} \right) = \sum_{k=1}^{N} \frac{\partial^2 \psi}{\partial {x^k}^2} \quad .$$

Using the material developed a few pages ago, you can easily verify that, in polar coordinates, $\{(\rho, \phi)\}$,

$$\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \frac{\partial \psi}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} \quad ,$$

and in spherical coordinates, $\{(r, \theta, \phi)\}$,

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial \psi}{\partial r} \right] + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left[\sin(\theta) \frac{\partial \psi}{\partial \theta} \right] + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 \psi}{\partial \phi^2} \quad .$$

?►Exercise 9.19: *Using the material developed a few pages ago, derive the above polar and spherical coordinate formulas for the Laplacian.*

Of course, using the even more general formulas for divergence and gradient given in section 9.5, you can derive more general formulas for the Laplacian for arbitrary two- and three-dimensional spaces.

?►*Exercise 9.20:* *Let ψ be a scalar field on a two-dimensional space with orthogonal coordinates system and associated scaling factors*

$$\{(x^1, x^2)\} \qquad \text{and} \qquad \{h_1, h_2\} \quad .$$

Show that

$$\nabla^2 \psi \;=\; \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial x^1} \left[\frac{h_2}{h_1} \frac{\partial \psi}{\partial x^1} \right] + \frac{\partial}{\partial x^2} \left[\frac{h_1}{h_2} \frac{\partial \psi}{\partial x^2} \right] \right\} \quad .$$

?►*Exercise 9.21:* *Let ψ be a scalar field on a three-dimensional space with orthogonal coordinates system and associated scaling factors*

$$\{(x^1, x^2, x^3)\} \qquad \text{and} \qquad \{h_1, h_2, h_3\} \quad .$$

Show that

$$\nabla^2 \psi \;=\; \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x^1} \left[\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial x^1} \right] + \frac{\partial}{\partial x^2} \left[\frac{h_1 h_3}{h_2} \frac{\partial \psi}{\partial x^2} \right] + \frac{\partial}{\partial x^3} \left[\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial x^3} \right] \right\} \quad .$$

It turns out that Laplace's equation

$$\nabla^2 \psi \;=\; 0$$

and Poisson's equation

$$\nabla^2 \psi \;=\; \text{some known nonzero function}$$

are two of the fundamental partial differential equations of physics. Their solutions describe both "potentials of conservative vector fields" and "the steady state behavior of various phenomenon". We'll discuss one occasionally useful approach to solving these equations in a moment, and more general methods next term.

The Laplacian also appears in many other fundamental equations of physics. This includes the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \nabla^2 \psi \;=\; 0 \quad ,$$

the heat equation

$$\frac{\partial \psi}{\partial t} - \kappa \nabla^2 \psi \;=\; 0 \quad ,$$

and the Schrödinger wave equation

$$i\hbar \frac{\partial \psi}{\partial t} \;=\; -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \quad .$$

As a side note, it's worth mentioning that the Laplacian of a vector field can also be defined. If the vector field

$$\mathbf{F} \;=\; \sum_{k=1}^{N} F_k \, \mathbf{e}_k$$

has sufficiently differentiable components (the F_k's), then, using Cartesian coordinates,

$$\nabla^2 \mathbf{F} \;=\; \sum_{k=1}^{N} \left(\nabla^2 F^k \right) \mathbf{e}_k \quad .$$

9.7 Multidimensional Differential Operators under Radial Symmetry

Many problems in physics (and mathematics and engineering) involve radially symmetric scalar or vector fields in an N-dimensional Euclidean space. So let us restrict ourselves to an N-dimensional Euclidean space $\mathcal{E}$ to consider such things. Let us even assume that we've chosen some point to be the origin $\boldsymbol{O}$. Remember, in such cases we can define the corresponding position vector for each point

$$\mathbf{r}(\boldsymbol{x}) = \overrightarrow{\boldsymbol{O}\boldsymbol{x}} \quad .$$

Along with this position vector, we have the corresponding radial distance (distance from the origin)

$$r = r(\boldsymbol{x}) = \left\| \overrightarrow{\boldsymbol{O}\boldsymbol{x}} \right\| \quad ,$$

and the unit radial vector at each point

$$\widehat{\mathbf{r}} = \widehat{\mathbf{r}}(\boldsymbol{x}) = \frac{\mathbf{r}(\boldsymbol{x})}{r(\boldsymbol{x})} \quad .$$

A scalar field Ψ and a vector field $\mathbf{F}$ on $\mathcal{E}$ are said to be radially symmetric if (and only if) there are functions ψ and f on $(0, \infty)$ such that

$$\Psi(\boldsymbol{x}) = \psi(r) \qquad \text{and} \qquad \mathbf{F}(\boldsymbol{x}) = f(r)\,\widehat{\mathbf{r}}(\boldsymbol{x}) \quad .$$

Arfken, Weber and Harris somewhat discuss these functions and their gradients, divergences, curls, and Laplacians (mainly for the case with $N = 3$). However, you can better develop (and may already have done so) this material yourself in the homework (problems *O*, *P* and *Y* in of *Homework Handout VIII*). In particular, you will show (or have shown) that,

Lemma 9.1
If Ψ is a radially symmetric scalar field on an N-dimensional Euclidean space,

$$\Psi(\boldsymbol{x}) = \psi(r) \qquad \text{where} \quad r = \left\| \overrightarrow{\boldsymbol{O}\boldsymbol{x}} \right\| \quad ,$$

then

$$\nabla^2 \Psi = \psi''(r) + \frac{N-1}{r}\psi'(r) \quad .$$

An immediate consequence is that Laplace's equation reduces to the simple (and easily solved) ordinary differential equation

$$\psi''(r) + \frac{N-1}{r}\psi'(r) = 0$$

when we can assume radial symmetry.