10

Multidimensional Integration

These sections continue the development of multidimensional calculus. Here we concentrate on integration.

The main goals in this chapter are to develop the basics of "potential theory", as well as the classical theorems of Green, Gauss and Stokes, along with appropriate generalizations and related results. Much of our work will concern the relations between certain multidimensional integrals and the classical differential operator (gradient, divergence and curl) discussed earlier.

A warning: Where possible, I like to use $\iint$ for integrals over two-dimensional things and $\iiint$ for integrals over three-dimensional regions. Some others, including Arfken, Weber and Harris, tend to just use $\int$ no matter what is the dimension of the space being integrated over. When discussing results that apply to arbitrary-dimensioned spaces, just using $\int$ is reasonable; otherwise, in my opinion, it's just lazy.

10.1 Integrals over Curves/Paths/Lines/Contours

Basic Definitions

Let C be some "reasonably smooth" curve in space.[1,2] For convenience, we'll assume it is oriented with a "starting point" and an "ending point". Following historically established conventions, we'll let

$$s = \text{arclength along the curve from the starting point},$$

$$\boldsymbol{r} = \text{position on the curve}$$

and

$$\mathbf{T} = \frac{d\boldsymbol{r}}{ds} = \text{the unit tangent vector in the direction of travel of } C \text{ at } \boldsymbol{r}.$$

Now, let Φ be any (reasonably continuous) scalar or vector field on C. (This field may depend on $\mathbf{T}$, the unit vector in the "positive direction" at each point on the curve.) We can define the "integral of Φ over C" pretty well the same way we defined "the integral of a function over an interval" in elementary calculus — via Riemann sums. For each positive integer K, we

[1] While we usually visualize curves as being in two- or three-dimensional Euclidean space, most of what we'll discuss in this section does not require that the space be Euclidean.

[2] You may want to quickly review the material on curves in sections 8.4 and 8.6.

set

$$\Delta s \;=\; \Delta s_K \;=\; \frac{\text{total arclength of } \mathcal{C}}{K} \quad ,$$

and then, starting starting with

$$\boldsymbol{r}_0 \;=\; \text{“the starting point”} \quad ,$$

successively we pick the points $\boldsymbol{r}_0$, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, …, $\boldsymbol{r}_K$ on $\mathcal{C}$ separated by arclength Δs (so $\boldsymbol{r}_K$ ends up being the “ending point” of $\mathcal{C}$). The corresponding K^{th} Riemann sum S_K is then given by

$$\sum_{k=0}^{K-1} \Phi(\boldsymbol{r}_k)\,\Delta s \quad .$$

The *integral of* Φ *over* $\mathcal{C}$, denoted $\int_{\mathcal{C}} \Phi\, ds$, is then defined as the limit of these Riemann sums as $\Delta s \to 0$ (equivalently, as $K \to \infty$),

$$\int_{\mathcal{C}} \Phi\, ds \;=\; \lim_{\Delta s \to 0} \sum_{k=0}^{K-1} \Phi(\boldsymbol{r}_k)\,\Delta s \quad . \tag{10.1}$$

In particular, given a scalar field ψ on $\mathcal{C}$ and a vector field $\mathbf{F}$ on $\mathcal{C}$, the above defines the integrals[3]

$$\int_{\mathcal{C}} \psi\, ds \qquad , \qquad \int_{\mathcal{C}} \psi \mathbf{T}\, ds \quad ,$$

$$\int_{\mathcal{C}} \mathbf{F}\, ds \qquad , \qquad \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\, ds \qquad \text{and} \qquad \int_{\mathcal{C}} \mathbf{F}\times\mathbf{T}\, ds \quad .$$

Sometimes these sorts of integral are referred to as either path integrals, contour integrals, or line integrals.

The definition given by equation (10.1) is essentially the same as given for the definite integral in elementary calculus. The only differences is that, here, we have a curve instead of a straight interval, and we have a somewhat more general notion of just what our function Φ might be. Consequently, everything (that should have been) learned in elementary calculus applies here.

To compute one of these integrals, it is usually best to convert the basic definition formula to an equivalent definite integral formula based on some parametrization for $\mathcal{C}$. So assume

$$\boldsymbol{r}(t) \qquad \text{where} \quad t_0 \le t \le t_1$$

is some parametrization for $\mathcal{C}$. Remember, this means that $\boldsymbol{r}(t)$ traces out $\mathcal{C}$ from “start” to “end” as t goes from t_0 to t_1. Remember, also, that

$$\begin{aligned} \frac{d\boldsymbol{r}}{dt} \;&=\; \text{“velocity of } \boldsymbol{r}(t)\text{”} \\ &=\; \text{“speed”} \times \text{“the unit vector in the direction of travel”} \;=\; \frac{ds}{dt}\mathbf{T} \quad , \end{aligned}$$

and that

$$\frac{ds}{dt} \;=\; \left\| \frac{d\boldsymbol{r}}{dt} \right\| \quad .$$

[3] Of course, for $\int_{\mathcal{C}} \mathbf{F}\times\mathbf{T}\, ds$ the space must be three-dimensional. Also, the integrals $\int_{\mathcal{C}} \psi\mathbf{T}\, ds$ and $\int_{\mathcal{C}} \mathbf{F}\, ds$ may be problemmatic if the space is not Euclidean.

So, in the integrals, we can use the substitutions

$$ds = \frac{ds}{dt}\,dt = \left\|\frac{d\boldsymbol{r}}{dt}\right\| dt \qquad \text{and} \qquad \mathbf{T}\,ds = \frac{d\boldsymbol{r}}{dt}\,dt = \text{“}d\boldsymbol{r}\text{”} \quad .$$

With these substitutions, the above integrals become the following integral formulas. If you actually write out these integrals in terms of the corresponding coordinate formulas, you will see that they are simply “elementary calculus” definite integrals. Also, for some, we will give a standard “shorthand” way of denoting the integral which is supposed to remind you how to compute it.

$$\int_C \psi\,ds = \int_{t_0}^{t_1} \psi\big(\boldsymbol{r}(t)\big)\frac{ds}{dt}\,dt = \int_{t_0}^{t_1} \psi\big(\boldsymbol{r}(t)\big)\left\|\frac{d\boldsymbol{r}}{dt}\right\| dt \tag{10.2}$$

$$\int_C \psi\mathbf{T}\,ds = \int_{t_0}^{t_1} \psi\big(\boldsymbol{r}(t)\big)\frac{d\boldsymbol{r}}{dt}\,dt = \int_C \psi\,d\boldsymbol{r} \tag{10.3}$$

$$\int_C \mathbf{F}\,ds = \int_{t_0}^{t_1} \mathbf{F}\big(\boldsymbol{r}(t)\big)\left\|\frac{d\boldsymbol{r}}{dt}\right\| dt \tag{10.4}$$

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_{t_0}^{t_1} \mathbf{F}\big(\boldsymbol{r}(t)\big)\cdot\frac{d\boldsymbol{r}}{dt}\,dt = \int_C \mathbf{F}\cdot d\boldsymbol{r} \tag{10.5}$$

$$\int_C \mathbf{F}\times\mathbf{T}\,ds = \int_{t_0}^{t_1} \mathbf{F}\big(\boldsymbol{r}(t)\big)\times\frac{d\boldsymbol{r}}{dt}\,dt = \int_C \mathbf{F}\times d\boldsymbol{r} \tag{10.6}$$

For us, (10.2), (10.5) and (10.6) will be the types of integrals of greatest interest. Integrals of the form (10.5) will be of especial interest since these describe total work done by a force field $\mathbf{F}$ in moving an object along a path given by C.

Do observe that, if $\psi = 1$, then (10.2) gives

$$\int_C ds = \int_C \frac{ds}{dt}\,dt = \text{“total arclength of } C\text{”} \quad .$$

With a little thought, you should also see that, with $\psi = 1$, (10.3) gives

$$\int_C \mathbf{T}\,ds = \int_C d\boldsymbol{r} = \overrightarrow{\boldsymbol{r}(t_0)\,\boldsymbol{r}(t_1)} = \text{“total change in position in traveling over } C\text{”} \quad ,$$

at least when the curve is in a Euclidean space.

Various Notes on Computing Line Integrals

1. From our derivations and definitions, it is clear that the values of these integrals do not depend on the choice of parametrization, so long as the parametrization “follows the orientation” of the curve. In fact, in most applications involving integrals of the forms

$$\int_C \psi\,ds \qquad \text{and} \qquad \int_C \mathbf{F}\,ds$$

the orientation of the curve is not relevant.

2. Remember that the curve C has exactly two orientations, and that, if $\mathbf{T}$ is a unit vector at a point $\boldsymbol{x}$ on C pointing in the positive direction according to one orientation, then $-\mathbf{T}$ points in the positive direction according to the other orientation. Clearly then, given an oriented curve C and letting $-C$ be the same curve with the opposite orientation, then

$$\int_{-C} [\text{whatever}]\, d\boldsymbol{r} \;=\; -\int_{C} [\text{whatever}]\, d\boldsymbol{r} \quad .$$

So, in practice, if you can use a "nice" parametrization which traces out the curve in the wrong direction — just multiply the resulting integral by -1.

3. It should also be obvious from our basic definition that, if C is the union of two subcurves C_1 and C_2, with the start of one being at the end of the other, then you can break up any integral over C into the sum of integrals over C_1 and C_2,

$$\int_{C_1 \cup C_2} \;=\; \int_{C_1} \;+\; \int_{C_2} \quad ,$$

using whatever convenient parametrizations for C_1 and C_2 seems convenient.

4. Be careful about one thing: Just because C and Γ are two curves starting at the same point and ending at the same point, do NOT assume they yield the same integrals. In general,

$$\int_{C} \Phi\, ds \;\neq\; \int_{\Gamma} \Phi\, ds \quad .$$

Certain exceptions to this are of special interest and will be discussed in the next section.

Coordinate Formulas for Computing Line Integrals

If you are actually forced to *compute* one of these integrals, first choose a convenient coordinate system

$$\left\{ x^1,\, x^2,\, \ldots,\, x^N \right\}$$

with associated scaling factors and tangent vectors and normalized tangent vectors,

$$\{ h_1,\, h_2,\, \ldots,\, h_N \} \quad , \quad \{ \boldsymbol{\varepsilon}_1,\, \boldsymbol{\varepsilon}_2,\, \ldots,\, \boldsymbol{\varepsilon}_N \} \quad \text{and} \quad \{ \mathbf{e}_1,\, \mathbf{e}_2,\, \ldots,\, \mathbf{e}_N \} \quad .$$

Then choose a convenient parametrization for the curve C,

$$\boldsymbol{r}(t) \;\sim\; \left(x^1(t), x^2(t),\, \ldots,\, x^N(t) \right) \qquad \text{for} \quad t_0 < t < t_1 \quad .$$

Then write everything in the corresponding integral

$$\int_{t_0}^{t_1} [\cdots]\, dt$$

in terms of the coordinate system you wish to use, replacing each x^i with its formula from the parametrization $x^i(t)$, and then evaluate the resulting integral using elementary calculus methods.

The integral formulas involving $d\boldsymbol{r}$ end up being the simpler ones. Recalling the formulas from the section on motion, section 9.1, we have

$$d\boldsymbol{r} \;=\; \frac{d\boldsymbol{r}}{dt}\,dt \;=\; \sum_{i=1}^{N} \frac{dx^i}{dt}\frac{\partial \boldsymbol{r}}{\partial x^i}\,dt \;=\; \sum_{i=1}^{N} \frac{dx^i}{dt} h_i \mathbf{e}_i\,dt \quad . \tag{10.7}$$

Thus, if $\mathbf{F}$ is a vector field with coordinate formula

$$\sum_{i=1}^{N} F^i\big(x^1, x^2, \ldots, x^N\big)\,\mathbf{e}_i \quad ,$$

then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\boldsymbol{r} \;&=\; \int_{t=t_0}^{t_1} \mathbf{F}\cdot\frac{d\boldsymbol{r}}{dt}\,dt \\ &=\; \int_{t=t_0}^{t_1} \left[\sum_{i=1}^{N} F^i\,\mathbf{e}_i\right]\cdot\left[\sum_{j=1}^{N} h_j\frac{dx^j}{dt}\mathbf{e}_j\right] dt \;=\; \int_{t=t_0}^{t_1} \sum_{i=1}^{N}\sum_{j=1}^{N} F^i h_j \frac{dx^j}{dt}\,\mathbf{e}_i\cdot\mathbf{e}_j\,dt \quad .\end{aligned}$$

and (assuming $N = 3$)

$$\begin{aligned}\int_C \mathbf{F}\times d\boldsymbol{r} \;&=\; \int_{t=t_0}^{t_1} \mathbf{F}\times\frac{d\boldsymbol{r}}{dt}\,dt \\ &=\; \int_{t=t_0}^{t_1} \left[\sum_{i=1}^{3} F^i\,\mathbf{e}_i\right]\times\left[\sum_{j=1}^{3} h_j\frac{dx^j}{dt}\mathbf{e}_j\right] dt \;=\; \int_{t=t_0}^{t_1} \sum_{i=1}^{3}\sum_{j=1}^{3} F^i h_j \frac{dx^j}{dt}\,\mathbf{e}_i\times\mathbf{e}_j\,dt \quad .\end{aligned}$$

If we are intelligent and chose an orthogonal coordinate system, the first reduces to

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} \;=\; \int_{t=t_0}^{t_1} \sum_{i=1}^{N} F^i h_i \frac{dx^i}{dt}\,dt \quad .$$

If, in addition, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is right handed at each point, then the second becomes

$$\int_C \mathbf{F}\times d\boldsymbol{r} \;=\; \int_{t=t_0}^{t_1} \left[\left(F^2 h_3\frac{dx^3}{dt} - F^3 h_2\frac{dx^2}{dt}\right)\mathbf{e}_1 \;-\; \cdots\right] dt \quad .$$

For ds we can use the above or go back to our formulas from the section on motion, section 9.1, obtaining, in general

$$ds \;=\; \frac{ds}{dt}\,dt \;=\; \left\|\frac{d\boldsymbol{r}}{dt}\right\|\,dt \;=\; \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{dx^i}{dt}\frac{dx^j}{dt}\,h_i h_j\,\mathbf{e}_i\cdot\mathbf{e}_j}\;dt \tag{10.8}$$

$$=\; \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{dx^i}{dt}\frac{dx^j}{dt}\,g_{ij}}\;dt \tag{10.8$'$}$$

If the coordinate system is orthogonal, the last reduces to

$$ds \;=\; \sqrt{\sum_{i=1}^{N}\left(h_i\,\frac{dx^i}{dt}\right)^2}\;dt \tag{10.9}$$

or, equivalently

$$ds \;=\; \sqrt{\sum_{i=1}^{N} g_{ii} \left(\frac{dx^i}{dt}\right)^2}\; dt \quad . \tag{10.9$'$}$$

Be warned that notation is habitually abused. Many authors will "cancel out the dt's" and write formula (10.7) as

$$d\boldsymbol{r} \;=\; \sum_{j=1}^{N} h_j\, \mathbf{e}_j\, dx^j$$

or worse. In addition, many authors will "cancel out the dt's" and then square both sides of, say, expression (10.8′) or (10.9) and state

$$ds^2 \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}\, dx^i\, dx^j \qquad \text{or} \qquad ds^2 \;=\; \sum_{i=1}^{N} \left(h_i\, dx^i\right)^2 \quad .$$

without having the courtesy to explain that ds^2 "means" $(ds)^2$, not $d(s^2)$, and that they are omiting the parametrization variable (t, for us).

10.2 A Little Potential Theory

Conservative Vector Fields

A vector field $\mathbf{F}$ is said to be *conservative* (in a region $\mathcal{R}$) if and only if there is a sufficiently smooth and "well-defined" scalar field ψ such that

$$\mathbf{F} \;=\; -\nabla\psi \qquad \text{everywhere in} \quad \mathcal{R} \quad . \tag{10.10}$$

The scalar field ψ is called a (scalar) potential for $\mathbf{F}$. (Precisely what is meant by the scalar field being "well defined" is discussed at the end of this section – see page 10–11.)

If $\mathbf{F}(\boldsymbol{x})$ represents some force field acting on an object at position $\boldsymbol{x}$, then the potential at that point, $\psi(\boldsymbol{x})$, describes the "potential energy" in the object due to the force field. The negative sign in the above definition is there to ensure that this potential energy decreases as the force "does its work" on the object. (Other texts may or may not require this negative sign.)

There are two quick observations that should be made:

1. The potential function is not unique. Since the gradient of a constant scalar field is the zero vector, we can add any constant value C to a potential function ψ for a given $\mathbf{F}$ and the resulting scalar field $\phi = \psi + C$ will also be a valid potential function for $\mathbf{F}$.[4]

2. If $\mathbf{F}$ is a divergence-free and conservative vector field (with potential function ψ), then

   $$0 \;=\; \nabla\cdot\mathbf{F} \;=\; \nabla\cdot(-\nabla\psi) \;=\; -\nabla^2\psi \quad .$$

   Consequently, to find a formula for the vector field $\mathbf{F}$, it suffices to solve Laplace's equation

   $$\nabla^2\psi \;=\; 0$$

[4] It is also easy to show that if ϕ and ψ are both potentials for the same vector field, then $\phi = \psi + C$ for some constant C.

for the potential function, and then compute $\mathbf{F} = -\nabla\psi$. (Typically, there will also be some additional conditions that ψ will need to satisfy on the region's boundary.)

?►***Exercise 10.1:*** *Assume an object of mass m is under the influence of some force field $\mathbf{F}$, and let $\boldsymbol{x}(t)$ be the position of the object at time t (so $\mathbf{F}(\boldsymbol{x}(t))$ is the force acting on the object at time t). The object's "kinetic energy" K (energy in its motion) is given by*

$$K(t) \;=\; \frac{1}{2}m\,\|\mathbf{v}(t)\|^2 \qquad \text{where} \quad \mathbf{v} \;=\; \frac{d\boldsymbol{x}}{dt} \quad .$$

Assume $\mathbf{F}$ is conservative with potential ψ . Show that the "total energy" of the object

$$E(t) \;=\; K(t) \;+\; \psi(\boldsymbol{x}(t))$$

remains constant by showing that $E'(t) = 0$. (Don't forget that $\mathbf{F} = m\mathbf{a}$!)

Conservative Vector Fields and "Path-Independence" of Integrals

Let's assume a vector field $\mathbf{F}$ is conservative with potential function ψ in some region $\mathcal{R}$, and let's consider the path integral

$$\int_C \mathbf{F}\cdot d\boldsymbol{r}$$

where C is an oriented curve in region $\mathcal{R}$ with starting point $\boldsymbol{x}_S$ and ending point $\boldsymbol{x}_E$. Take any sufficiently smooth parametrization

$$\boldsymbol{r}(t) \qquad \text{with} \quad t_S \le t \le t_E$$

of this curve (tracing it out in the correct direction, so that $\boldsymbol{r}(t_S) = \boldsymbol{x}_S$ and $\boldsymbol{r}(t_E) = \boldsymbol{x}_E$), and observe that, by the general definition of gradient (formula (9.11) on page 9–17),

$$\mathbf{F}\cdot\frac{d\boldsymbol{r}}{dt} \;=\; -\nabla\psi\cdot\frac{d\boldsymbol{r}}{dt} \;=\; -\frac{d}{dt}\big[\psi\big(\boldsymbol{r}(t)\big)\big] \quad .$$

So,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\boldsymbol{r} \;&=\; -\int_{t_0}^{t_1}\frac{d}{dt}\big[\psi\big(\boldsymbol{r}(t)\big)\big]\,dt \\ &=\; -\big\{\psi\big(\boldsymbol{r}(t_E)\big) \;-\; \psi\big(\boldsymbol{r}(t_S)\big)\big\} \;=\; -\{\psi(\boldsymbol{x}_E) \;-\; \psi(\boldsymbol{x}_S)\} \quad .\end{aligned}$$

That is,

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} \;=\; -\big\{\psi\big(\text{end of } C\big) \;-\; \psi\big(\text{start of } C\big)\big\} \quad . \tag{10.11}$$

In this case, the value of $\int_C \mathbf{F}\cdot d\boldsymbol{r}$ does not depend on the actual path between the starting and ending points of the curve. If Γ is any other curve in region $\mathcal{R}$ with the same starting ending points as C , then

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} \;=\; \int_\Gamma \mathbf{F}\cdot d\boldsymbol{r} \quad .$$

In cases like this, where the specific choice of the curve C between $\boldsymbol{x}_S$ and $\boldsymbol{x}_E$ is irrelevant in determining the value of the integral $\int_C \mathbf{F}\cdot d\boldsymbol{r}$, we refer to the integral as being *path independent*, and, on occasion, we may even denote the integral by

$$\int_{\boldsymbol{x}_S}^{\boldsymbol{x}_E} \mathbf{F}\cdot d\boldsymbol{r} \quad .$$

In the above, we've shown that

$$\mathbf{F} \text{ is conservative on } \mathcal{R} \quad \Longrightarrow \quad \int_C \mathbf{F}\cdot d\boldsymbol{r} \text{ is path independent whenever } C \text{ is in } \mathcal{R} \quad .$$

Conversely, if you know

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} \text{ is path independent whenever } C \text{ is in } \mathcal{R} \quad ,$$

then you can take any point $\boldsymbol{x}_0$ in the region, and define a scalar field ψ on $\mathcal{R}$ by

$$\psi(\boldsymbol{x}) \;=\; -\int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \mathbf{F}\cdot d\boldsymbol{r} \quad .$$

With just a little careful analysis, you can differentiate this integral and show that

$$-\nabla\psi \;=\; \mathbf{F} \qquad \text{everywhere in} \quad \mathcal{R} \quad .$$

So $\mathbf{F}$ is conservative with potential ψ. Thus, we actually have

$$\mathbf{F} \text{ is conservative on } \mathcal{R} \quad \Longleftrightarrow \quad \int_C \mathbf{F}\cdot d\boldsymbol{r} \text{ is path independent whenever } C \text{ is in } \mathcal{R} \quad . \tag{10.12}$$

This is useful. There are many situations where the work done by a force field $\mathbf{F}$ is observed to be path independent. By the above, we know that it can then be mathematically described as the negative of the gradient of some scalar field. This can simplify finding a formula for this force field.

While on the subject of "path-independence", it is worth observing that "path-independence" is equivalent to the vanishing of corresponding integrals over "closed" curves. A *closed* curve C is a simply a finite loop on which any point can be considered as both the starting and the ending point. If C is such a loop (with any orientation), then we can take any two points $\boldsymbol{x}_S$ and $\boldsymbol{x}_E$ on the curve, and let C_1 and C_2 be the two subcurves with C_1 being the piece going from $\boldsymbol{x}_S$ to $\boldsymbol{x}_E$ in the positive direction, and C_2 being the piece continuing from $\boldsymbol{x}_E$ to $\boldsymbol{x}_S$ in the positive direction. (`Draw your own picture below`)

Note that, with the implied orientations, C is "$C_1 + C_2$", while C_1 and $-C_2$ are both oriented curves from $\boldsymbol{x}_S$ to $\boldsymbol{x}_E$. Thus, if integrals of $\mathbf{F}$ are path independent in the region $\mathcal{R}$ and C is in $\mathcal{R}$, then

$$\int_{C_1} \mathbf{F}\cdot d\boldsymbol{r} \;=\; \int_{-C_2} \mathbf{F}\cdot d\boldsymbol{r} \;=\; -\int_{C_2} \mathbf{F}\cdot d\boldsymbol{r} \quad ,$$

and thus

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} \;=\; \int_{C_1} \mathbf{F}\cdot d\boldsymbol{r} \;+\; \int_{C_2} \mathbf{F}\cdot d\boldsymbol{r} \;=\; -\int_{C_2} \mathbf{F}\cdot d\boldsymbol{r} \;+\; \int_{C_2} \mathbf{F}\cdot d\boldsymbol{r} \;=\; 0 \quad .$$

Conversely, suppose we know

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} \;=\; 0$$

whenever C is a closed curve in $\mathcal{R}$. Let Γ_1 and Γ_2 be any two curves in $\mathcal{R}$ from one point $\boldsymbol{x}_S$ to another $\boldsymbol{x}_E$ and observe that "$C = \Gamma_1 - \Gamma_2$" is a closed curve. By assumption, then,

$$\int_{\Gamma_1} \mathbf{F}\cdot d\boldsymbol{r} \;-\; \int_{\Gamma_2} \mathbf{F}\cdot d\boldsymbol{r} \;=\; \int_C \mathbf{F}\cdot d\boldsymbol{r} \;=\; 0 \quad ,$$

which tells us that we have "path independence",

$$\int_{\Gamma_1} \mathbf{F}\cdot d\boldsymbol{r} \;=\; \int_{\Gamma_2} \mathbf{F}\cdot d\boldsymbol{r} \;=\; \int_{\boldsymbol{x}_S}^{\boldsymbol{x}_E} \mathbf{F}\cdot d\boldsymbol{r} \quad .$$

In summary, we have:

Theorem 10.1 (the big theorem on conservative vector fields)
Let $\mathbf{F}$ be a vector field on a region $\mathcal{R}$. If any one of the following statements holds, then they all hold:

1. *$\mathbf{F}$ is conservative on $\mathcal{R}$; that is, there is a well-defined scalar field ψ such that*

   $$\mathbf{F} \;=\; -\nabla\psi \qquad \text{on} \quad \mathcal{R} \quad .$$

2. *For any curve C in $\mathcal{R}$, the value of*

   $$\int_C \mathbf{F}\cdot d\boldsymbol{r}$$

   depends only on the starting and ending points of C.

3. *If C is any closed curve in $\mathcal{R}$, then*

   $$\int_C \mathbf{F}\cdot d\boldsymbol{r} \;=\; 0 \quad .$$

Finding the Potential, If Possible

What all the above does not tell us is how to determine when a particular vector field formula

$$\mathbf{F}(x^1, x^2, \ldots, x^N) \;=\; \sum_{k=1}^{N} F^k\big(x^1, x^2, \ldots, x^N\big)\, \mathbf{e}_k$$

gives a conservative vector field. One approach is to attempt to compute all possible path integrals to see if we have "path-independence" — but this is hardly practical. A more realistic approach is to attempt to find a scalar field formula $\psi(x^1, x^2, \ldots, x^N)$ such that

$$\mathbf{F} \;=\; -\nabla\psi \quad .$$

Assuming our space is Euclidean and we are using a Cartesian coordinate system, this vector field equation can be written explicitly as

$$\sum_{k=1}^{N} F^k\big(x^1, x^2, \ldots, x^N\big)\, \mathbf{e}_k \;=\; -\sum_{k=1}^{N} \frac{\partial \psi}{\partial x^k}\mathbf{e}_k \quad .$$

This means we need to find a single formula $\psi(x^1, x^2, \ldots, x^N)$ satisfying

$$\frac{\partial \psi}{\partial x^k} \;=\; -F^k\big(x^1, x^2, \ldots, x^N\big) \qquad \text{for} \quad k = 1, 2, \ldots, N \quad .$$

This is do-able.[5] In fact, you may have done it in calculus (finding the potential of a conservative vector field!) and in differential equations (solving an exact first-order differential equation), so we won't discuss it in detail here.

Still, attempting to find ψ can be time consuming. What's more, you may go through several pages of computations before discovering that there is no solution (hence, either $\mathbf{F}$ is not conservative, or you made a mathematical error somewhere). Fortunately, there is a relatively simple test for determining if a given $\mathbf{F}$ is conservative. The three-dimensional version is inspired by the observation we made a while back that

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi \;=\; \mathbf{0} \quad .$$

Thus, if $\mathbf{F}$ is conservative (and differentiable), then letting ψ be any potential for $\mathbf{F}$,

$$\boldsymbol{\nabla} \times \mathbf{F} \;=\; -\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi \;=\; \mathbf{0} \quad .$$

This immediately gives us that

$$\boldsymbol{\nabla} \times \mathbf{F} \;\neq\; \mathbf{0} \quad \Longrightarrow \quad \mathbf{F} \text{ is NOT conservative} \quad .$$

If the region $\mathcal{R}$ has "no holes through it"[6] (and $\mathbf{F}$ is differentiable), then you can use Stoke's theorem (to be discussed later) to show that

$$\boldsymbol{\nabla} \times \mathbf{F} \;=\; \mathbf{0} \quad \Longrightarrow \quad \mathbf{F} \text{ is conservative} \quad .$$

However, if the region $\mathcal{R}$ has a hole or $\mathbf{F}$ is not differentiable in then it is possible for $\mathbf{F}$ to not be conservative in the entire region even though $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$. (An apparent potential ψ can still be derived, but it isn't single valued — see *Comments on the Potentials Being "Well Defined"* starting on page 10–11.)

In summary, here is what can be said:

Theorem 10.2 (Test for Conservatism in Three-dimensional Space)
Let $\mathbf{F}$ be a differentiable vector field on a three-dimensional region $\mathcal{R}$. Then

1. *$\mathbf{F}$ is not conservative if $\boldsymbol{\nabla} \times \mathbf{F} \neq \mathbf{0}$.*
2. *If $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ and $\mathcal{R}$ has "no holes through it", then $\mathbf{F}$ is conservative.*

[5] First integrate $\partial\psi/\partial x^1 = -F^1$ with respect to x^1, plug the result into $\partial\psi/\partial x^2 = -F^2$ and integrate with respect to x^2, … .

[6] What this really means is that every closed loop in $\mathcal{R}$ is the boundary of some surface in $\mathcal{R}$.

Extending the "curl test for conservativism" is easily done, at least when the space is Euclidean and the coordinate system is Cartesian:

?►***Exercise 10.2:*** *Assuming the standard two-dimensional Cartesian coordinate system on a Euclidean plane, let*

$$\mathbf{F}(x, y) = F^1(x, y)\,\mathbf{i} + F^2(x, y)\,\mathbf{j}$$

be the coordinate/component formula for some vector field $\mathbf{F}$ *. Verify that the "curl test for conservativism" given above reduces to*

$$\mathbf{F} \text{ is conservative on } \mathcal{R} \iff \frac{\partial F^1}{\partial y} = \frac{\partial F^2}{\partial x} \quad \text{throughout } \mathcal{R}$$

(provided $\mathcal{R}$ *is a region in the plane "with no holes").*

?►***Exercise 10.3:*** *Assume* $\mathcal{R}$ *is a region in an* N*-dimensional Euclidean space, and let*

$$\mathbf{F}(x^1, x^2, \ldots, x^N) = \sum_{k=1}^{N} F^k\big(x^1, x^2, \ldots, x^N\big)\,\mathbf{e}_k$$

be the coordinate/component formula for a vector field $\mathbf{F}$ *using a Cartesian coordinate system with corresponding basis. Show that, if* $\mathbf{F}$ *is conservative on* $\mathcal{R}$*, then its components must satisfy*

$$\frac{\partial F^j}{\partial x^k} = \frac{\partial F^k}{\partial x^j} \qquad \text{for all} \quad j \neq k \tag{10.13}$$

throughout the region $\mathcal{R}$ *.*

In the general test for conservativism in an arbitrary dimensional space, equation system (10.13) replaces the $\nabla \times \mathbf{F} = \mathbf{0}$ equation.

Comments on the Potentials Being "Well Defined"

According to our big theorem, we have three basic definitions for "$\mathbf{F}$ is conservative in a region $\mathcal{R}$":

1. $\mathbf{F} = -\nabla\psi$ on $\mathcal{R}$ for some "well-defined" scalar field ψ.
2. $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ is path independent for each curve C in $\mathcal{R}$.
3. $\int_C \mathbf{F} \cdot d\boldsymbol{r} = 0$ for each closed curve C in $\mathcal{R}$.

We started with the first simply because it was fairly easy to see how the second followed from the first. However, the second and third definitions could be considered "more fundamental" since they directly correspond to a physical quantity (i.e., "work") and not to some mathematical construction (i.e., the "potential").

The reason I raise this point is that there is a complication that can arise with the first definition if we are not careful about insisting ψ be "well defined". To be precise, we need ψ to be "single valued". To see the issue more clearly, consider the two-dimensional situation where

we have polar coordinates $\{(\rho, \phi)\}$. Then ϕ is a scalar field on the plane (excluding the origin) and we can define

$$\mathbf{F} = -\nabla\phi \quad .$$

At the chosen origin $\boldsymbol{O}$, of course, ϕ is clearly not well defined, so let's use the plane minus the origin for our region $\mathcal{R}$. There is still a problem, however, because, if $(\rho, \phi) = (\rho_0, \phi_0)$ are the polar coordinates for some point in $\mathcal{R}$, then so are $(\rho, \phi) = (\rho_0, \phi_0 + 2\pi)$. This means ϕ is not single valued on $\mathcal{R}$. Still, the computations leading to equation (10.11) remain valid with $\mathbf{F} = -\nabla\phi$, and they give us

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = -\big\{\phi\big(\text{end of } C\big) - \phi\big(\text{start of } C\big)\big\}$$

provided we choose the values of ϕ so that $\phi(\boldsymbol{r})$ varies continuously as $\boldsymbol{r}$ moves along the curve. In particular, if C is the unit circle oriented counterclockwise and we choose the starting point to be $\boldsymbol{r}_0 \sim (\rho, \phi) = (1, 0)$, then, as $\boldsymbol{r}$ travels around this circle, $\phi(\boldsymbol{r})$ will go from 0 to 2π. Thus,

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = -\big\{\phi\big(\text{end of } C\big) - \phi\big(\text{start of } C\big)\big\} = -\{2\pi - 0\} = -2\pi$$

even though this curve starts and ends at the same point, $\boldsymbol{r}_0$. This shows that integrals of the form $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ for this vector field $\mathbf{F}$ are not, in general, path independent even though $\mathbf{F} = -\nabla\phi$. And this, in turn, is why we do not want to consider this vector field as being "conservative on $\mathcal{R}$" even though $\mathbf{F} = -\nabla\phi$.

On the other hand, you should be able to convince yourself that, if the region $\mathcal{R}$ is further restricted so that curves cannot wrap around the origin (where ϕ is not defined) — say, by cutting out what's normally called the negative X–axis — then

$$\mathbf{F} = -\nabla\phi \qquad \text{with} \quad -\pi < \phi < \pi$$

is a conservative vector field on this more restricted region.

10.3 Surface Integrals

Basic Definitions

Let's now turn our attention to a (fairly) arbitrary two-dimensional space $\mathcal{S}$. It won't hurt to visualize it as a surface "floating" in a three-dimensional Euclidean space $\mathcal{E}$. However, for much of what we will be doing, $\mathcal{S}$ does not have to be such a surface. For example, it could be the two-dimensional Euclidean plane or even the surface of a Klein bottle.

An integral over the surface $\mathcal{S}$ of a scalar field ψ (or vector field $\mathbf{F}$) can be defined much the same way we defined integrals over curves, only instead of partitioning a curve into pieces of length Δs, we partition the surface into a "two-dimensional grid" of tiny subsurfaces with ΔA_{jk} being the area of the $(j, k)^{\text{th}}$ subsurface". Corresponding to this grid of subsurfaces is a grid of points $\{\boldsymbol{p}_{j,k}\}$ with $\boldsymbol{p}_{j,k}$ being a point in the $(j, k)^{\text{th}}$ subsurface.

Try sketching the appropriate picture yourself!

Then we define

$$\iint_{\mathcal{S}} \psi \, dA \;=\; \lim_{\Delta A \to 0} \sum_j \sum_k \psi(\boldsymbol{p}_{jk}) \, \Delta A_{j,k} \quad , \tag{10.14}$$

where the "$\Delta A \to 0$" indicates that we keep taking grids with smaller and smaller subsurfaces.

From this definition, it should be clear that

$$\iint_{\mathcal{S}} dA \;=\; \text{total surface area of } \mathcal{S} \quad ,$$

and, if

$$\psi(\boldsymbol{p}) \;=\; \text{the planar density (amount/unit area) of something at } \boldsymbol{p} \text{ in } \mathcal{S} \quad ,$$

then

$$\iint_{\mathcal{S}} \psi \, dA \;=\; \text{total amount of that something in } \mathcal{S} \quad .$$

The "dA" in the integral is often called the (infinitesimal) element of area. For computations, it usually needs to be converted to the appropriate formula involving the coordinates being used. If the surface happens to be in the XY–plane, then we have the well-known Cartesian and polar formulas

$$dA \;=\; dx\,dy \qquad \text{and} \qquad dA \;=\; \rho\,d\rho\,d\phi \quad .$$

Coordinate Formulas for the Element of Area
General Case

To derive the formula for dA for more general surfaces, let $\{(u, v)\}$ be any coordinate system for $\mathcal{S}$. This coordinate system has corresponding scaling factors $\{h_u, h_v\}$ along with corresponding unit tangent vectors $\{\mathbf{e}_u, \mathbf{e}_v\}$ at each point in $\mathcal{S}$.

We can partition our region by choosing (small) positive values Δu and Δv, and using a grid of points whose u-coordinates and v-coordinates differ by Δu and Δv, respectively. The region corresponding to a given point $\boldsymbol{p}_{jk} \sim (u_j, v_k)$ in this grid is the region bounded by the coordinate curves passing through this point,

$$u \;=\; u_j \qquad \text{and} \qquad v \;=\; v_k \quad ,$$

along with the coordinate curves corresponding to "increasing u by Δu and v by Δv",

$$u \;=\; u_j \,+\, \Delta u \qquad \text{and} \qquad v \;=\; v_k \,+\, \Delta v \quad ,$$

as illustrated in figure 10.1a.

Now let $\boldsymbol{p} = \boldsymbol{p}_{jk} \sim (u_j, v_k)$, and let Δs be the arclength along the curve $v = v_k$ from $\boldsymbol{p}$ to where $u = u_i + \Delta u$. Then

$$\frac{\Delta s}{\Delta u} \;\approx\; \left.\frac{ds}{du}\right|_{\boldsymbol{p}} \;=\; \left\| \left.\frac{d\boldsymbol{x}}{du}\right|_{\boldsymbol{p}} \right\| \;=\; \|\boldsymbol{\varepsilon}_u(\boldsymbol{p})\| \;=\; h_u(\boldsymbol{p}) \quad .$$

Hence, letting $h_u = h_u(\boldsymbol{p})$,

$$\Delta s \;\approx\; h_u \Delta u \quad .$$

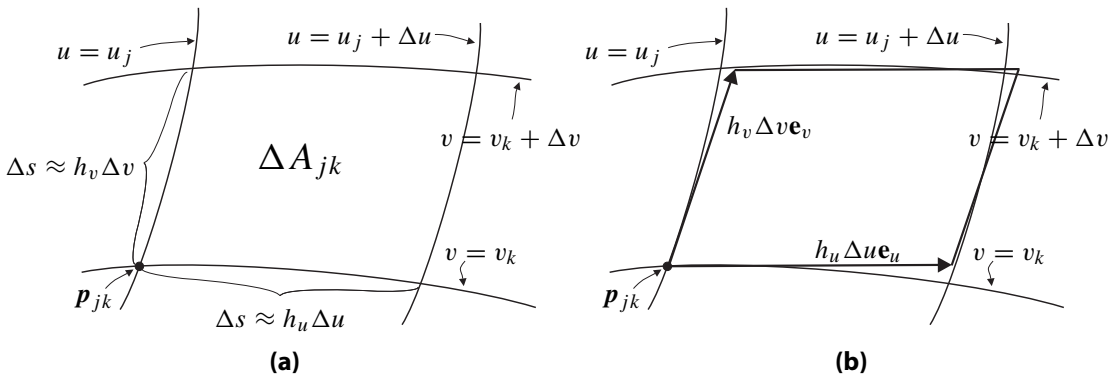

Figure 10.1: (a) A small region corresponding to changing the coordinates u and v by Δu and Δv, respectively. (b) The approximating parallelogram generated by tangent vectors. In each figure, $h_u = h_u(\boldsymbol{p}_{jk})$ and $h_v = h_v(\boldsymbol{p}_{jk})$.

That is, as illustrated in figure 10.1a,

$$\text{“arclength along } v = v_k \text{ from } \boldsymbol{p} \text{ to where } u = u_j + \Delta u\text{”} \;\approx\; h_u \Delta u \quad .$$

Likewise, letting $h_v = h_v(\boldsymbol{p})$,

$$\text{“arclength along } u = u_j \text{ from } \boldsymbol{p} \text{ to where } v = v_k + \Delta v\text{”} \;\approx\; h_v \Delta v \quad .$$

Clearly, then, if we take the unit tangents $\mathbf{e}_u$ and $\mathbf{e}_v$ at $\boldsymbol{p}$, the area ΔA_{jk} of the region bounded by the curves in figure 10.1a is closely approximated by the area of the parallelogram in figure 10.1b generated by the vectors $h_u \Delta u \mathbf{e}_u$ and $h_v \Delta v \mathbf{e}_v$. In particular, if $\mathcal{S}$ is a surface in a three-dimensional Euclidean space, then we can use the vector formula for such parallelograms, obtaining

$$\Delta A_{jk} \;\approx\; \|(h_u \Delta u \mathbf{e}_u) \times (h_v \Delta v \mathbf{e}_v)\| \;=\; \|(h_u \mathbf{e}_u) \times (h_v \mathbf{e}_v)\| \, \Delta u \Delta v$$

Thus,

$$\sum_j \sum_k \phi(\boldsymbol{p}_{jk}) \, \Delta A_{j,k} \;\approx\; \sum_j \sum_k \phi(\boldsymbol{p}_{jk}) \, \|(h_u \mathbf{e}_u) \times (h_v \mathbf{e}_v)\| \, \Delta u \Delta v \quad . \tag{10.15}$$

By carefully examining the maximum possible error in using

$$\|(h_u \mathbf{e}_u) \times (h_v \mathbf{e}_v)\| \, \Delta u \Delta v \qquad \text{for} \qquad \Delta A_{jk} \quad ,$$

it can be confirmed that the difference between the two sides of approximation (10.15) shrinks to zero as Δu and Δv shrink to zero. And from this, it follows that the $\{(u, v)\}$ coordinate formula for the element of area is

$$dA \;=\; \|(h_u \mathbf{e}_u) \times (h_v \mathbf{e}_v)\| \, du \, dv \quad . \tag{10.16}$$

For reference, let us observe that this can be written in any of the following equivalent ways:

$$dA \;=\; \|\mathbf{e}_u \times \mathbf{e}_v\| \, h_u h_v \, du \, dv \quad , \tag{10.17a}$$

$$dA \;=\; \|\boldsymbol{\varepsilon}_u \times \boldsymbol{\varepsilon}_v\| \, du \, dv \tag{10.17b}$$

and

$$dA = \left\| \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \right\| du\, dv \quad . \tag{10.17c}$$

These formulas are valid using any reasonable coordinate system $\{(u, v)\}$ for the two-dimensional space. Let us, however, consider two cases of particular interest.

Orthogonal Two-Dimensional Coordinate Systems

If $\{(u, v)\}$ is an orthogonal coordinate system on the surface $\mathcal{S}$, then the approximating parallelogram in figure 10.1b is actually a rectangle, and $\{\mathbf{e}_u, \mathbf{e}_v\}$ is orthogonal. Hence,

$$\|\mathbf{e}_u \times \mathbf{e}_v\| = 1$$

and formula (10.17a) reduces to

$$dA = h_u h_v\, du\, dv \quad . \tag{10.18}$$

!►***Example 10.1:*** *Using polar coordinates*

$$dA = h_\rho h_\phi\, d\rho\, d\phi = \rho\, d\rho\, d\phi \quad .$$

?►***Exercise 10.4 (latitude and longitude on a sphere):*** *Let $\{(\theta, \phi)\}$ be the "latitude and longitude" coordinates for a sphere $\mathcal{S}$ of radius R in a three-dimensional Euclidean space (i.e., $\{(r, \theta, \phi)\}$ is the standard spherical coordinates, with the origin in the center of the sphere and with $\theta = 0$ being the "positive z–axis"). Show that, for this sphere,*

$$dA = R^2 \sin(\theta)\, d\theta\, d\phi \quad .$$

Surfaces Given by Graphs

Let us now consider the case where the surface $\mathcal{S}$ is a surface in a three-dimensional Euclidean space which is the graph of some function

$$z = f(u, v)$$

where $\{(x, y, z)\}$ is a standard Cartesian coordinate system and $\{(u, v)\}$ is some orthogonal coordinate system on the XY–plane.

As we have already noted several times, the $\{(u, v)\}$ coordinate system on the XY–plane automatically imposes a corresponding coordinate system on the surface $\mathcal{S}$. To reduce confusion, let us temporarily call this the $\{(\mu, \nu)\}$ coordinate system for $\mathcal{S}$. That is,

$$\boldsymbol{p} \sim (\mu, \nu) \text{ in } \mathcal{S} \iff \boldsymbol{p} \sim (u, v, f(u, v)) \quad \text{with} \quad u = \mu \text{ and } v = \nu \quad .$$

Applying the chain rule, we have

$$\begin{aligned} \boldsymbol{\varepsilon}_\mu = \frac{\partial \boldsymbol{x}}{\partial \mu} &= \frac{\partial u}{\partial \mu}\frac{\partial \boldsymbol{x}}{\partial u} + \frac{\partial v}{\partial \mu}\frac{\partial \boldsymbol{x}}{\partial v} + \frac{\partial z}{\partial \mu}\frac{\partial \boldsymbol{x}}{\partial z} \\ &= \frac{\partial u}{\partial \mu}\boldsymbol{\varepsilon}_u + \frac{\partial v}{\partial \mu}\boldsymbol{\varepsilon}_v + \left[\frac{\partial f}{\partial u}\frac{\partial u}{\partial \mu} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \mu}\right]\mathbf{k} \\ &= 1 \cdot h_u \mathbf{e}_u + 0 \cdot \boldsymbol{\varepsilon}_v + \left[\frac{\partial f}{\partial u} \cdot 1 + \frac{\partial f}{\partial v} \cdot 0\right]\mathbf{k} = h_u \mathbf{e}_u + \frac{\partial f}{\partial u}\mathbf{k} \quad . \end{aligned}$$

By similar computations,

$$\boldsymbol{\varepsilon}_v \;=\; h_v\mathbf{e}_v \;+\; \frac{\partial f}{\partial v}\mathbf{k} \quad .$$

Combining this with formula (10.17b) for dA and the fact that $\{(u, v, z\}$ is an orthogonal system, we get

$$\begin{aligned} dA \;&=\; \left\|\boldsymbol{\varepsilon}_\mu \times \boldsymbol{\varepsilon}_v\right\| d\mu\, d\nu \\ &=\; \left\|\left(h_u\mathbf{e}_u \;+\; \frac{\partial f}{\partial u}\mathbf{k}\right) \times \left(h_v\mathbf{e}_v \;+\; \frac{\partial f}{\partial v}\mathbf{k}\right)\right\| d\mu\, d\nu \\ &=\; \left\|h_uh_v\underbrace{\mathbf{e}_u \times \mathbf{e}_v}_{\pm\mathbf{k}} + h_v\frac{\partial f}{\partial u}\underbrace{\mathbf{k} \times \mathbf{e}_v}_{\pm\mathbf{e}_u} + h_u\frac{\partial f}{\partial v}\underbrace{\mathbf{e}_u \times \mathbf{k}}_{\pm\mathbf{e}_v} + \frac{\partial f}{\partial u}\frac{\partial f}{\partial v}\underbrace{\mathbf{k} \times \mathbf{k}}_{\mathbf{0}}\right\| d\mu\, d\nu \\ &=\; \sqrt{[h_uh_v]^2 + \left[h_u\frac{\partial f}{\partial v}\right]^2 + \left[h_v\frac{\partial f}{\partial u}\right]^2}\, d\mu\, d\nu \quad . \end{aligned}$$

But $(\mu, \nu) = (u, v)$. So the element of area for a surface given as the graph of f is

$$dA \;=\; \sqrt{[h_uh_v]^2 + \left[h_u\frac{\partial f}{\partial v}\right]^2 + \left[h_v\frac{\partial f}{\partial u}\right]^2}\, du\, dv \quad . \tag{10.19}$$

!►*Example 10.2:* *If $\{(u, v)\}$ is the standard Cartesian system $\{(x, y)\}$, then formula (10.19) reduces to*

$$dA \;=\; \sqrt{1 + \left[\frac{\partial f}{\partial x}\right]^2 + \left[\frac{\partial f}{\partial y}\right]^2}\, dx\, dy \quad .$$

In particular, if the surface is the graph of $z = x^2 + y^2$,

$$dA \;=\; \sqrt{1 + \left[\frac{\partial}{\partial x}\left[x^2\right]\right]^2 + \left[\frac{\partial}{\partial y}\left[y^2\right]\right]^2}\, dx\, dy \;=\; \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy \quad .$$

?►*Exercise 10.5:* *Show that, using polar coordinates, formula (10.19) reduces to*

$$dA \;=\; \sqrt{\rho^2 + \rho^2\left[\frac{\partial f}{\partial \rho}\right]^2 + \left[\frac{\partial f}{\partial \phi}\right]^2}\, d\rho\, d\phi \quad ,$$

and that, if the surface is the graph of $z = \rho^2$,

$$dA \;=\; \sqrt{1 + 4\rho^2}\,\rho\, d\rho\, d\phi \quad .$$

Using Jacobians

Let (u, v) and (σ, τ) be two coordinate systems for some surface $\mathcal{S}$. The two corresponding *Jacobians* of the change of coordinates are the determinants of the 2×2 matrices you might naturally construct from the partial derivatives,

$$\frac{\partial(\sigma, \tau)}{\partial(u, v)} \;=\; \begin{vmatrix} \dfrac{\partial\sigma}{\partial u} & \dfrac{\partial\sigma}{\partial v} \\[2ex] \dfrac{\partial\tau}{\partial u} & \dfrac{\partial\tau}{\partial v} \end{vmatrix} \;=\; \begin{vmatrix} \dfrac{\partial\sigma}{\partial u} & \dfrac{\partial\tau}{\partial u} \\[2ex] \dfrac{\partial\sigma}{\partial v} & \dfrac{\partial\tau}{\partial v} \end{vmatrix}$$

and

$$\frac{\partial(u,v)}{\partial(\sigma,\tau)} = \begin{vmatrix} \dfrac{\partial u}{\partial \sigma} & \dfrac{\partial u}{\partial \tau} \\ \dfrac{\partial v}{\partial \sigma} & \dfrac{\partial v}{\partial \tau} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial \sigma} & \dfrac{\partial v}{\partial \sigma} \\ \dfrac{\partial u}{\partial \tau} & \dfrac{\partial v}{\partial \tau} \end{vmatrix} \quad .$$

I probably should have mentioned them sooner.

Computing Elements of Area

In section 5.6 we saw that the N-dimensional volume of a hyper-parallelepiped generated by a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ is given by

$$\left| \det \begin{bmatrix} v_{1,1} & v_{2,1} & v_{3,1} & \cdots & v_{N,1} \\ v_{1,2} & v_{2,2} & v_{3,2} & \cdots & v_{N,2} \\ v_{1,3} & v_{2,3} & v_{3,3} & \cdots & v_{N,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{1,N} & v_{2,N} & v_{3,N} & \cdots & v_{N,N} \end{bmatrix} \right|$$

where the $v_{j,k}$'s are the components of $\mathbf{v}_j$ with respect to some orthonormal basis (see, in particular, formula (5.13) on page 5–17).

Let's use this to find the area ΔA of the parallelogram in figure 10.1a assuming that the $\{(\sigma, \tau)\}$ coordinate system is orthogonal. This parallelogram is generated by the vectors

$$\Delta u\, \boldsymbol{\varepsilon}_u \qquad \text{and} \qquad \Delta v\, \boldsymbol{\varepsilon}_v$$

Rewriting these in terms of the normalized local basis corresponding to the $\{(\sigma, \tau)\}$ system,

$$\begin{aligned} \Delta u \boldsymbol{\varepsilon}_u &= \Delta u \left[\frac{\partial \boldsymbol{x}}{\partial \sigma}\frac{\partial \sigma}{\partial u} + \frac{\partial \boldsymbol{x}}{\partial \tau}\frac{\partial \tau}{\partial u} \right] \\ &= \cdots \\ &= \Delta u \frac{\partial \sigma}{\partial u} h_\sigma \mathbf{e}_\sigma + \Delta u \frac{\partial \tau}{\partial u} h_\tau \mathbf{e}_\tau \end{aligned}$$

and

$$\begin{aligned} \Delta v \boldsymbol{\varepsilon}_v &= \cdots \\ &= \Delta v \frac{\partial \sigma}{\partial v} h_\sigma \mathbf{e}_\sigma + \Delta v \frac{\partial \tau}{\partial v} h_\tau \mathbf{e}_\tau \quad . \end{aligned}$$

So,

$$\Delta A = \left| \det \begin{bmatrix} \Delta u \dfrac{\partial \sigma}{\partial u} h_\sigma & \Delta v \dfrac{\partial \sigma}{\partial v} h_\sigma \\ \Delta u \dfrac{\partial \tau}{\partial u} h_\tau & \Delta v \dfrac{\partial \tau}{\partial v} h_\tau \end{bmatrix} \right| = h_\sigma h_\tau \left| \det \begin{bmatrix} \dfrac{\partial \sigma}{\partial u} & \dfrac{\partial \sigma}{\partial v} \\ \dfrac{\partial \tau}{\partial u} & \dfrac{\partial \tau}{\partial v} \end{bmatrix} \right| \Delta u \Delta v \quad .$$

That is,

$$\Delta A = h_\sigma h_\tau \left| \frac{\partial(\sigma,\tau)}{\partial(u,v)} \right| \Delta u \Delta v \quad .$$

Taking the limits, we get

$$dA = h_\sigma h_\tau \left| \frac{\partial(\sigma,\tau)}{\partial(u,v)} \right| du\, dv \quad . \tag{10.20}$$

Change of Coordinates in Integration

As a side note, let us observe that, if $\{(\sigma, \tau)\}$ is an orthogonal coordinate system, then equations (10.18) and (10.20) give us

$$h_\sigma h_\tau \, d\sigma \, d\tau \;=\; dA \;=\; h_\sigma h_\tau \left| \frac{\partial(\sigma, \tau)}{\partial(u, v)} \right| du \, dv \quad ,$$

which, in turn, gives us the "change of coordinates" expression for double integrals,

$$d\sigma \, d\tau \;=\; \left| \frac{\partial(\sigma, \tau)}{\partial(u, v)} \right| du \, dv \quad . \tag{10.21}$$

10.4 Volume Integrals

Now let $\mathcal{R}$ be a solid region in a three-dimensional space. By now, you should be able to write a short treatise on defining the integral of a scalar field (or vector field) over $\mathcal{R}$. I will usually use dV to denote the (infinitesimal) element of volume.[7] In your treatise (if you write it) you should naturally point out that

$$\iiint_{\mathcal{R}} dV \;=\; \text{total volume of } \mathcal{R} \quad ,$$

and, if

$$\phi(\boldsymbol{x}) \;=\; \text{the density (amount/unit volume) of something at } \boldsymbol{x} \text{ in } \mathcal{R} \quad ,$$

then

$$\iiint_{\mathcal{R}} \phi \, dV \;=\; \text{total amount of that something in } \mathcal{R} \quad .$$

You should also convince yourself that, if $\{(x^1, x^2, x^3)\}$ is any orthogonal coordinate system for the space, then the coordinate surfaces partition the volume into "somewhat rectangular" boxes with "square" corners. Because of this, arguments similar (actually, simpler) to those given for finding the element of area in terms of a two-dimensional orthogonal system, lead to the formula for the element of volume being

$$dV \;=\; h_1 h_2 h_3 \, dx^1 \, dx^2 \, dx^3 \quad . \tag{10.22}$$

This is just the three-dimensional version of formula (10.18) on page 10–15. Don't forget that it requires an orthogonal coordinate system.

It should also be clear that the discussion regarding Jacobians could be extended to three-dimensions (and even higher dimensions). In particular, if $\{(\sigma, \tau, \xi)\}$ is an orthogonal coordinate system and $\{(u, v, w)\}$ is any other coordinate system for our space, then

$$dV \;=\; h_\sigma h_\tau h_\xi \left| \frac{\partial(\sigma, \tau, \xi)}{\partial(u, v, w)} \right| du \, dv \, dw \quad . \tag{10.23}$$

[7] Arfken, Weber and Harris seem to prefer $d\tau$ or $d^3\boldsymbol{r}$. You will also find d^3x, sometimes.

and

$$d\sigma\, d\tau\, d\xi \;=\; \left|\frac{\partial(\sigma,\tau,\xi)}{\partial(u,v,w)}\right| du\, dv\, dw \tag{10.24}$$

where

$$\frac{\partial(\sigma,\tau,\xi)}{\partial(u,v,w)} \;=\; \begin{vmatrix} \dfrac{\partial\sigma}{\partial u} & \dfrac{\partial\sigma}{\partial v} & \dfrac{\partial\sigma}{\partial w} \\ \dfrac{\partial\tau}{\partial u} & \dfrac{\partial\tau}{\partial v} & \dfrac{\partial\tau}{\partial w} \\ \dfrac{\partial\xi}{\partial u} & \dfrac{\partial\xi}{\partial v} & \dfrac{\partial\xi}{\partial w} \end{vmatrix} \quad .$$

?►***Exercise 10.6:*** *Using formula (10.22), find dV in terms of each of the standard three-dimensional coordinate systems for Euclidean space: Cartesian, cylindrical and spherical.*

10.5 Normal Vector Fields

Many of the big integral theorems involve "unit normal vector fields", so let's introduce these objects.

By the way, just in case you want to tie the material here with previous material: Recall how the "normal" in 'orthonormal' meant 'unit length'? Well, that is *not* what "normal" means here. Here, "normal" actually means "orthogonal to".

General Definitions and Concepts

Suppose $\mathcal{S}$ is an $(N-1)$-dimensional subspace in an N-dimensional space $\mathcal{R}$. For example, we could have:

$\mathcal{S}$ is a curve (one-dimensional) in a plane $\mathcal{E}$ (two-dimensional).

or

$\mathcal{S}$ is a surface (two-dimensional) in a three-dimensional Euclidean space $\mathcal{E}$.

For much of what we will be discussing, these are the situations of interest. Still, much of what we will discuss will also apply if $\mathcal{S}$ is an $(N-1)$-dimensional "hypersurface" in any N-dimensional space.

Normal Vectors

Now let $\boldsymbol{p}$ be a (smooth) point on $\mathcal{S}$. A *normal* vector to $\mathcal{S}$ at $\boldsymbol{p}$ is a vector perpendicular to $\mathcal{S}$. More precisely, it is a vector in the tangent space of $\mathcal{R}$ at $\boldsymbol{p}$ which is orthogonal to each vector in the tangent space of $\mathcal{S}$ at $\boldsymbol{p}$. If you visualize $\mathcal{S}$ as being either a curve in a plane or a surface in three-dimensional space, you can see that there will be exactly two unit normal vectors at $\boldsymbol{p}$, pointing in opposite directions. In general, no matter what is the dimension of the spaces, there will always be exactly two unit normal vectors at $\boldsymbol{p}$, pointing in opposite directions. If this is obvious to you, skip to the next paragraph. Otherwise, let

$$\mathcal{T}_{\mathcal{S},\boldsymbol{p}} \;=\; \text{vector space of vectors tangent to } \mathcal{S} \text{ at } \boldsymbol{p} \quad ,$$

and

$$\mathcal{T}_{\mathcal{E},\boldsymbol{p}} \;=\; \text{vector space of vectors ``tangent'' to } \mathcal{E} \text{ at } \boldsymbol{p} \quad .$$

Because $\mathcal{S}$ has dimension $N-1$, $\mathcal{T}_{\mathcal{S},\boldsymbol{p}}$ will have an orthonormal basis of $N-1$ vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{N-1}\}$. These vectors are also in $\mathcal{T}_{\mathcal{E},\boldsymbol{p}}$, but since $\mathcal{E}$ is N-dimensional, we can find another vector $\mathbf{n}$ so that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{N-1}, \mathbf{n}\}$ is an orthonormal basis for $\mathcal{T}_{\mathcal{E},\boldsymbol{p}}$. Clearly, then, a vector $\mathbf{v}$ in $\mathcal{T}_{\mathcal{E},\boldsymbol{p}}$ is orthogonal to all vectors in $\mathcal{T}_{\mathcal{S},\boldsymbol{p}}$ if and only if

$$\mathbf{v} \;=\; \alpha\mathbf{n} \qquad \text{for some} \quad \alpha \in \mathbb{R} \quad .$$

Consequently, there will be exactly two unit normal vectors to $\mathcal{S}$ at $\boldsymbol{p}$, namely, $\mathbf{n}$ and $-\mathbf{n}$.

Normal Vector Fields

A *normal vector field* on $\mathcal{S}$ is simply a vector field $\mathbf{v}$ on $\mathcal{S}$ in which $\mathbf{v}(\boldsymbol{p})$ is a normal vector to $\mathcal{S}$ at each (smooth) point $\boldsymbol{p}$ in $\mathcal{S}$. Unless otherwise indicated, always assume that a given normal vector field varies smoothly over $\mathcal{S}$.

In practice, our main interests are in *unit normal vector fields*, commonly denoted by $\mathbf{n}$, in which every vector has unit length. Such a vector field is commonly used to define a "positive" direction through $\mathcal{S}$, with, naturally, the direction of $\mathbf{n}(\boldsymbol{p})$ defining the "positive" direction through $\mathcal{S}$ at position $\boldsymbol{p}$.

Try visualizing all possible unit normal vector fields on some curve or surface — draw a few pictures. Chances are that you could find exactly two possible choices for the unit normal vector field. This corresponds to the fact that most surfaces (and hypersurfaces) have two sides. Remember, there are exactly two unit normals at each point, and from this it naturally follows that there are exactly two possible choices for a unit normal vector field on all of $\mathcal{S}$, *provided a unit normal vector field can be defined on all of* $\mathcal{S}$. It turns out that there are surfaces (and hypersurfaces) on which a unit normal vector field cannot be defined (a möbius strip, for example).

If a unit normal vector field $\mathbf{n}$ can be defined on all of a surface (or hypersurface) $\mathcal{S}$, then we say $\mathcal{S}$ is *orientable*. A particular choice of $\mathbf{n}$ makes $\mathcal{S}$ an *oriented* surface (or hypersurface) and is sometimes called the *orientation* for $\mathcal{S}$. (Strictly speaking, this terminology can be applied to curves in the plane as well, but that would be confusing since we've already defined "orientation" for a curve in terms of "direction of travel".)

Standard Conventions

There are few standard conventions normally assumed regarding orientation:

1. If $\mathcal{S}$ is a *closed* surface — that is, $\mathcal{S}$ completely encloses some volume in space — then $\mathbf{n}$ is almost always chosen to be the *outward pointing* unit normal vector field on $\mathcal{S}$, which, of course, means that each $\mathbf{n}(\boldsymbol{p})$ points away from the volume.

   While this was stated as if $\mathcal{S}$ were a two-dimensional surface enclosing a three-dimensional volume, this convention applies no matter what the dimensions are. $\mathcal{S}$ can be a closed curve enclosing some region in the plane. It can also be a 86-dimensional hypersurface enclosing some 87-dimensional "hypervolume".

2. Suppose $\mathcal{S}$ is a two-dimensional orientable surface that is not closed but has a boundary that is a curve $\mathcal{C}$. Then the surface $\mathcal{S}$ and curve $\mathcal{C}$ are oriented together so that the following holds at each (smooth) point $\boldsymbol{p}$ on the boundary $\mathcal{C}$:

If

$$\mathbf{T} \;=\; \text{the unit tangent at } \boldsymbol{p} \text{ in the positive direction along } \mathcal{C}$$

and

$$\mathbf{n} \;=\; \text{the unit normal to } \mathcal{S} \text{ at } \boldsymbol{p},$$

then $\mathbf{T} \times \mathbf{n}$ points "away" from the surface $\mathcal{S}$.

This convention doesn't actually specify which orientation to choose for $\mathcal{S}$; it simply ensures that the orientations of the surface and its boundary are matched for the standard phrasing of Stokes' and related theorems.

3. If $\mathcal{S}$ is a region in the XY–plane then the last convention is usually applied with $\mathbf{n} = \mathbf{k}$. Thus the bounding curve is oriented "counterclockwise".

Integrals with Normal Vector Fields

Let $\mathcal{S}$ be an oriented, two-dimensional surface and $\mathbf{F}$ a vector field on $\mathcal{S}$. Two integrals often arising in physics are

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dA \qquad \text{and} \qquad \iint_{\mathcal{S}} \mathbf{F} \times \mathbf{n}\, dA \quad .$$

Some texts (including Arfken, Weber and Harris) use the shorthand $d\boldsymbol{\sigma} = \mathbf{n}\, dA$. So the above two integrals would be written

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\boldsymbol{\sigma} \qquad \text{and} \qquad \iint_{\mathcal{S}} \mathbf{F} \times d\boldsymbol{\sigma} \quad .$$

Since the dot product is defined no matter what dimension our space is, we have analogs of the first integral over curves

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds$$

and over higher-dimensional hypersurfaces

$$\int \cdots \int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dA$$

where, here, dA is the higher-dimensional analog of the infinitesimal element of area.

A Brief Physical Interpretation of $\mathbf{F} \cdot \mathbf{n}$

Let us consider the flow of some liquid in some three-dimensional region. This flow can be described by a vector field $\mathbf{F}$ where, for each point $\boldsymbol{p}$ in the region,

$$\frac{\mathbf{F}(\boldsymbol{p})}{\|\mathbf{F}(\boldsymbol{p})\|} \;=\; \text{direction of the flow at } \boldsymbol{p}$$

and

$$\|\mathbf{F}(\boldsymbol{p})\| \;=\; \text{the "flow rate density" at } \boldsymbol{p} \quad .$$

By "flow rate density at $\boldsymbol{p}$", I mean the value F such that, if $\mathbf{F}$ were constant and $\Delta\mathcal{S}$ were a small, flat, imaginary surface of area ΔA oriented perpendicular to the flow, then $F\Delta A$ would be the mass per unit time of the liquid flowing through $\Delta\mathcal{S}$.[8]

If $\mathbf{n}$ is a unit vector at $\boldsymbol{p}$, then $\mathbf{F}(\boldsymbol{p})\cdot\mathbf{n}$ is the component of the "flow" in the direction of $\mathbf{n}$ at point $\boldsymbol{p}$. That is, assuming a constant flow $\mathbf{F}$ over a small, flat, imaginary surface $\Delta\mathcal{S}$ of area ΔA oriented by $\mathbf{n}$, then $\mathbf{F}\cdot\mathbf{n}\Delta A$ would be the mass per unit time of the liquid flowing through $\Delta\mathcal{S}$ in the general direction of $\mathbf{n}$. This value is negative if the flow is generally in the opposite direction of $\mathbf{n}$. Consequently, if $\mathcal{S}$ is an (imaginary) oriented surface with unit normal vector field $\mathbf{n}$ indicating the "positive direction" through the surface, then

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dA \;=\; \text{net mass of liquid flowing in the positive direction through } \mathcal{S} \text{ per unit time} \quad .$$

In particular, if $\mathcal{S}$ is a closed surface enclosing a solid region $\mathcal{R}$ and $\mathbf{n}$ is the outward pointing unit normal vector field on $\mathcal{S}$, then

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dA \;=\; \text{net mass of liquid flowing out of region } \mathcal{R} \text{ per unit time} \quad .$$

For this last case, since we usually expect the "flow in" to equal the "flow out" of a region, we would normally expect that

$$\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dA \;=\; 0 \quad .$$

10.6 The Basic Fundamental Theorem of Multidimensional Calculus

The big integral theorems of field theory — Gauss's, Stokes' and Green's — are all multidimensional analogs of the fundamental theorem of calculus you learned in elementary calculus,

$$\int_a^b f'(s)\,ds \;=\; f(b) \;-\; f(a) \quad ,$$

which holds whenever f is continuous on $[a,b]$ and at least piecewise differentiable on (a,b). In addition, the "Green's identities" that will play a role in partial differential equations (and will help tie that theory to the theory of self-adjoint operators) is really nothing more that a multidimensional analog of the elementary "integration by parts" formula, which, itself, followed from the fundamental theorem and the product rule for differentiation.

In this section, we will derive a basic fundamental theorem for multidimensional calculus from whence all the above follows. We begin with an integral theorem for a fairly simple region, and then extend that result to more general regions. The extension will be easy; its the obtaining of the first result for the simple regions that requires fairly careful analysis.

[8] If you think about it, $\mathbf{F}(\boldsymbol{p})$ is directly related to the density and velocity of the liquid flowing past $\boldsymbol{p}$.

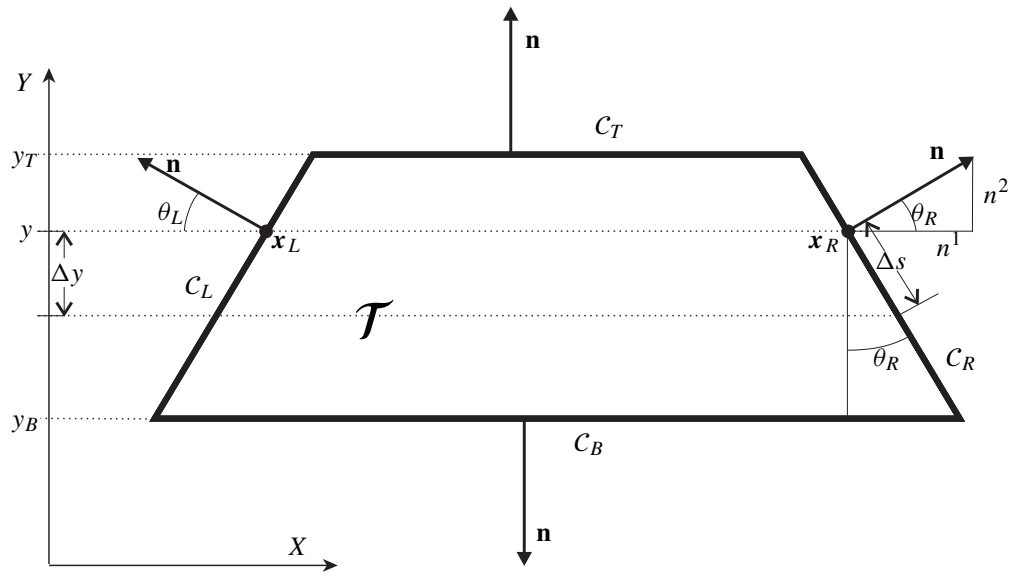

Figure 10.2: Trapezoidal region for lemma 10.3.

The Basic Lemma

We now restrict ourselves to a Euclidean plane on which we've already established a Cartesian coordinate system $\{(x, y)\}$. We will state the main result, and spend the rest of this subsection verifying the claim.

Lemma 10.3 (Integrating a derivative on a trapezoidal region)
Suppose $\mathcal{T}$ is some trapezoidal region in the plane with the parallel sides being parallel to the X–axis (as in figure 10.2). Let C be the curve bounding the region, and let

$$\mathbf{n} \;=\; n^1\,\mathbf{i} \;+\; n^2\,\mathbf{j}$$

be the outward pointing unit normal vector field on C. Then, for any sufficiently differentiable scalar field ψ on $\mathcal{T}$,

$$\iint_{\mathcal{T}} \frac{\partial \psi}{\partial x}\, dA \;=\; \int_C \psi\, n^1\, ds \quad .$$

To verify this claim, first divide the boundary, C, into the straight lines making up the top, bottom, left and right sides of the trapezoid,

$$C \;=\; C_T \;+\; C_B \;+\; C_R \;+\; C_L \quad ,$$

as indicated in figure 10.2. Observe that $n^1 = 0$ on the top and bottom. So

$$\int_{C_T} \psi\, n^1\, ds \;=\; 0 \;=\; \int_{C_T} \psi\, n^1\, ds$$

and

$$\int_C \psi\, n^1\, ds \;=\; \int_{C_R} \psi\, n^1\, ds \;+\; \int_{C_L} \psi\, n^1\, ds \quad . \tag{10.25}$$

As also indicated in the figure, we can parametrize C_R and C_L with respect to the y-coordinate by setting

$$\boldsymbol{x}_R(y) \;=\; \text{the point on } C_R \text{ with } y \text{ as the Y-coordinate} \;\sim\; \big(x_R(y),\, y\big)$$

and

$$\boldsymbol{x}_L(y) \;=\; \text{the point on } C_L \text{ with } y \text{ as the Y-coordinate} \;\sim\; \big(x_L(y),\, y\big) \quad .$$

(Explicit formulas for $x_R(y)$ and $x_L(y)$ could be derived easily enough, but won't be needed.)

Letting y_B and y_T be the y-coordinates of the bottom and top, respectively, we then have

$$\begin{aligned}\iint_{\mathcal{T}} \frac{\partial \psi}{\partial x}\, dA \;&=\; \int_{y=y_B}^{y_T} \left[\int_{x=x_L(y)}^{x_R(y)} \frac{\partial \psi}{\partial x}\, dx \right] dy \\ &=\; \int_{y=y_B}^{y_T} \big[\psi\big(x_R(y),\, y\big) \;-\; \psi\big(x_L(y),\, y\big) \big]\, dy \quad .\end{aligned}$$

For convenience, let's write this as

$$\iint_{\mathcal{T}} \frac{\partial \psi}{\partial x}\, dA \;=\; I_R \;-\; I_L \tag{10.26}$$

where

$$I_R \;=\; \int_{y_B}^{y_T} \psi\big(\boldsymbol{x}_R(y)\big)\, dy \qquad \text{and} \qquad I_L \;=\; \int_{y_B}^{y_T} \psi\big(\boldsymbol{x}_L(y)\big)\, dy \quad .$$

Now observe (see figure 10.2) that, on C_R, any change in y is related to a corresponding change in arclength s by

$$\Delta y \;=\; \cos(\theta_R)\; \Delta s$$

where θ_R is the angle between the outward normal $\mathbf{n}$ on C_R and $\mathbf{i}$. But on C_R (again, see figure 10.2)

$$n^1 \;=\; \mathbf{n} \cdot \mathbf{i} \;=\; \cos(\theta_R)\, \|\mathbf{n}\|\, \|\mathbf{i}\| \;=\; \cos(\theta_R) \quad .$$

Thus,

$$\Delta y \;=\; \cos(\theta_R)\, \Delta s \;=\; n^1 \Delta s \quad ,$$

which clearly tells that "$dy = n^1\, ds$"; that is,

$$I_R \;=\; \int_{y_B}^{y_T} \psi\big(\boldsymbol{x}_R(y)\big)\, dy \;=\; \int_{C_R} \psi\, n^1\, ds \quad .$$

On C_L, a similar analysis yields

$$I_L \;=\; \int_{y_B}^{y_T} \psi\big(\boldsymbol{x}_L(y)\big)\, dy \;=\; -\int_{C_L} \psi\, n^1\, ds \quad ,$$

with the negative sign arising because, on C_L,

$$n^1 \;=\; \mathbf{n} \cdot \mathbf{i} \;=\; \cos(\theta_L + \pi) \;=\; -\cos(\theta_L) \quad .$$

Thus,

$$I_R \;-\; I_L \;=\; \int_{C_R} \psi\, n^1\, ds \;-\; \left[-\int_{C_L} \psi\, n^1\, ds \right] \;=\; \int_{C_R} \psi\, n^1\, ds \;+\; \int_{C_L} \psi\, n^1\, ds \quad .$$

Combining this with equation (10.26) and equation (10.25), we get

$$\begin{aligned}\iint_{\mathcal{T}} \frac{\partial \psi}{\partial x}\, dA &= I_R - I_L \\ &= \int_{C_R} \psi\, n^1\, ds + \int_{C_L} \psi\, n^1\, ds = \int_C \psi\, n^1\, ds \quad ,\end{aligned}$$

finally verifying the lemma.

By the way, if you recall the requirements for using the fundamental theorem of calculus (which we used to integrate $\partial\psi/\partial x$ to get equation (10.26)), you will realize that for ψ to be "sufficiently differentiable", it suffices that ψ be continuous on $\mathcal{T}$ with its boundary, and at least "piecewise differentiable" inside $\mathcal{T}$.

Extending the Lemma

Now suppose ψ is any sufficiently differentiable scalar field on some bounded region $\mathcal{R}$ in the plane having some curve C as its boundary.

For each $\Delta y > 0$, we can approximate the region $\mathcal{R}$ as a bunch of horizontal trapezoidal subregions — $\mathcal{T}_1$, $\mathcal{T}_2$, ... — each of height Δy (as in figure 10.3). For each $\mathcal{T}_k$, let C_k be the curve enclosing $\mathcal{T}_k$, and let C_{kL} and C_{kR} be the left and right sides of $\mathcal{T}_k$. Also, let

$$\mathbf{n} = n^1\,\mathbf{i} + n^2\,\mathbf{j}$$

denote the outward pointing unit normal vector field wherever appropriate.

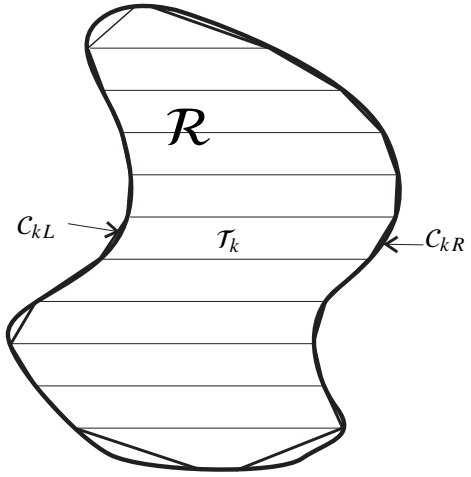

Figure 10.3: A region $\mathcal{R}$ approximated as a union of horizontal trapezoidal regions

Applying the lemma just verified (lemma 10.3), along with the observation that $n^1 = 0$ on the horizontal lines, we clearly have

$$\begin{aligned}\iint_{\mathcal{R}} \frac{\partial \psi}{\partial x}\, dA &= \lim_{\Delta y \to 0} \sum_k \iint_{\mathcal{T}_k} \frac{\partial \psi}{\partial x}\, dA \\ &= \lim_{\Delta y \to 0} \sum_k \int_{C_k} \psi\, n^1\, ds \\ &= \lim_{\Delta y \to 0} \sum_k \left[\int_{C_{kL}} \psi\, n^1\, ds + \int_{C_{kR}} \psi\, n^1\, ds \right] = \int_C \psi\, n^1\, ds \quad .\end{aligned}$$

Cutting out the middle leaves

$$\iint_{\mathcal{R}} \frac{\partial \psi}{\partial x}\, dA = \int_C \psi\, n^1\, ds \quad .$$

It should also be clear that:

1. By very similar arguments (or turning the paper sideways),

$$\iint_{\mathcal{R}} \frac{\partial \psi}{\partial y}\, dA = \int_C \psi\, n^2\, ds \quad .$$

2. Very similar analysis can be done with higher-dimensional regions.

This immediately yields

Theorem 10.4 (Basic fundamental theorem of multidimensional calculus)
Assume $\mathcal{R}$ is a bounded region in Euclidean space, and ψ is a sufficiently differentiable scalar field on $\mathcal{R}$.[9] *Using a Cartesian coordinate system corresponding to basis $\{\mathbf{i}, \mathbf{j}, \ldots\}$, and letting*

$$\mathbf{n} = n^1 \mathbf{i} + n^2 \mathbf{j} + \cdots$$

be the outward pointing unit vector field on the boundary of $\mathcal{R}$, then:

1. *If $\mathcal{R}$ is a two-dimensional region in the plane with bounding curve $\mathcal{C}$, then*

$$\iint_{\mathcal{R}} \frac{\partial \psi}{\partial x}\, dA = \int_{\mathcal{C}} \psi\, n^1\, ds$$

and

$$\iint_{\mathcal{R}} \frac{\partial \psi}{\partial y}\, dA = \int_{\mathcal{C}} \psi\, n^2\, ds \quad .$$

2. *If $\mathcal{R}$ is a three-dimensional region with bounding surface $\mathcal{S}$, then*

$$\iiint_{\mathcal{R}} \frac{\partial \psi}{\partial x}\, dV = \iint_{\mathcal{S}} \psi\, n^1\, dA \quad ,$$

$$\iiint_{\mathcal{R}} \frac{\partial \psi}{\partial y}\, dV = \iint_{\mathcal{S}} \psi\, n^2\, dA$$

and

$$\iiint_{\mathcal{R}} \frac{\partial \psi}{\partial z}\, dV = \iint_{\mathcal{S}} \psi\, n^3\, dA \quad .$$

3. *If $\mathcal{R}$ is an N-dimensional region with bounding $(N-1)$-dimensional hypersurface $\mathcal{S}$, then, using the higher-dimensional analogs of dV and dA,*

$$\int\cdot\cdot\int_{\mathcal{R}} \frac{\partial \psi}{\partial x^k}\, dV = \int\cdot\int_{\mathcal{S}} \psi\, n^k\, dA \qquad \text{for} \quad k = 1, 2, \ldots, N \quad .$$

10.7 The Classical Theorem of Gauss

As an almost immediate corollary of the above basic fundamental theorem of multidimensional calculus (theorem 10.4), we have:

Theorem 10.5 (Gauss's theorem (all three versions))
Let $\mathcal{R}$ be a bounded region in N-dimensional Euclidean space with boundary $\mathcal{S}$, and suppose ψ and $\mathbf{F}$ are sufficiently differentiable scalar and vector fields on $\mathcal{R}$. Then, letting $\mathbf{n}$ denote

[9] To be precise, assume ψ is continuous on $\mathcal{R}$ with its boundary, and is piecewise differentiable inside $\mathcal{R}$.

the outward pointing unit normal vector field on $\mathcal{S}$, *and letting* dV *and* dA *denote the* N*-dimensional element of volume and corresponding* $(N-1)$*-dimensional element of surface area,*

$$\int\cdots\int_{\mathcal{R}} \nabla\psi \, dV \;=\; \int\cdot\int_{\mathcal{S}} \psi \, \mathbf{n} \, dA \tag{10.27}$$

and

$$\int\cdots\int_{\mathcal{R}} \nabla\cdot\mathbf{F} \, dV \;=\; \int\cdot\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n} \, dA \quad . \tag{10.28}$$

Also, if $N = 3$,

$$\iiint_{\mathcal{R}} \nabla\times\mathbf{F} \, dV \;=\; \iint_{\mathcal{S}} \mathbf{n}\times\mathbf{F} \, dA \quad . \tag{10.29}$$

To prove the above equations, just write out the integrals in Cartesian coordinates and apply the basic fundamental theorem of multidimensional calculus (theorem 10.4).

?►*Exercise 10.7:* *Do the proof for equations (10.27) and (10.28)*

Notes:

1. Gauss's theorem with just equation (10.28) is also known as the *divergence theorem*.

2. Suppose a vector field $\mathbf{F}$ describes the flow of some liquid through some region $\mathcal{R}$ in space. Recall that, if $\mathcal{R}_0$ is any subregion and $\mathcal{S}_0$ is its boundary, then

   $$\iint_{\mathcal{S}_0} \mathbf{F}\cdot\mathbf{n} \, dA \;=\; \text{net mass of liquid flowing out of region } \mathcal{R}_0 \text{ per unit time} \quad .$$

   Combine this with equation (10.28), and we have

   $$\iiint_{\mathcal{R}_0} \nabla\cdot\mathbf{F} \, dV \;=\; \text{net mass of liquid flowing out of region } \mathcal{R}_0 \text{ per unit time} \quad .$$

   Consequently, if $\mathcal{R}_0$ is a region where $\nabla\cdot\mathbf{F} > 0$, then more liquid is flowing out of that region than is flowing in; that is, we can view that region as a *source* for the liquid. On the other hand, if $\mathcal{R}_0$ is a region where $\nabla\cdot\mathbf{F} < 0$, then more liquid is flowing into that region than is flowing out. Such a region is often referred to as a *sink*.

   In many applications, of course, we expect the flow into any region to exactly balance the flow out of the region. By the above, we see that this will only be possible if

   $$\nabla\cdot\mathbf{F} \;=\; 0 \quad .$$

10.8 The Classical Theorems of Green and Stokes

Green's Theorem

Green's theorem can be viewed as an application of the two-dimensional divergence theorem to finding the work done by a force field in moving something around a closed curve in the plane.

So let $\mathcal{S}$ be a bounded region in a Euclidean plane, and let $\mathcal{C}$ be the closed curve (or curves, if $\mathcal{S}$ has holes) bounding $\mathcal{S}$.[10] Orient $\mathcal{C}$ as described in section 10.5 (so the outside bounding curve is oriented counterclockwise and any inner curves bounding holes are oriented clockwise), and consider

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\boldsymbol{r}$$

where $\mathbf{F}$ is any sufficiently differentiable vector field on $\mathcal{S}$. Remember that here

$$d\boldsymbol{r} = \mathbf{T}\, ds$$

where

$$\mathbf{T} = \text{unit tangent to } \mathcal{C} \text{ in the positive direction along } \mathcal{C} \quad .$$

Also remember that, by the orientation,

$$\mathbf{T} \times \mathbf{k} = \text{a vector (orthogonal to both } \mathbf{T} \text{ and } \mathbf{k}\text{) pointing away from the region}$$

Clearly, that "vector (orthogonal to both $\mathbf{T}$ and $\mathbf{k}$) pointing away from the region" must be the outward pointing unit normal $\mathbf{n}$ at that point. Thus, $\{\mathbf{T}, \mathbf{k}, \mathbf{n}\}$ is a righthanded basis. Thus also,

$$\mathbf{T} = \mathbf{k} \times \mathbf{n} \quad ,$$

which means, in the path integral over $\mathcal{C}$,

$$d\boldsymbol{r} = \mathbf{T}\, ds = \mathbf{k} \times \mathbf{n}\, ds \quad .$$

Now let $\{(x, y)\}$ be Cartesian coordinates with corresponding orthonormal basis $\{\mathbf{i}, \mathbf{j}\}$. Both $\mathbf{n}$ and $\mathbf{F}$ can be written in coordinate/component form

$$\mathbf{n} = n^1 \mathbf{i} + n^2 \mathbf{j} \qquad \text{and} \qquad \mathbf{F} = F^1(x, y)\, \mathbf{i} + F^2(x, y)\, \mathbf{j} \quad .$$

Using the component form for $\mathbf{n}$ in the above formula for $d\boldsymbol{r}$, we get

$$d\boldsymbol{r} = \mathbf{k} \times \mathbf{n}\, ds = \cdots = \left[-n^2 \mathbf{i} + n^1 \mathbf{j}\right] ds \quad .$$

Thus,

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\boldsymbol{r} &= \int_{\mathcal{C}} \left[F^1 \mathbf{i} + F^2 \mathbf{j}\right] \cdot \left[-n^2 \mathbf{i} + n^1 \mathbf{j}\right] ds \\
&= \int_{\mathcal{C}} \left(F^2 n^1 - F^1 n^2\right) ds \\
&= \int_{\mathcal{C}} \left[F^2 \mathbf{i} - F^1 \mathbf{j}\right] \cdot \left[n^1 \mathbf{i} + n^2 \mathbf{j}\right] ds = \int_{\mathcal{C}} \left[F^2 \mathbf{i} - F^1 \mathbf{j}\right] \cdot \mathbf{n}\, ds
\end{aligned}$$

[10] Since our "regions" here are two-dimensional (and we are leading to Stoke's theorem), we are changing notation slightly and denoting these regions by $\mathcal{S}$ instead of $\mathcal{R}$.

Cutting out the middle and then applying the divergence theorem, we continue:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\boldsymbol{r} &= \int_C \left[F^2\,\mathbf{i} - F^1\,\mathbf{j}\right]\cdot \mathbf{n}\,ds \\ &= \iint_S \nabla\cdot\left[F^2\,\mathbf{i} - F^1\,\mathbf{j}\right] dA = \iint_S \left[\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right] dA \quad .\end{aligned}$$

This gives us the classical version of Green's theorem:

Theorem 10.6 (Green's theorem — version 1)
Let S be a bounded region in a Euclidean plane with boundary curve C oriented in the standard way, and let $\{(x, y)\}$ be Cartesian coordinates for the plane with corresponding orthonormal basis $\{\mathbf{i}, \mathbf{j}\}$. Assume, further, that $\mathbf{F} = F^1\mathbf{i} + F^2\mathbf{j}$ is a sufficiently differentiable vector field on S.[11] *Then*

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} = \iint_S \left[\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right] dA \quad . \tag{10.30}$$

The classical Green's theorem (above) assumes $\mathbf{F}$ is "two-dimensional" and S is a region in "the plane". Suppose, instead that we have a "three-dimensional" vector field

$$\mathbf{F} = F^1\,\mathbf{i} + F^2\,\mathbf{j} + F^3\,\mathbf{k} \quad ,$$

but with the region of interest S still being in some plane in a three-dimensional Euclidean space. Let $\mathbf{n}$ now be a normal vector field for this planar region. Keep in mind that we can choose a coordinate system to simplify our computations. In particular, let's assume we have chosen a Cartesian coordinate system $\{(x, y, z)\}$ with corresponding orthonormal basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ so that

$$\mathbf{k} = \mathbf{n} \qquad \text{and} \qquad S \text{ is in the } XY\text{–plane} \quad .$$

The classical Green's theorem tells us that

$$\int_C \left[F^1\,\mathbf{i} + F^2\,\mathbf{j}\right]\cdot d\boldsymbol{r} = \iint_S \left[\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right] dA \quad .$$

Notice, however, that

$$\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y} = (\nabla\times\mathbf{F})\cdot\mathbf{k} = (\nabla\times\mathbf{F})\cdot\mathbf{n} \quad .$$

Moreover, since C is in the XY–plane, any tangent vector to it will be orthogonal to $\mathbf{k}$. Hence

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} = \int_C \left[F^1\,\mathbf{i} + F^2\,\mathbf{j} + F^3\,\mathbf{k}\right]\cdot d\boldsymbol{r} = \int_C \left[F^1\,\mathbf{i} + F^2\,\mathbf{j}\right]\cdot d\boldsymbol{r} \quad .$$

Combining the last three sets of equations, we obtain

$$\begin{aligned}\int_C \mathbf{F}\cdot d\boldsymbol{r} &= \int_C \left[F^1\,\mathbf{i} + F^2\,\mathbf{j}\right]\cdot d\boldsymbol{r} \\ &= \iint_S \left[\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right] dA = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dA \quad .\end{aligned}$$

[11] i.e., $\mathbf{F}$ is continuous on S with its boundary, and is piecewise differentiable inside S.

Keeping track of the orientations, this gives us a slightly more general version of Green's theorem:

Theorem 10.7 (Green's theorem — version 2)
Let S be a bounded planar region in three-dimensional Euclidean space with bounding curve C. Let S be oriented by a unit normal vector field $\mathbf{n}$, and assume the bounding curve C is oriented in the standard way (see Standard Conventions*, page 10–20). Then, for any sufficiently differentiable vector field $\mathbf{F}$ on S,*

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} \;=\; \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA \quad . \tag{10.31}$$

Some additional comments:

1. This version of Green's theorem is almost Stoke's theorem, and it pretty much tells us what the curl of a vector field tells us. To see this, pick any point $\boldsymbol{p}$ and consider a small circle C about that point in some plane with normal $\mathbf{n}$. Assume the circle is small enough that the value of $\|\mathbf{F}\|$ is fairly constant in the disk enclosed by C. Observe that:
   (a) The left side of equation (10.31) mainly depends on the angle between $\mathbf{n}$ and $\nabla \times \mathbf{F}$ at $\boldsymbol{p}$, with the value being (approximately) maximum when $\mathbf{n}$ is chosen to point in the direction of $\nabla \times \mathbf{F}$ at $\boldsymbol{p}$, and is (approximately) zero when the two vectors are orthogonal. (We talk of approximate maximums and zeroes since $\mathbf{F}$ might not be constant.)
   (b) The integral on the right side of equation (10.31) is a measure of the extent to which $\mathbf{F}$ is directed in the positive direction "around $\boldsymbol{p}$" along circles in the plane perpendicular to $\mathbf{n}$.

   Thus, equation (10.31) shows that the curl of $\mathbf{F}$ at $\boldsymbol{p}$ is telling us that $\mathbf{F}$ is basically "wrapping" around $\boldsymbol{p}$ in the oriented plane whose unit normal points in the direction of the curl of $\mathbf{F}$ at that point. If $\mathbf{F}$ describes a flow of liquid, then $\nabla \times \mathbf{F}$ gives the rotational component of the flow at each point. If $\mathbf{F}$ is a force field, then $\nabla \times \mathbf{F}$ gives the part of force that can drive particles constrained to short closed paths about each point, with the greatest part being in planes generally orthogonal to the curl.
2. For possible future reference, let us note that "sufficiently differentiable" in these theorems means just the same as was meant in the previous theorems: The functions are continuous on the region with its boundary, and are at least piecewise differentiable inside the region.

Stokes' Theorem

For Stokes' theorem, we can simply take the second version of Green's theorem, and allow the two-dimensional region to be any oriented surface.

Theorem 10.8 (Stokes' theorem)
Let S be a bounded oriented surface in three-dimensional Euclidean space with bounding curve C. Let $\mathbf{n}$ be the unit normal vector field orienting S, and assume the bounding curve C is oriented in the standard way (see Standard Conventions*, page 10–20). Then, for any sufficiently differentiable vector field $\mathbf{F}$ on S,*

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} \;=\; \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA \quad . \tag{10.32}$$

PROOF: The basic idea in proving Stokes's theorem is simple. Pick a convenient grid of closely spaced points on the surface S and connect enough of them with straight line segments so that we get a surface S_T that closely approximates S but which consists of triangular regions. On each triangular region, let $\mathbf{n}$ be the unit normal vector field that points in the same general direction as the unit normal vector field on the portion of S that triangle approximates. Since triangles are automatically planar, the second version of Green's theorem applies, giving us

$$\iint_{S_T} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA \;=\; \sum_k \iint_{\mathcal{T}_k} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA \;=\; \sum_k \int_{C_k} \mathbf{F} \cdot d\boldsymbol{r}$$

where $\mathcal{T}_k$ is the k^{th} triangular region and C_k is its bounding curve. Each C_k, in turn, is made up of three straight lines, and each of these straight lines will either be a common boundary between two triangles, or will be part of the boundary of S_T, C_T. If it is a common boundary between two triangles, say $\mathcal{T}_j$ and $\mathcal{T}_k$, then its orientation as part of the boundary of $\mathcal{T}_j$ will be opposite that as a part of the boundary of $\mathcal{T}_k$ (draw the picture!). Consequently, the part of

$$\int_{C_j} \mathbf{F} \cdot d\boldsymbol{r} \qquad \text{and} \qquad \int_{C_k} \mathbf{F} \cdot d\boldsymbol{r}$$

due to the common boundary will cancel out. Taking into account all the cancellations due to integral pairs over common boundary pieces leaves us with just the line integrals over the pieces of C_T, the bounding curve of C_T,

$$\sum_k \int_{C_k} \mathbf{F} \cdot d\boldsymbol{r} \;=\; \int_{C_T} \mathbf{F} \cdot d\boldsymbol{r} \quad .$$

Thus,

$$\iint_{S_T} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA \;=\; \sum_k \int_{C_k} \mathbf{F} \cdot d\boldsymbol{r} \;=\; \int_{C_T} \mathbf{F} \cdot d\boldsymbol{r} \quad .$$

By repeatedly rechoosing our grid points so that the triangles get smaller and smaller, and taking the limit, we can get equation (10.32). ∎

As a final comment, let us simply observe that if $\nabla \times \mathbf{F} = \mathbf{0}$ in some three-dimensional region $\mathcal{R}$, then Stokes theorem immediately tells us that

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} \;=\; 0$$

for every closed curve C in $\mathcal{R}$. This finishes the verification that, for a vector field $\mathbf{F}$ defined on a three-dimensional region $\mathcal{R}$,

$$\mathbf{F} \text{ is conservative on } \mathcal{R} \quad \Longleftrightarrow \quad \nabla \times \mathbf{F} \;=\; \mathbf{0} \qquad \text{everywhere in } \mathcal{R} \quad .$$

(See the discussion of "Potential Theory" in section 10.2.)

10.9 The General Divergence and Curl

Now that we've verified the classical Divergence and Stokes' Theorems, we can turn our attention to more general definitions of the divergence and curl inspired by these theorems, and to deriving the more general formulas for the divergence and gradient given in section 9.5.

Coordinate-Free Definitions

The basic approach is to go backwards from our development of the divergence and curl of vector fields in Euclidean space. Instead of defining these by formulas and then proving the integral theorems of Gauss and Stokes (which includes the divergence theorem), we will define the divergence and curl in terms of the divergence and Stokes' theorems. That is:

1. The *divergence of* $\mathbf{F}$, denoted by either $\operatorname{div}(\mathbf{F})$ of $\nabla \cdot \mathbf{F}$ is *defined* to be the scalar field such that
$$\int\cdots\int_{\mathcal{R}} \nabla \cdot \mathbf{F}\, dV \;=\; \int\cdot\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dA \tag{10.33}$$
whenever $\mathcal{R}$ is an N-dimensional subregion enclosed by the $(N-1)$-dimensional (hyper)surface $\mathcal{S}$ with outward pointing unit normal vector field $\mathbf{n}$.

2. (Assuming a three-dimensional space.) The *curl of* $\mathbf{F}$, denoted by either $\mathbf{curl}(\mathbf{F})$ of $\nabla \times \mathbf{F}$ is *defined* to be the vector field such that
$$\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA \;=\; \int_{\mathcal{C}} \mathbf{F} \cdot d\boldsymbol{r} \tag{10.34}$$
whenever $\mathcal{S}$ is an bounded oriented surface with normal vector field $\mathbf{n}$ and bounding (oriented) curve $\mathcal{C}$, and with the orientations of $\mathcal{S}$ and $\mathcal{C}$ being related in the standard manner (see *Standard Conventions* on page 10–20).

There are two advantages to the above definitions:

1. They do not require that the N-dimensional space be Euclidean. The above define the divergence and the curl of a vector field in nonEuclidean spaces as well as Euclidean.

2. They are coordinate free.

On the other hand, these definitions do not give us explicit instructions on how to actually compute the divergence or curl of a given vector field. In fact, there may even be some concern as to whether the above truly defines anything. What guarantee do we have that either the claimed scalar field satisfying equation (10.33) or the claimed vector field satisfying equation (10.34) exist? And if they do exist, are they unique?

We will address these concerns by using the above coordinate-free definition of the divergence to derive what must be the formula for $\nabla \cdot \mathbf{F}$ when the space is two-dimensional, and then briefly comment on how to completely verify that the corresponding divergence theorem holds in general. From this derivation and discussion, it should be fairly clear how the corresponding derivation and discussion follows in higher-dimensional spaces. Better yet, I'll be able to tell you the results without you wondering "How did he get that?"

Before starting on the derivation, however, we will take a moment to discuss some basic facts concerning certain limits of integrals, and concerning the coordinate formulas for computing

surface and volume integrals. These facts will be used in our derivation, and may be of more general application, later.

Some Integral Limit Facts

In our derivation, we will have limits of the form

$$\lim_{\Delta u \to 0} \frac{1}{\Delta u} \int_{u=a}^{a+\Delta u} \psi(u)\, du$$

and of the form

$$\lim_{\Delta u \to 0^+} \lim_{\Delta v \to 0^+} \frac{1}{\Delta u\, \Delta v} \int_{u=a}^{a+\Delta u} \int_{v=b}^{b+\Delta v} \psi(u, v)\, du\, dv$$

where ψ and ψ are "suitably continuous" real-valued functions. Let's observe here that these expressions simplify considerably.

First, consider the integral

$$\int_{u=a}^{a+\Delta u} \psi(u)\, du$$

where a is some fixed real number, Δu is some small positive value, and ψ is a real-valued function that is continuous around a. Recall that

$$\int_{u=a}^{a+\Delta u} \psi(u)\, du \;=\; \Big[\text{Average value of } \psi \text{ on } (a, a+\Delta u)\Big] \times \Delta u$$

Thus, as long as ψ is at least continuous at a, we must have

$$\begin{aligned} \frac{1}{\Delta u} \int_{u=a}^{a+\Delta u} \psi(u)\, du \;&=\; \text{average value of } \psi \text{ on } (a, a+\Delta u) \\ &\to\; \psi(a) \quad \text{as} \quad \Delta u \to 0^+ \quad . \end{aligned}$$

That is,

$$\lim_{\Delta u \to 0^+} \frac{1}{\Delta u} \int_{u=a}^{a+\Delta u} \psi(u)\, du \;=\; \psi(a) \quad .$$

Repeating these arguments in two dimensions gives us

$$\lim_{\Delta u \to 0^+} \lim_{\Delta v \to 0^+} \frac{1}{\Delta u\, \Delta v} \int_{u=a}^{a+\Delta u} \int_{v=b}^{b+\Delta v} \psi(u, v)\, du\, dv \;=\; \psi(a, b) \quad .$$

Divergence in Two Dimensions

Now we can begin deriving the formula for the divergence in a two-dimensional space, assuming we have an *orthogonal* coordinate system. To avoid index overload (which makes the derivation look more confusing than it is), we will denote that orthogonal coordinate system and its associated scaling factors and unit tangent vectors by

$$\{(u, v)\} \qquad , \qquad \{h_u, h_v\} \qquad \text{and} \qquad \{\mathbf{e}_u, \mathbf{e}_v\} \quad .$$

Let

$$\mathbf{F} \;=\; F^u \mathbf{e}_u \;+\; F^v \mathbf{e}_v$$

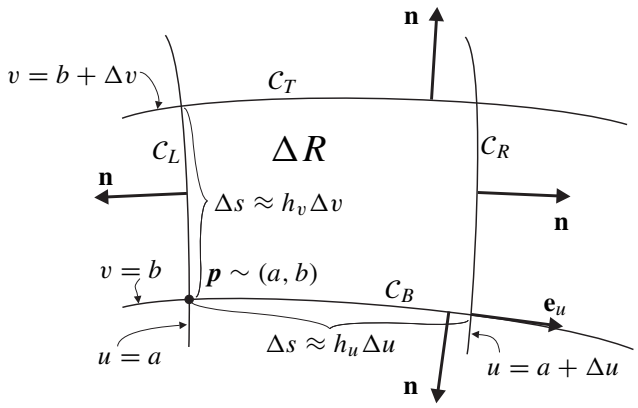

Figure 10.4: The region for finding $\nabla \cdot \mathbf{F}$ at position $\boldsymbol{p}$.

be any "suitably differentiable" vector field. Remember, since the components and basis vectors vary with position, we can express $\mathbf{F}$ more explicitly as

$$\mathbf{F}(u, v) \;=\; F^u(u, v)\, \mathbf{e}_u(u, v) \;+\; F^v(u, v)\, \mathbf{e}_v(u, v) \quad .$$

Our task is now to derive the coordinate formula for the divergence $\nabla \cdot \mathbf{F}$ using the coordinate-free definition given above.

First, we need a small region for the integrals. So pick any fixed point

$$\boldsymbol{p} \;\sim\; (a, b)$$

where $\mathbf{F}$ is suitably smooth, along with arbitrary small positive values Δu and Δv. Let $\Delta\mathcal{R}$ be the small rectangularish region bounded by the coordinate curves given by

$$u \;=\; a \quad , \quad u \;=\; a \;+\; \Delta u \quad , \quad v \;=\; b \quad , \quad v \;=\; b \;+\; \Delta v$$

as illustrated in figure 10.4. By the coordinate-free definition of divergence, $\nabla \cdot \mathbf{F}$ satisfies

$$\iint_{\Delta\mathcal{R}} \nabla \cdot \mathbf{F}\, dA \;=\; \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds \tag{10.35}$$

where $\mathcal{C}$ is the bounding curve and $\mathbf{n}$ is the outward-pointing unit normal vector field on $\mathcal{C}$. This curve is naturally partitioned into the four sides of the rectangle $\mathcal{C}_L$, $\mathcal{C}_R$, $\mathcal{C}_T$ and $\mathcal{C}_B$, where $\mathcal{C}_L$ is the "left side", etcetera.

Now consider $\mathcal{C}_R$ carefully: On this curve, u is the constant value $a + \Delta u$, and v varies from b to $b + \Delta v$. This means $\mathbf{e}_v$ is tangent to this curve, and, hence, it is $\mathbf{e}_u$ which is orthogonal to this curve at each point. So $\mathbf{e}_u$ is a unit normal vector field on $\mathcal{C}_R$. Moreover, since we've chosen $\Delta u > 0$, the direction of $\mathbf{e}_u$ (which is the direction of increasing u) is "to the right" in figure 10.4, the same general direction as the outward normal on $\mathcal{C}_R$. So $\mathbf{e}_u$ is the outward pointing unit normal on $\mathcal{C}_R$, and thus

$$\mathbf{F} \cdot \mathbf{n} \;=\; \mathbf{F} \cdot \mathbf{e}_u \;=\; F^u \qquad \text{on} \quad \mathcal{C}_R \quad .$$

On the left side, $\mathcal{C}_L$, $\mathbf{e}_u$ is also a unit normal vector field pointing "to the right" in figure 10.4. But here, "to the right" is the direction leading *into* region $\Delta\mathcal{R}$. So

$$\mathbf{F} \cdot \mathbf{n} \;=\; \mathbf{F} \cdot (-\mathbf{e}_u) \;=\; -F^u \qquad \text{on} \quad \mathcal{C}_L \quad .$$

By very similar arguments, we see that

$$\mathbf{F}\cdot\mathbf{n} \;=\; \mathbf{F}\cdot\mathbf{e}_v \;=\; F^v \qquad \text{on} \quad C_T \quad ,$$

and

$$\mathbf{F}\cdot\mathbf{n} \;=\; \mathbf{F}\cdot(-\mathbf{e}_v) \;=\; -F^v \qquad \text{on} \quad C_B \quad .$$

Combining the above observations with equation (10.35), we get

$$\begin{aligned}
\iint_{\Delta\mathcal{R}} \boldsymbol{\nabla}\cdot\mathbf{F}\,dA \;&=\; \int_C \mathbf{F}\cdot\mathbf{n}\,ds \\
&=\; \int_{C_R} \mathbf{F}\cdot\mathbf{n}\,ds \;+\; \int_{C_L} \mathbf{F}\cdot\mathbf{n}\,ds \;+\; \int_{C_T} \mathbf{F}\cdot\mathbf{n}\,ds \;+\; \int_C \mathbf{F}\cdot\mathbf{n}\,ds \\
&=\; \int_{C_R} F^u\,ds \;-\; \int_{C_L} F^u\,ds \;+\; \int_{C_T} F^v\,ds \;-\; \int_C F^v\,ds \quad .
\end{aligned}$$

which we rewrite as

$$\begin{aligned}
\iint_{\Delta\mathcal{R}} \boldsymbol{\nabla}\cdot\mathbf{F}\,dA \;=\; &\left[\int_{C_R} F^u\,ds \;-\; \int_{C_L} F^u\,ds\right] \\
&+\; \left[\int_{C_T} F^v\,ds \;-\; \int_C F^v\,ds\right] \quad .
\end{aligned} \tag{10.36}$$

To determine the next steps, let's write out the integral on the right in terms of the coordinates,

$$\iint_{\Delta\mathcal{R}} \boldsymbol{\nabla}\cdot\mathbf{F}\,dA \;=\; \int_{u=a}^{a+\Delta u}\int_{v=b}^{b+\Delta v} \boldsymbol{\nabla}\cdot\mathbf{F}\,h_u h_v\,du\,dv \quad .$$

One of the "limits of integrals" facts we just discussed applies, giving us

$$\begin{aligned}
&\lim_{\Delta u\to 0^+}\lim_{\Delta v\to 0^+} \frac{1}{\Delta u\,\Delta v}\iint_{\Delta\mathcal{R}} \boldsymbol{\nabla}\cdot\mathbf{F}\,dA \\
&\qquad =\; \lim_{\Delta u\to 0^+}\lim_{\Delta v\to 0^+} \frac{1}{\Delta u\,\Delta v}\int_{u=a}^{a+\Delta u}\int_{v=b}^{b+\Delta v} (\boldsymbol{\nabla}\cdot\mathbf{F})\,h_u h_v\,du\,dv \;=\; [(\boldsymbol{\nabla}\cdot\mathbf{F})\,h_u h_v]\big|_{\boldsymbol{p}} \quad .
\end{aligned}$$

This assumes the divergence and the scaling factors are continuous. Combining this with equation (10.36) yields

$$\begin{aligned}
[(\boldsymbol{\nabla}\cdot\mathbf{F})\,h_u h_v]\big|_{\boldsymbol{p}} \;&=\; \lim_{\Delta u\to 0^+}\lim_{\Delta v\to 0^+} \frac{1}{\Delta u\,\Delta v}\iint_{\Delta\mathcal{R}} \boldsymbol{\nabla}\cdot\mathbf{F}\,dA \\
&=\; \lim_{\Delta u\to 0^+}\lim_{\Delta v\to 0^+} \frac{1}{\Delta u\,\Delta v}\left[\int_{C_R} F^u\,ds \;-\; \int_{C_L} F^u\,ds\right] \\
&\qquad +\; \lim_{\Delta u\to 0^+}\lim_{\Delta v\to 0^+} \frac{1}{\Delta u\,\Delta v}\left[\int_{C_T} F^v\,ds \;-\; \int_{C_B} F^v\,ds\right] \quad .
\end{aligned} \tag{10.37}$$

The path before us is clear. We must evaluate the limits on the right side of the above equation.

Rewriting the integral over C_L in terms of the coordinates, we see that

$$\int_{C_L} F^u\,ds \;=\; \int_{v=b}^{b+\Delta v} F^u(a,v)\,h_v(a,v)\,dv \quad .$$

For notational simplicity, let

$$\left[F^u h_v\right]\Big|_{(a,v)} \;=\; F^u(a,v)\,h_v(a,v)$$

So,

$$\lim_{\Delta v\to 0^+}\frac{1}{\Delta v}\int_{C_L} F^u\,ds \;=\; \lim_{\Delta v\to 0^+}\frac{1}{\Delta v}\int_{v=b}^{b+\Delta v}\left[F^u h_v\right]\Big|_{(a,v)}\,dv \;=\; \left[F^u h_v\right]\Big|_{(a,b)} \quad .$$

Replacing $u = a$ with $u = a + \Delta u$ in the above computations yields the corresponding computations involving the integral over C_R :

$$\lim_{\Delta v\to 0^+}\frac{1}{\Delta v}\int_{C_R} F^u\,ds \;=\; \lim_{\Delta v\to 0^+}\frac{1}{\Delta v}\int_{v=b}^{b+\Delta v}\left[F^u h_v\right]\Big|_{(a+\Delta u,v)}\,dv \;=\; \left[F^u h_v\right]\Big|_{(a+\Delta u,b)} \quad .$$

Thus,

$$\begin{aligned}
&\lim_{\Delta u\to 0^+}\lim_{\Delta v\to 0^+}\frac{1}{\Delta u\,\Delta v}\left[\int_{C_R} F^u\,ds \;-\; \int_{C_L} F^u\,ds\right] \\
&\qquad = \lim_{\Delta u\to 0^+}\frac{1}{\Delta u}\left[\lim_{\Delta v\to 0^+}\frac{1}{\Delta v}\int_{C_R} F^u\,ds \;-\; \lim_{\Delta v\to 0^+}\frac{1}{\Delta v}\int_{C_L} F^u\,ds\right] \\
&\qquad = \lim_{\Delta u\to 0^+}\frac{1}{\Delta u}\left[\left[F^u h_v\right]\Big|_{(a+\Delta u,b)} \;-\; \left[F^u h_v\right]\Big|_{(a,b)}\right] \\
&\qquad = \lim_{\Delta u\to 0^+}\frac{[F^u h_v]|_{(a+\Delta u,b)} \;-\; [F^u h_v]|_{(a,b)}}{\Delta u} \quad .
\end{aligned}$$

But this last limit is just a partial derivative with respect to u . So,

$$\lim_{\Delta u\to 0^+}\lim_{\Delta v\to 0^+}\frac{1}{\Delta u\,\Delta v}\left[\int_{C_R} F^u\,ds \;-\; \int_{C_L} F^u\,ds\right] \;=\; \frac{\partial}{\partial u}\left[F^u h_v\right]\Big|_{(a,b)} \quad .$$

Unsurprisingly, very similar analysis for the integrals over the other sides yields

$$\lim_{\Delta u\to 0^+}\lim_{\Delta v\to 0^+}\frac{1}{\Delta u\,\Delta v}\left[\int_{C_T} F^v\,ds \;-\; \int_{C_B} F^v\,ds\right] \;=\; \frac{\partial}{\partial v}\left[F^v h_u\right]\Big|_{(a,b)} \quad .$$

Plugging in these last two results into equation (10.37), we finally obtain

$$[(\nabla\cdot\mathbf{F})\,h_u h_v]|_{\boldsymbol{p}} \;=\; \frac{\partial}{\partial u}\left[F^u h_v\right]\Big|_{(a,b)} \;+\; \frac{\partial}{\partial v}\left[F^v h_u\right]\Big|_{(a,b)}$$

Since $\boldsymbol{p} \sim (a, b)$ was arbitrary, it need not be specified. Then dividing out the scaling factors on the left, we have the general two-dimensional orthogonal coordinate formula for the divergence,

$$\nabla\cdot\mathbf{F} \;=\; \frac{1}{h_u h_v}\left\{\frac{\partial}{\partial u}\left[F^u h_v\right] \;+\; \frac{\partial}{\partial v}\left[F^v h_u\right]\right\} \quad .$$

Let us be honest about what we have actually derived. We have derived what must be the single (orthogonal) coordinate formula for $\nabla\cdot\mathbf{F}$ *if* there is any hope of the divergence theorem being true in general. However, we have not actually verified the divergence theorem in general. To do that, we could first partition our region using coordinate curves into small rectangularish regions corresponding to coordinate step sizes Δu and Δv. Summing up over all the little

regions, observing that all the line integrals not involving the boundary of the entire region cancel out, and making use of equation (10.37), we could then verify that the desired divergence theorem really does hold, *provided the partial derivatives are continuous*.[12] I will leave the details as an exercise for anyone brave enough to try it. But it does hold in general. Since this is a significant fact, let's state it, along with the formula just derived (but using $\{(x^1, x^2)\}$ instead of $\{(u, v)\}$ for the coordinates).

Theorem 10.9 (Two-dimensional divergence)
Let $\mathbf{F}$ *be a suitably differentiable vector field on a two-dimensional region. Then there is a unique scalar field, called the divergence of* $\mathbf{F}$ *and denoted by* $\nabla \cdot \mathbf{F}$ *(or* $\text{div}(\mathbf{F})$*), such that*

$$\iint_{\mathcal{R}} \nabla \cdot \mathbf{F}\, dA \;=\; \int_C \mathbf{F} \cdot \mathbf{n}\, ds$$

whenever $\mathcal{R}$ *is a bounded subregion enclosed by a curve* C *with outward pointing unit normal vector field* $\mathbf{n}$*.*

Moreover, if

$$\{(x^1, x^2)\} \qquad , \qquad \{h_1, h_2\} \qquad \text{and} \qquad \{\mathbf{e}_1, \mathbf{e}_2\}$$

is any orthogonal coordinate system for the region with associated scaling factors and unit tangents, then

$$\nabla \cdot \mathbf{F} \;=\; \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial x^1} \left[F^1 h_2 \right] \;+\; \frac{\partial}{\partial x^2} \left[F^2 h_1 \right] \right\} \quad . \tag{10.38}$$

Divergence in Higher Dimensions

The three-dimensional analog of theorem 10.9 is

Theorem 10.10 (Three-dimensional divergence)
Let $\mathbf{F}$ *be a suitably differentiable vector field on a three-dimensional region. Then there is a unique scalar field, called the divergence of* $\mathbf{F}$ *and denoted by* $\nabla \cdot \mathbf{F}$ *(or* $\text{div}(\mathbf{F})$*), such that*

$$\iiint_{\mathcal{R}} \nabla \cdot \mathbf{F}\, dV \;=\; \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dA$$

whenever $\mathcal{R}$ *is a bounded subregion enclosed by a surface* $\mathcal{S}$ *with outward pointing unit normal vector field* $\mathbf{n}$*.*

Moreover, if

$$\{(x^1, x^2, x^3)\} \qquad , \qquad \{h_1, h_2, h_3\} \qquad \text{and} \qquad \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \quad .$$

is any orthogonal coordinate system for the region with associated scaling factors and unit tangents, then

$$\nabla \cdot \mathbf{F} \;=\; \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x^1} \left[F^1 h_2 h_3 \right] \;+\; \frac{\partial}{\partial x^2} \left[F^2 h_1 h_3 \right] \;+\; \frac{\partial}{\partial x^3} \left[F^3 h_1 h_2 \right] \right\} \quad . \tag{10.39}$$

[12] Bear in mind that we derived our results using formulas for limits of integrals that assumed continuity.

The proof is a fairly straightforward extension of what we discussed in the two-dimensional case. In deriving the formula, we first assume

$$\iiint_{\mathcal{R}} \nabla \cdot \mathbf{F}\, dV \;=\; \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, dA \quad ,$$

just as we assumed the two-dimensional analog in deriving the formula (10.38). Since the left side is a three-dimensional volume integral, we end up with $h_1h_2h_3$ instead of h_1h_2 in the denominator of our coordinate formula. And because the right side will involve a sum of surface integrals over the sides of a three-dimensional box, instead of path integrals around a rectangle, we get h_jh_k instead h_j in the partial derivatives.

Knowing this, it may not surprise you that the above generalizes to any dimension. In particular, the four-dimensional analog of formula (10.38) is

$$\nabla \cdot \mathbf{F} \;=\; \frac{1}{h_1h_2h_3h_4} \left\{ \frac{\partial}{\partial x^1}\left[F^1h_2h_3h_4\right] \;+\; \frac{\partial}{\partial x^2}\left[F^2h_1h_3h_4\right] \right.$$
$$\left. +\; \frac{\partial}{\partial x^3}\left[F^3h_1h_2h_4\right] \;+\; \frac{\partial}{\partial x^4}\left[F^4h_1h_2h_3\right] \right\} \quad .$$

I'll leave the details to you.

The Curl and Stokes' Theorem

Discussion to be added

Theorem 10.11 (General Curl)
Let $\mathbf{F}$ *be a suitably differentiable vector field in a three-dimensional region. Then there is a unique scalar field, called the curl of* $\mathbf{F}$ *and denoted by* $\nabla \times \mathbf{F}$ *(or* $\mathbf{curl}(\mathbf{F})$*), such that*

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA \;=\; \int_{\mathcal{C}} \mathbf{F} \cdot d\boldsymbol{r}$$

whenever $\mathcal{S}$ *is an bounded oriented surface with normal vector field* $\mathbf{n}$ *and bounding (oriented) curve* $\mathcal{C}$*, and with the orientations of* $\mathcal{S}$ *and* $\mathcal{C}$ *being related in the standard manner (see* Standard Conventions *on page 10–20).*

Moreover, if

$$\{(x^1, x^2, x^3)\} \qquad , \qquad \{h_1, h_2, h_3\} \qquad \text{and} \qquad \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

is any orthogonal coordinate system for the region with associated scaling factors and unit tangents, then

$$\nabla \times \mathbf{F} \;=\; \frac{1}{h_1h_2h_3} \begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial x^1} & \dfrac{\partial}{\partial x^2} & \dfrac{\partial}{\partial x^3} \\ h_1F^1 & h_2F^2 & h_3F^3 \end{vmatrix}$$

$$
\begin{aligned}
= \; & \frac{1}{h_2 h_3}\left(\frac{\partial}{\partial x^2}\left[h_3 F^3\right] \; - \; \frac{\partial}{\partial x^3}\left[h_2 F^2\right]\right)\mathbf{e}_1 \\
& - \; \frac{1}{h_1 h_3}\left(\frac{\partial}{\partial x^1}\left[h_3 F^3\right] \; - \; \frac{\partial}{\partial x^3}\left[h_1 F^1\right]\right)\mathbf{e}_2 \\
& + \; \frac{1}{h_1 h_2}\left(\frac{\partial}{\partial x^1}\left[h_2 F^2\right] \; - \; \frac{\partial}{\partial x^2}\left[h_1 F^1\right]\right)\mathbf{e}_3 \quad .
\end{aligned}
$$

Answers to Selected Exercises

Chapter 13

3a. $a_k = \dfrac{4-k}{k} a_{k-2}$ for $k \geq 2$, $y(x) = a_0 y_1(x) + a_1 y_2(x)$ where $y_1 = 1 + x^2$ and $y_2 = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{(1+2m)(1-2m)} x^{2m+1}$ **3b.** $r^2 - 3r + 2 = 0$; $r_1 = 2$, $r_2 = 1$;

For $r = r_1$, $a_1 = 0$, $a_k = \dfrac{-1}{(k+1)k} a_{k-2}$; $y(x) = a y_1(x)$ with

$$y_1(x) = x^2 \left[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 - \cdots \right] = x^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} = x \sin(x)$$

For $r = r_2$, $a_1 = 0$; $a_k = \dfrac{-1}{k(k-1)} a_{k-2}$; $y(x) = a y_2(x)$ with

$$y_2(x) = x \left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \right] = x \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} = x \cos(x)$$

3c. $4r^2 - 4r + 1 = 0$; $r_1 = r_2 = \dfrac{1}{2}$;

For $r = r_1$: $a_k = \dfrac{1}{k^2} a_{k-1}$; $y(x) = a y_1(x)$ with

$$y_1(x) = \sqrt{x} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} x^k$$

No "second" value of r **3d.** $r^2 - 4 = 0$; $r_1 = 2$, $r_2 = -2$;

For $r = r_1$: $a_k = \dfrac{-4}{k(k+4)} a_{k-1}$; $y(x) = a y_1(x)$ with

$$y_1(x) = x^2 \left[1 - \frac{4}{5}x + \frac{4^2}{(2)(6 \cdot 5)}x^2 - \frac{4^3}{(3 \cdot 2)(7 \cdot 6 \cdot 5)}x^3 + \cdots \right] = x^2 \sum_{k=0}^{\infty} \frac{(-4)^k 4!}{k!(k+4)!} x^k$$

For $r = r_2$, the recursion formula blows up **3e.** $r^2 - 7r + 10 = 0$; $r_1 = 5$, $r_2 = 2$;

For $r = r_1$, $a_1 = 0$, $a_k = \dfrac{9(k+2)}{k} a_{k-2}$; $y(x) = a y_1(x)$ with

$$y_1(x) = x^2 \left[1 + \frac{9 \cdot 4}{2}x^2 + \frac{9^2 \cdot 6}{2}x^4 + \frac{9^3 \cdot 8}{2}x^6 - \cdots \right] = x^5 \sum_{m=0}^{\infty} 9^m (m+1) x^{2m}$$

For $r = r_2$, $a_1 = 0$; $a_k = \dfrac{9(k-1)}{k-3} a_{k-2}$; $y(x) = a y_2(x)$ with

$$y_2(x) = x^2 \left[1 - 9x^2 - 3 \cdot 9^2 x^4 - 5 \cdot 9^3 x^6 + \cdots \right] = x^2 \sum_{m=0}^{\infty} 9^m (1-2m) x^{2m}$$

3f. $r^2 - 2r + 1 = 0$; $r_1 = r_2 = 1$;

For $r = r_1$: $a_k = \dfrac{k-2}{k} a_{k-1}$; $y(x) = a y_1(x)$ with

$y_1(x) \;=\; x - x^2$

No "second" value of r **3g.** $2r^2 + 3r + 1 = 0$; $r_1 = -1/2$, $r_2 = -1$;

For $r = r_1$: $a_1 = 0$, $a_k = \dfrac{2(k-2)}{k(2k+1)} a_{k-2}$; $y(x) = a y_1(x)$ with

$y_1(x) \;=\; x^{-1/2}$

For $r = r_2$: $a_1 = 0$, $a_k = \dfrac{2k-5}{k(2k-1)} a_{k-2}$; $y(x) = a y_2(x)$ with

$$y_2(x) = x^{-1}\left[1 - \frac{1}{2\cdot 3}x^2 - \frac{1}{4\cdot 2\cdot 7}x^4 - \frac{1}{6\cdot 4\cdot 2\cdot 11}x^6 + \cdots\right] \;=\; x^{-1}\sum_{m=0}^{\infty} \frac{-1}{2^m m!(4m-1)} x^{2m}$$

3h. $r^2 - 6r + 9 = 0$; $r_1 = r_2 = 3$;

For $r = r_1$: $a_k = \dfrac{2}{k} a_{k-1}$; $y(x) = a y_1(x)$ with

$$y_1(x) \;=\; x^3\left[1 + \frac{2}{1}x + \frac{2^2}{2}x^2 + \frac{2^3}{3\cdot 2}x^3 + \cdots\right] = x^3 \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = x^3 e^{2x}$$

No "second" value of r **3i.** $3r^2 - 7r + 2 = 0$; $r_1 = 2$, $r_2 = 1/3$;

For $r = r_1$: $a_k = \dfrac{k+1}{k} a_{k-1}$; $y(x) = a y_1(x)$ with

$$y_1(x) = x^2\left[1 + 2x + 3x^2 + 4x^3 + \cdots\right] = x^2 \sum_{k=0}^{\infty} (k+1)x^k$$

For $r = r_2$: $a_k = \dfrac{3k-2}{3k-5} a_{k-1}$; $y(x) = a y_1(x)$ with

$$y_1(x) = \sqrt[3]{x}\left[1 - \frac{1}{2}x - \frac{4}{2}x^2 - \frac{7}{2}x^3 + \cdots\right] = \sqrt[3]{x} \sum_{k=0}^{\infty} \frac{2-3k}{2} x^k$$

3j. $r^2 - 2r = 0$; $r_1 = 2$, $r_2 = 0$;

For $r = r_1$: $a_k = \dfrac{k-3}{k(k+2)} a_{k-1}$; $y(x) = a y_1(x)$ with

$y_1(x) \;=\; x^2 - \dfrac{2}{3}x^3 + \dfrac{1}{12}x^4$

For $r = r_2$, the recursion formula blows up **3k.** $4r^2 - 4r + 1 = 0$; $r_1 = r_2 = 1/2$;

For $r = r_1$: $a_k = -\dfrac{2k-1}{k^2} a_{k-1}$; $y(x) = a y_1(x)$ with

$$y_1(x) = \sqrt{x}\left[1 - x + \frac{3}{4}x^2 - \frac{5\cdot 3}{9\cdot 4}x^3 + \cdots\right]$$

$$= \sqrt{x}\left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1}{(k!)^2} x^k\right]$$

No "second" value of r **3l.** $r^2 = 0$; $r_1 = r_2 = 0$;

For $r = r_1$: $a_1 = 0$, $a_2 = 0$, $a_k = \dfrac{k-6}{k^2} a_{k-3}$; $y(x) = a y_1(x)$ with

$y_1(x) \;=\; 1 - \dfrac{1}{3}x^3$

No "second" value of r **3m.** $9r^2 - 1 = 0$; $r_1 = 1/3$, $r_2 = -1/3$;

For $r = r_1$: $a_k = -a_{k-1}$; $y(x) = a y_1(x)$ with

$$y_1(x) \;=\; x^{1/3}\left[1 - x + x^2 - x^3 + \cdots\right] \;=\; x^{1/3}\sum_{k=0}^{\infty} (-1)^k x^k \;=\; x^{1/3}(1+x)^{-1}$$

For $r = r_2$: $a_k = -a_{k-1}$; $y(x) = a y_2(x)$ with

$$y_2(x) = x^{-1/3}\left[1 - x + x^2 - x^3 + \cdots\right] \;=\; x^{-1/3}\sum_{k=0}^{\infty} (-1)^k x^k \;=\; x^{-1/3}(1+x)^{-1}$$

4a. $a_k = \dfrac{2k-4-\lambda}{k(k-1)} a_{k-2}$ **4b.** $(-\infty, \infty)$ **4d i.** 1 **4d ii.** x **4d iii.** $1 - 2x^2$

4d iv. $x - \frac{2}{3}x^3$ **4d v.** $1 - 4x^2 + \frac{4}{3}x^4$ **4d vi.** $x - \frac{4}{3}x^3 + \frac{4}{15}x^5$

5a. $a_k = \frac{k^2 - 3k - \lambda + 2}{k(k-1)} a_{k-2}$ **5c i.** $\lambda = m(m+1)$ **5d i.** $\lambda_0 = 0$, $p_0(x) = 1$, and $s_0(x) = \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}$ **5d ii.** $\lambda_1 = 2$, $p_1(x) = x$, and $s_1(x) = \sum_{n=0}^{\infty} \frac{1}{1-2n} x^{2n}$

5d iii. $\lambda_2 = 6$ and $p_2(x) = 1 - 3x^2$ **5d iv.** $\lambda_3 = 12$ and $p_3(x) = x - \frac{5}{3}x^3$

5d v. $\lambda_4 = 20$ and $p_4(x) = 1 - 10x^2 + \frac{35}{3}x^4$ **5d vi.** $\lambda_5 = 30$ and $p_5(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$ **5e i.** $(-1, 1)$ **5e ii.** $R = 1$ **6a.** $a_k = \frac{(k-2)^2 - \lambda}{k(k-1)} a_{k-2}$

6c i. $\lambda = m^2$ **6d i.** $\lambda_0 = 0$, $p_0(x) = 1$ **6d ii.** $\lambda_1 = 1$, $p_1(x) = x$ **6d iii.** $\lambda_2 = 4$ and $p_2(x) = 1 - 2x^2$ **6d iv.** $\lambda_3 = 9$ and $p_3(x) = x - \frac{4}{3}x^3$ **6d v.** $\lambda_4 = 16$ and $p_4(x) = 1 - 8x^2 + 8x^4$ **6d vi.** $\lambda_5 = 25$ and $p_5(x) = x - 4x^3 + \frac{16}{5}x^5$ **6e i.** $(-1, 1)$ **6e ii.** $R = 1$ **7a.** Reg. sing. pts.: $0, 2, -2$; No irreg. sing. pt. ; $R = 2$

7b. No Reg. sing. pt. ; Irreg. sing. pt.: 0 ; $R = 2$

7c. Reg. sing. pt.: 1 ; Irreg. sing. pt.: 0 ; $R = 1$

7d. Reg. sing. pt.: $3, 4$; No irreg. sing. pt. ; $R = 1$

7e. Reg. sing. pt.: 4 ; Irreg. sing. pt.: 3 ; $R = 1$

7f. Reg. sing. pt.: 0 ; No irreg. sing. pt. ; $R = \infty$

7g. Reg. sing. pts.: $0, \frac{1}{2}, -\frac{1}{2}, i, -i$; No irreg. sing. pts. ; $R = \frac{1}{2}$

7h. Reg. sing. pts.: $2i, -2i$; No irreg. sing. pt. ; $R = 2$

8a. $r^2 - r = 0$; $r_1 = 1$, $r_2 = 0$;

For $r = r_1$: $a_k = -\frac{1}{k} a_{k-1}$; $y(x) = a y_1(x)$ with

$$y_1(x) = (x-3)\left[1 - \frac{1}{2}(x-3) + \frac{1}{2 \cdot 1}(x-3)^2 - \frac{1}{3 \cdot 2 \cdot 1}(x-3)^3 + \cdots\right]$$

$$= (x-3)\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}(x-3)^k = (x-3)e^{-(x-3)}$$

For $r = r_2$, the recursion formula blows up **8b.** $r^2 + r = 0$; $r_1 = 0$, $r_2 = -1$;

For $r = r_1$: $a_k = -\frac{1}{(k+1)k} a_{k-2}$; $y(x) = a y_1(x)$ with

$$y_1(x) = 1 - \frac{1}{3!}(x+2)^2 + \frac{1}{5!}(x+2)^4 - \frac{1}{7!}(x+2)^6 + \cdots = \frac{\sin(x+2)}{x+2}$$

For $r = r_2$: $a_k = -\frac{1}{k(k-1)} a_{k-2}$; $y(x) = a y_2(x)$ with

$$y_2(x) = \frac{1}{x+2}\left[1 - \frac{1}{2!}(x+2)^2 + \frac{1}{4!}(x+2)^4 - \frac{1}{6!}(x+2)^6 + \cdots\right] = \frac{\cos(x+2)}{x+2}$$

8c. $4r^2 - 4r + 1 = 0$; $r_1 = r_2 = \frac{1}{2}$;

For $r = r_1$: $a_k = -\frac{1}{k^2}a_{k-1}$; $y(x) = ay_1(x)$ with

$y_1(x) = \sqrt{x-1}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(k!)^2}(x-1)^k$

No "second" value of r **8d.** $r^2 + 2r - 3 = 0$; $r_1 = 1$, $r_2 = -3$;

For $r = r_1$: $a_k = \frac{-1}{k+4}a_{k-1}$; $y(x) = ay_1(x)$ with

$y_1(x) = (x-3)\left[1 - \frac{1}{5}(x-3) + \frac{1}{6\cdot 5}(x-3)^2 - \frac{1}{7\cdot 6\cdot 5}(x-3)^3 + \cdots\right]$

$= (x-3)\sum_{k=0}^{\infty}\frac{(-1)^k 4!}{(k+4)!}(x-3)^k$

For $r = r_2$: $a_k = -\frac{1}{k}a_{k-1}$; $y(x) = ay_2(x)$ with

$y_2(x) = (x-3)^{-3}\left[1 - (x-3) + \frac{1}{2}(x-3)^2 - \frac{1}{3\cdot 2}(x-3)^3 + \cdots\right]$

$= (x-3)^{-3}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(x-3)^k = (x-3)^{-3}e^{-(x-3)}$

Private Notes