17

Residue Theory

"Residue theory" is basically a theory for computing integrals by looking at certain terms in the Laurent series of the integrated functions about appropriate points on the complex plane. We will develop the basic theorem by applying the Cauchy integral theorem and the Cauchy integral formulas along with Laurent series expansions of functions about the singular points. We will then apply it to compute many, many integrals that cannot be easily evaluated otherwise. Most of these integrals will be over subintervals of the real line.

17.1 Basic Residue Theory

The Residue Theorem

Suppose f is a function that, except for isolated singularities, is single-valued and analytic on some simply-connected region $\mathcal{R}$. Our initial interest is in evaluating the integral

$$\oint_{C_0} f(z)\,dz \quad .$$

where C_0 is a circle centered at a point z_0 at which f may have a pole or essential singularity. We will assume the radius of C_0 is small enough that no other singularity of f is on or enclosed by this circle. As usual, we also assume C_0 is oriented counterclockwise.

In the region right around z_0, we can express $f(z)$ as a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \quad ,$$

and, as noted earlier somewhere, this series converges uniformly in a region containing C_0. So

$$\oint_{C_0} f(z)\,dz = \oint_{C_0} \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k\,dz = \sum_{k=-\infty}^{\infty} a_k \oint_{C_0} (z-z_0)^k\,dz \quad .$$

But we've seen

$$\oint_{C_0} (z-z_0)^k\,dz \qquad \text{for} \quad k = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots$$

before. You can compute it using the Cauchy integral theorem, the Cauchy integral formulas, or even (as you did way back in exercise 14.14 on page 14–17) by direct computation after parameterizing C_0. However you do it, you get, for any integer k,

$$\oint_{C_0} (z - z_0)^k \, dz = \begin{cases} 0 & \text{if } k \neq -1 \\ i2\pi & \text{if } k = -1 \end{cases} \quad .$$

So, the above integral of f reduces to

$$\oint_{C_0} f(z) \, dz = \sum_{k=-\infty}^{\infty} a_k \begin{Bmatrix} 0 & \text{if } k \neq -1 \\ i2\pi & \text{if } k = -1 \end{Bmatrix} = i2\pi a_{-1} \quad .$$

This shows that the a_{-1} coefficient in the Laurent series of a function f about a point z_0 completely determines the value of the integral of f over a sufficiently small circle centered at z_0. This quantity, a_{-1}, is called the *residue* of f at z_0, and is denoted by

$$a_{-1,z_0} \qquad \text{or} \qquad \text{Res}_{z_0}(f) \qquad \text{or} \qquad \cdots$$

In fact, it seems that every author has their own notation. We will use $\text{Res}_{z_0}(f)$.

If, instead of integrating around the small circle C_0, we were computing

$$\oint_C f(z) \, dz$$

where C is any simple, counterclockwise oriented loop in $\mathcal{R}$ that touched no point of singularity of f but did enclose points z_0, z_1, z_2, … at which f could have singularities, then, a consequence of Cauchy's integral theorem (namely, theorem 15.5 on page 15–7) tells us that

$$\oint_C f(z) \, dz = \sum_k \oint_{C_k} f(z) \, dz$$

where each C_k is a counterclockwise oriented circle centered at z_k small enough that no other point of singularity for f is on or enclosed by this circle. Combined with the calculations done just above, we get the following:

Theorem 17.1 (Residue Theorem)
Let f be a single-valued function on a region $\mathcal{R}$, and let C be a simple loop oriented counterclockwise. Assume C encircles a finite set of points $\{z_0, z_1, z_2, \ldots\}$ at which f might not be analytic. Assume, further, that f is analytic at every other point on or enclosed by C. Then

$$\oint_C f(z) \, dz = i2\pi \sum_k \text{Res}_{z_k}(f) \quad . \tag{17.1}$$

In practice, most people just write equation (17.1) as

$$\oint_C f(z) \, dz = i2\pi \times \big[\text{sum of the enclosed residues}\big] \quad .$$

The residue theorem can be viewed as a generalization of the Cauchy integral theorem and the Cauchy integral formulas. In fact, many of the applications you see of the residue theorem can be done nearly as easily using theorem 15.5 (the consequence of the Cauchy integral theorem used above) along with the Cauchy integral formulas.

Computing Residues

Remember, what we are now calling the residue of a function f at z_0 is simply the value of a_{-1} in the Laurent series expansion of f,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \quad ,$$

right around z_0, and, for this a_{-1} to be nonzero, f must have either a pole or essential singularity at z_0. So the discussion about such singularities in section 16.3 applies both for identifying where a function may have residues and for computing the residues.

The basic approach to computing the residue at z_0 is be to simply find the above Laurent series. Then

$$\mathrm{Res}_{z_0}(f) = a_{-1} \quad .$$

This may be necessary if f has an essential singularity at z_0.

If f has a pole of finite order, say, of order M, then we can use formula (16.8) on page 16–14 for a_{-1},

$$\mathrm{Res}_{z_0}(f) = a_{-1} = \frac{1}{(M-1)!} \frac{d^{M-1}}{dz^{M-1}} \Big[(z - z_0)^M f(z) \Big] \Big|_{z=z_0} \quad . \tag{17.2}$$

If the pole is simple (i.e., $M = 1$), this simplifies to

$$\mathrm{Res}_{z_0}(f) = a_{-1} = (z - z_0) f(z)|_{z=z_0} \quad . \tag{17.3}$$

Often, we may notice that

$$f(z) = \frac{g(z)}{z - z_0}$$

for some function g which is analytic and nonzero at z_0. In this case, we clearly have a simple pole, and formula (17.3), above, clearly reduces to

$$\mathrm{Res}_{z_0}(f) = g(z_0) \quad .$$

This will make computing residues very easy in many cases.

More generally, from our earlier discussion of poles, we know that if

$$f(z) = \frac{g(z)}{(z - z_0)^M} \tag{17.4}$$

for some function g which is analytic and nonzero at z_0, then f has a pole of order M at z_0. In this case, formula (17.2) reduces to

$$\mathrm{Res}_{z_0}(f) = \frac{1}{(M-1)!} g^{(M-1)}(z_0) \quad . \tag{17.5}$$

Keep in mind that, if $g(z_0) = 0$, then the pole of

$$f(z) = \frac{g(z)}{(z - z_0)^M}$$

has order less than M, and a little more work will be needed to determine the precise order of the pole and the corresponding residue.

17.2 "Simple" Applications

The main application of the residue theorem is to compute integrals we could not compute (or don't want to compute) using more elementary means. We will consider some of the common cases involving single-valued functions not having poles on the curves of integration. Later, we will add poles and deal with multi-valued functions.

Integrals of the form $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$

Suppose we have an integral over $(0, 2\pi)$ of some formula involving $\sin(\theta)$ and $\cos(\theta)$, say,

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} \quad .$$

We can convert this to an integral about the unit circle by using the substitution

$$z = e^{i\theta} \quad .$$

Note that z does go around the unit circle in the counterclockwise direction as θ goes from 0 to 2π. Under this substitution, we have

$$dz = d\big[e^{i\theta}\big] = ie^{i\theta}\, d\theta = iz\, d\theta \quad .$$

So,

$$d\theta = \frac{1}{iz}\, dz = -iz^{-1}\, dz \quad .$$

For the sines and cosines, we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + {}^1\!/_z}{2} = \frac{z + z^{-1}}{2}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - {}^1\!/_z}{2i} = \frac{z - z^{-1}}{2i} \quad .$$

These substitutions convert the original integral to an integral of some function of z over the unit circle, which can then be evaluated by finding the enclosed residues and applying the residue theorem.

!►*Example 17.1:* *Let's evaluate*

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} \quad .$$

Letting C denote the unit circle and applying the substitution $z = e^{i\theta}$, as described above, we get

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} &= \oint_C \frac{-iz^{-1}\, dz}{2 + \frac{1}{2}\left[z + z^{-1}\right]} \\
&= \oint_C \frac{2z}{2z} \cdot \frac{-iz^{-1}}{2 + \frac{1}{2}\left[z + z^{-1}\right]}\, dz \\
&= \oint_C \frac{-i2}{4z + \left[z^2 + 1\right]}\, dz = -2i \oint_C \frac{1}{z^2 + 4z + 1}\, dz \quad .
\end{aligned}$$

Letting

$$f(z) = \frac{1}{z^2 + 4z + 1}$$

and applying the residue theorem, the above becomes

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} = -2i \times i2\pi \times \big[\text{sum of the residues of } f(z) \text{ in the unit circle}\big]$$

$$= 4\pi \times \big[\text{sum of the residues of } f(z) \text{ in the unit circle}\big] \quad .$$

To find the necessary residues we must find where the denominator of $f(z)$ *vanishes,*

$$z^2 + 4z + 1 = 0 \quad .$$

Using the quadratic formula, these points are found to be

$$z_\pm = \frac{-4 \pm \sqrt{4^2 - 4}}{2} = -2 \pm \sqrt{3} \quad .$$

So

$$f(z) = \frac{1}{\left(z - \left[-2+\sqrt{3}\right]\right)\left(z - \left[-2-\sqrt{3}\right]\right)} \quad .$$

Now, since $\sqrt{3} \approx 1.7$ *,*

$$|z_+| = \left|-2+\sqrt{3}\right| \approx |-2+1.7| = 0.3$$

while

$$|z_-| = \left|-2-\sqrt{3}\right| \approx |-2-1.7| = 3.7 \quad .$$

Clearly, $-2 + \sqrt{3}$ *is the only singular point of* $f(z)$ *enclosed by the unit circle, and the singularity there is a simple pole. We can rewrite* $f(z)$ *as*

$$f(z) = \frac{g(z)}{z - \left[-2+\sqrt{3}\right]} \qquad \text{where} \quad g(z) = \frac{1}{z - \left[-2-\sqrt{3}\right]} \quad .$$

Thus,

$$\text{Res}_{z_+}[f] = g(z_+) = g(-2+\sqrt{3}) = \frac{1}{\left[-2+\sqrt{3}\right] - \left[-2-\sqrt{3}\right]} = \frac{1}{2\sqrt{3}} \quad .$$

Plugging this back into the last formula obtained for our integral, we get

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta} &= 4\pi \times \big[\text{sum of the residues of } f(z) \text{ in the unit circle}\big] \\
&= 4\pi \, \text{Res}_{z_+}[f] \\
&= 4\pi \frac{1}{2\sqrt{3}} \\
&= \frac{2\pi}{\sqrt{3}} \quad .
\end{aligned}$$

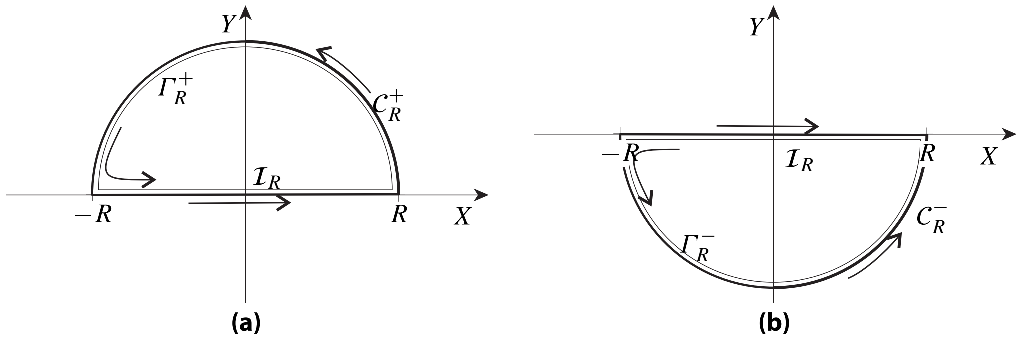

Figure 17.1: Closed curves for integrating on the X–axis when (a) $\Gamma_R = \Gamma_R^+ = \mathcal{I}_R + C_R^+$ and when (b) $\Gamma_R = \Gamma_R^- = -\mathcal{I}_R + C_R^-$.

Integrals of the form $\int_{-\infty}^{\infty} f(x)\, dx$

Now let's consider evaluating

$$\int_{-\infty}^{\infty} f(x)\, dx \quad ,$$

assuming that:

1. Except for a finite number of poles and/or essential singularities, f is a single-valued analytic function on the entire complex plane.
2. None of these singularities are on the X–axis.
3. One of the following holds:

   (a) $zf(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

   (b) $f(z) = g(z)e^{i\alpha z}$ where $\alpha > 0$ and

   $$g(z) \rightarrow 0 \qquad \text{as} \qquad |z| \rightarrow \infty \quad .$$

   (c) $f(z) = g(z)e^{-i\alpha z}$ where $\alpha > 0$ and

   $$g(z) \rightarrow 0 \qquad \text{as} \qquad |z| \rightarrow \infty \quad .$$

Under these assumptions, we can evaluate the integral by constructing, for each $R > 0$, a suitable closed loop Γ_R containing the interval $(-R, R)$. The integral over Γ_R is computed via the residue theorem, and R is allowed to go to ∞ . The conditions listed under the third assumption above ensure that the integral over that part of Γ_R which is not the interval $(-R, R)$ vanishes as $R \rightarrow \infty$, leaving us with the integral over $(-\infty, \infty)$.

The exact choice for the closed loop Γ_R depends on which of the three conditions under assumption 3 is known to hold. If either (3a) or (3b) holds, we take

$$\Gamma_R \;=\; \Gamma_R^+ \;=\; \mathcal{I}_R \;+\; C_R^+$$

where $\mathcal{I}_R$ is the subinterval $(-R, R)$ of the X–axis and C_R^+ is the semicirle in the upper half plane of radius R and centered at 0 (see figure 17.1a). In this case, with Γ_R^+ oriented counterclockwise, the direction of travel on the interval $\mathcal{I}_R$ is from $x = -R$ to $x = R$ (the normal 'positive' direction of travel on the X–axis) and, so,

$$\int_{\mathcal{I}_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx \quad .$$

Since we are letting $R \to \infty$ and there are only finitely many singularities, we can always assume that we've taken R large enough for Γ_R^+ to enclose all the singularities of f in the upper half plane (UHP). Then

$$\begin{aligned} i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big] &= \int_{\Gamma_R^+} f(z)\, dz \\ &= \int_{I_R} f(z)\, dz + \int_{C_R^+} f(z)\, dz \quad . \end{aligned}$$

Thus,

$$\begin{aligned} \int_{-R}^{R} f(x)\, dx &= \int_{I_R} f(z)\, dz \\ &= i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big] - \int_{C_R^+} f(z)\, dz \quad . \end{aligned} \tag{17.6}$$

To deal with the integral over C_R^+, first note that this curve is parameterized by

$$z = z(\theta) = Re^{i\theta} \qquad \text{where} \quad 0 \leq \theta \leq \pi \quad .$$

So "$dz = d[Re^{i\theta}] = i\,Re^{i\theta}\, d\theta$" and

$$\int_{C_R^+} f(z)\, dz = \int_0^{\pi} f\big(Re^{i\theta}\big)\, i\,Re^{i\theta}\, d\theta \quad .$$

Now assume assumption (3a) holds, and let

$$M(R) = \text{the maximum of } |zf(z)| \text{ when } |z| = R\,.$$

By definition, then,

$$\big|Re^{i\theta} f\big(Re^{i\theta}\big)\big| \leq M(R)$$

Moreover, it is easily verified that assumption (3a) (that $zf(z) \to 0$ as $|z| \to \infty$) implies that

$$M(R) \to 0 \quad \text{as} \quad R \to \infty \quad .$$

So

$$\begin{aligned} \lim_{R\to\infty} \left| \int_{C_R^+} f(z)\, dz \right| &= \lim_{R\to\infty} \left| \int_0^{\pi} f\big(Re^{i\theta}\big)\, i\,Re^{i\theta}\, d\theta \right| \\ &\leq \lim_{R\to\infty} \int_0^{\pi} \big| f\big(Re^{i\theta}\big)\, i\,Re^{i\theta} \big|\, d\theta \\ &\leq \lim_{R\to\infty} \int_0^{\pi} M(R)\, d\theta = \lim_{R\to\infty} M(R)\,\pi = 0 \quad . \end{aligned}$$

Thus,

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,dz \;=\; 0 \quad . \tag{17.7}$$

Under assumption (3b),

$$\int_{C_R^+} f(z)\,dz \;=\; \int_{C_R^+} g(z)e^{i\alpha z}\,dz$$

for some $\alpha > 0$. Now,

$$\left|e^{i\alpha z}\right| \;=\; \left|e^{i\alpha(x+iy)}\right| \;=\; \left|e^{i\alpha x}e^{-\alpha y}\right| \;=\; e^{-\alpha y} \quad ,$$

which goes to zero very quickly as $y \to \infty$. So it is certainly reasonable to expect equation (17.7) to hold under assumption (3b). And it does — but a rigorous verification requires a little more space than is appropriate here. Anyone interested can find the details in the proof of lemma 17.2 on page 17–14.

So, after letting $R \to \infty$, formula (17.6) for the integral of $f(x)$ on $(-R, R)$ becomes

$$\int_{-\infty}^{\infty} f(x)\,dx \;=\; i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big] \quad . \tag{17.8}$$

On the other hand, if it is assumption (3c) that holds, then

$$\int_{C_R^+} f(z)\,dz \;=\; \int_{C_R^+} g(z)e^{-i\alpha z}\,dz$$

for some $\alpha > 0$. In this case, though,

$$\left|e^{-i\alpha z}\right| \;=\; \left|e^{-i\alpha(x+iy)}\right| \;=\; \left|e^{-i\alpha x}e^{\alpha y}\right| \;=\; e^{\alpha y} \quad ,$$

which rapidly blows up as y gets large. So it is *not* reasonable to expect equation (17.7) to hold here. Instead, take

$$\Gamma_R \;=\; \Gamma_R^- \;=\; -\mathcal{I}_R \;+\; C_R^-$$

where C_R^- is the semicirle in the lower half plane of radius R and centered at 0 (see figure 17.1b). (Observe that, this time, the direction of travel on $\mathcal{I}_R$ is opposite to the normal 'positive' direction of travel on the X–axis.) Again, since we are letting $R \to \infty$ and there are only finitely many singularities, we can assume that we've taken R large enough for Γ_R^- to enclose all the singularities of f in the lower half plane (LHP). Then

$$\begin{aligned} i2\pi\big[\text{sum of the residues of } f \text{ in the LHP}\big] &= \int_{\Gamma_R^-} f(z)\,dz \\ &= \int_{-I_R} f(z)\,dz \;+\; \int_{C_R^-} f(z)\,dz \\ &= -\int_{I_R} f(z)\,dz \;+\; \int_{C_R^-} f(z)\,dz \quad . \end{aligned}$$

Thus,

$$\begin{aligned} \int_{-R}^{R} f(x)\,dx &= -\int_{-I_R} f(z)\,dz \\ &= -i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big] \;+\; \int_{C_R^+} f(z)\,dz \end{aligned}$$

But, as before, it can be shown that $|f(z)| \to 0$ fast enough as $|z| \to \infty$ (on the lower half plane) to ensure that

$$\lim_{R\to\infty} \int_{C_R^-} f(z)\,dz \;=\; 0 \quad .$$

So, after letting $R \to \infty$, the above formula for the integral of $f(x)$ on $(-R, R)$ becomes

$$\int_{-\infty}^{\infty} f(x)\,dx \;=\; -i2\pi\big[\text{sum of the residues of } f \text{ in the LHP}\big] \quad . \tag{17.9}$$

!►*Example 17.2:* *Let us evaluate the “Fourier integral”*

$$\int_{-\infty}^{\infty} \frac{e^{i2\pi x}}{1+x^2}\,dx \quad .$$

Here we have

$$f(z) \;=\; g(z)e^{i2\pi z} \qquad \textit{with} \quad g(z) \;=\; \frac{1}{1+z^2} \quad .$$

Clearly, $|g(z)| \to 0$ as $|z| \to \infty$ and $2\pi > 0$; so condition (3b) on page 17–6 holds. Thus, we will be applying equation (17.8), which requires the residues of f from the upper half plane.[1] *By inspection, we see that*

$$f(z) \;=\; \frac{e^{i2\pi z}}{1+z^2} \;=\; \frac{e^{i2\pi z}}{(z+i)(z-i)} \quad ,$$

which tells us that the only singularities of $f(z)$ are at $z = i$ and $z = -i$. Only i, though, is in the upper half plane, so we are only interested in the residue at i. Since we can write $f(z)$ as

$$f(z) \;=\; \frac{h(z)}{z-i} \qquad \textit{with} \quad h(z) \;=\; \frac{e^{i2\pi z}}{z+i}$$

(and $h(i) \neq 0$), we know $f(z)$ has a simple pole at i and

$$\text{Res}_i[f] \;=\; h(i) \;=\; \frac{e^{i2\pi i}}{i+i} \;=\; \frac{e^{-2\pi}}{2i} \quad .$$

Thus, applying equation (17.8)

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{i2\pi x}}{1+x^2}\,dx \;&=\; i2\pi\big[\textit{sum of the residues of } f \textit{ in the UHP}\big] \\ &=\; i2\pi\big[\,\text{Res}_i[f]\big] \\ &=\; i2\pi\left[\frac{e^{-2\pi}}{2i}\right] \;=\; \pi e^{-2\pi} \quad .\end{aligned}$$

[1] Rather than memorize that “condition (3b) implies that (17.8) is used”, keep in mind the derivation and the fact that you want the integral over one of the semicircles to vanish as $R \to \infty$. Write out the exponential in terms of x and y and see if this exponential is vanishing as $y \to +\infty$ or as $y \to -\infty$. Then “rederive” the residue-based formula for the integral of interest using the semicircle in the upper half plane if the exponential vanishes as $y \to +\infty$ and the semicircle in the lower half plane if the exponential vanishes as $y \to -\infty$. For our example

$$e^{i2\pi z} \;=\; e^{i2\pi(x+iy)} \;=\; e^{i2\pi}e^{-2\pi y} \;\to\; 0 \qquad \text{as} \quad y \to +\infty \quad .$$

So we are using the upper half plane.

Standard "Simple" Tricks
Using Real and Imaginary Parts

It is often helpful to observe that one integral of interest may be the real or imaginary part of another integral that may, possibly, be easier to evaluate. Do remember that

$$\int_a^b \operatorname{Re}[f(x)]\,dx = \operatorname{Re}\left[\int_a^b f(x)\,dx\right] \quad \text{and} \quad \int_a^b \operatorname{Im}[f(x)]\,dx = \operatorname{Im}\left[\int_a^b f(x)\,dx\right] \quad .$$

(If this isn't obvious, spend a minute to (re)derive it.) In this regard, it is especially useful to observe that

$$\cos(X) = \operatorname{Re}\left[e^{iX}\right] \quad \text{and} \quad \sin(X) = \operatorname{Im}\left[e^{iX}\right] \quad .$$

!►*Example 17.3:* *Consider*

$$\int_{-\infty}^{\infty} \frac{\cos(2\pi x)}{1+x^2}\,dx$$

Using the above observations and our answer from the previous exercise,

$$\int_{-\infty}^{\infty} \frac{\cos(2\pi x)}{1+x^2}\,dx = \int_{-\infty}^{\infty} \operatorname{Re}\left[\frac{e^{i2\pi x}}{1+x^2}\right]dx$$
$$= \operatorname{Re}\left[\int_{-\infty}^{\infty} \frac{e^{i2\pi x}}{1+x^2}\,dx\right] = \operatorname{Re}\left[\pi e^{-2\pi}\right] = \pi e^{-2\pi} \quad .$$

Clever Choice of Curve and Function

The main "trick" to applying residue theory in computing integrals of real interest (i.e., integrals that actually do arise in applications) as well as other integrals you may encounter (e.g., other integrals in assigned homework and tests) is to make clever choices for the functions and the curves so that you really can extract the value of desired integral from the integral over the closed curve used. Some suggestions, such as were given on page 17–6 for computing certain integrals on $(-\infty,\infty)$, can be given. In general, though, choosing the right curves and functions is a cross between an art and a skill that one just has to develop.

A good example of using both clever choices of functions and curves, and in using real and imaginary parts, is given in computing the Fresnel integrals (which arise in optics).

!►*Example 17.4:* *Consider the Fresnel integrals*

$$\int_0^{\infty} \cos\left(x^2\right)\,dx \quad \text{and} \quad \int_0^{\infty} \sin\left(x^2\right)\,dx \quad .$$

Rather than use $\cos\left(x^2\right)$ *and* $\sin\left(x^2\right)$ *directly, it is clever to use* e^{ix^2} *and the fact that*

$$e^{ix^2} = \cos\left(x^2\right) + i\sin\left(x^2\right) \quad .$$

So, if we can evaluate

$$\int_0^{\infty} e^{ix^2}\,dx \quad ,$$

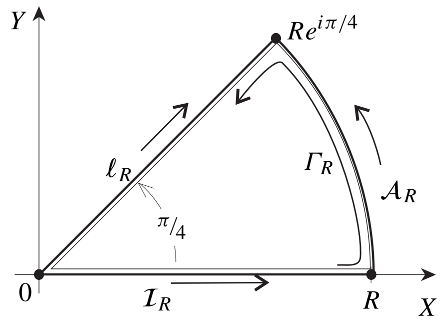

Figure 17.2: The closed curve for computing the Fresnel integrals.

then we can get the two integrals we want via

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \mathrm{Re}\big[e^{ix^2}\big]\,dx = \mathrm{Re}\left[\int_0^\infty e^{ix^2}\,dx\right]$$

and

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \mathrm{Im}\big[e^{ix^2}\big]\,dx = \mathrm{Im}\left[\int_0^\infty e^{ix^2}\,dx\right] \quad .$$

Now, to compute

$$\int_0^\infty e^{ix^2}\,dx \quad ,$$

we naturally choose the function $f(z) = e^{iz^2}$ *. The clever choice for the closed curve is sketched in figure 17.2. It is*

$$\Gamma_R = \mathcal{I}_R + \mathcal{A}_R - \ell_R$$

where, for any choice of $R > 0$ *,*

$$\begin{aligned} \mathcal{I}_R &= \text{the straight line on the real line from } z = 0 \text{ to } z = R \\ &= \text{the interval on } \mathbb{R} \text{ from } x = 0 \text{ to } x = R , \end{aligned}$$

$$\mathcal{A}_R = \text{the circular arc centered at } 0 \text{ starting at } z = R \text{ and going to } z = Re^{i\pi/4} ,$$

and

$$\ell_R = \text{the straight line from } z = 0 \text{ to } z = Re^{i\pi/4} .$$

Notice that the chosen function, $f(z) = e^{iz^2}$ *, is analytic on the entire complex plane. So there are no residues, and, for each* $R > 0$ *, we have*

$$\begin{aligned} 0 = \oint_{\Gamma_R} e^{iz^2}\,dz &= \int_{\mathcal{I}_R} e^{iz^2}\,dz + \int_{\mathcal{A}_R} e^{iz^2}\,dz - \int_{\ell_R} e^{iz^2}\,dz \\ &= \int_{x=0}^R e^{ix^2}\,dx + \int_{\mathcal{A}_R} e^{iz^2}\,dz - \int_{\ell_R} e^{iz^2}\,dz \quad . \end{aligned}$$

So,

$$\int_{x=0}^R e^{ix^2}\,dx = \int_{\ell_R} e^{iz^2}\,dz - \int_{\mathcal{A}_R} e^{iz^2}\,dz \quad .$$

Letting $R \to \infty$ *, this becomes*

$$\int_0^\infty e^{ix^2}\,dx = \lim_{R\to\infty} \int_{\ell_R} e^{iz^2}\,dz - \lim_{R\to\infty} \int_{\mathcal{A}_R} e^{iz^2}\,dz \quad . \tag{17.10}$$

Now, ℓ_R is parameterized by

$$z = z(r) = re^{i\pi/4} \qquad \text{with } r \text{ going from } 0 \text{ to } R \quad .$$

Using this,

$$iz^2 = ir^2 e^{i\pi/2} = ir^2 i = -r^2 \quad ,$$

$$dz = d\left[re^{i\pi/4}\right] = e^{i\pi/4}\,dr$$

and

$$\lim_{R\to\infty} \int_{\ell_R} e^{iz^2}\,dz = \lim_{R\to\infty} \int_0^R e^{-r^2} e^{i\pi/4}\,dr = e^{i\pi/4} \int_0^\infty e^{-r^2}\,dr \quad .$$

The value of the last integral is $\sqrt{\pi}/2$. This can be found by cheap tricks not involving residues (see the appendix on the integral of the basic Gaussian starting on page 17–16). Thus,

$$\lim_{R\to\infty} \int_{\ell_R} e^{iz^2}\,dz = e^{i\pi/4} \frac{\sqrt{\pi}}{2} \quad . \tag{17.11}$$

The curve $\mathcal{A}_R$ for the other integral in equation (17.10) is parameterized by

$$z = z(\theta) = Re^{i\theta} \qquad \text{with } \theta \text{ going from } 0 \text{ to } {}^{\pi}/_{4} \quad .$$

Using this,

$$iz^2 = iR^2 e^{i2\theta} = iR^2[\cos(2\theta) + i\sin(2\theta)] = iR^2\cos(2\theta) - R^2\sin(2\theta) \quad ,$$

$$dz = d\left[Re^{i\theta}\right] = iRe^{i\theta}\,d\theta$$

and

$$\lim_{R\to\infty} \int_{\mathcal{A}_R} e^{iz^2}\,dz = \lim_{R\to\infty} \int_0^{\pi/4} e^{iR^2\cos(2\theta) - R^2\sin(2\theta)} iRe^{i\theta}\,d\theta$$

$$= \lim_{R\to\infty} \int_0^{\pi/4} e^{iR^2\cos(2\theta)} e^{-R^2\sin(2\theta)} iRe^{i\theta}\,d\theta \quad .$$

Note that

$$\left|\int_0^{\pi/4} e^{iR^2\cos(2\theta)} e^{-R^2\sin(2\theta)} iRe^{i\theta}\,d\theta\right| \le \int_0^{\pi/4} \left|e^{iR^2\cos(2\theta)} e^{-R^2\sin(2\theta)} iRe^{i\theta}\right|\,d\theta$$

$$= \int_0^{\pi/4} e^{-R^2\sin(2\theta)} R\,d\theta \quad .$$

For each θ in $(0, {}^{\pi}/_{4}]$, the exponential in the last integral goes to zero much faster than R increases. So this integral "clearly" vanishes as $R \to \infty$ (see exercise 17.2 on page 17–16 for a rigorous verification). Thus,

$$\lim_{R\to\infty} \int_{\mathcal{A}_R} e^{iz^2}\,dz = 0 \quad . \tag{17.12}$$

Gathering together what we've derived (equations (17.10), (17.11) and (17.12)), we finally get

$$\begin{aligned}\int_0^\infty e^{ix^2}\,dx &= \lim_{R\to\infty}\int_{\ell_R} e^{iz^2}\,dz \;-\; \lim_{R\to\infty}\int_{\mathcal{A}_R} e^{iz^2}\,dz\\ &= e^{i\pi/4}\frac{\sqrt{\pi}}{2} \;+\; 0\\ &= \left[\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right]\frac{\sqrt{\pi}}{2}\\ &= \left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]\frac{\sqrt{\pi}}{2} \;=\; \frac{1}{2}\sqrt{\frac{\pi}{2}} \;+\; i\frac{1}{2}\sqrt{\frac{\pi}{2}} \quad .\end{aligned}$$

Consequently,

$$\int_0^\infty \cos\left(x^2\right)\,dx \;=\; \operatorname{Re}\left[\int_0^\infty e^{ix^2}\,dx\right] \;=\; \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

and

$$\int_0^\infty \sin\left(x^2\right)\,dx \;=\; \operatorname{Im}\left[\int_0^\infty e^{ix^2}\,dx\right] \;=\; \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad .$$

?►*Exercise 17.1:* *Consider the last example.*

a: *Why was the curve $\Gamma_R = \mathcal{I}_R + \mathcal{A}_R - \ell_R$, as illustrated in figure 17.2, such a clever choice of curves? (Consider what happened to the function e^{iz^2} on $\mathcal{I}_R$.)*

b: *Why would the curve $\Gamma_R = \mathcal{I}_R + C_R^+$ illustrated in figure 17.1a not be a clever choice for computing the Fresnel integrals?*

In the last example, as in a previous example, the closed curve constructed so the residue theorem can be applied included a piece of a circle of radius R, say, the $\mathcal{A}_R$ in our last example. Our formula for the desired integral then includes a term of the form

$$\lim_{R\to\infty}\int_{\mathcal{A}_R} f(z)\,dz \quad ,$$

and it is important that this limit be zero (or some other computable finite number). Otherwise, either this general approach fall apart, or we can show that the desired integral does not converge.

Keep in mind that the length of $\mathcal{A}_R$ increases as R increases; so the vanishing of the integrand is not enough to ensure the vanishing of the above limit. To take into account the increasing length of the curve, it is often a good idea to parameterize it using angular measurement,

$$z \;=\; z(\theta) \;=\; Re^{i\theta}$$

with θ limited to some fixed interval (α,β). Then, as we've seen in examples,

$$\begin{aligned}\left|\int_{\mathcal{A}_R} f(z)\,dz\right| &= \left|\int_\alpha^\beta f\left(Re^{i\theta}\right) i\,Re^{i\theta}\,d\theta\right|\\ &\leq \int_\alpha^\beta \left|f\left(Re^{i\theta}\right) i\,Re^{i\theta}\right|\,d\theta \;=\; \int_\alpha^\beta \left|f\left(Re^{i\theta}\right)\right| R\,d\theta \quad .\end{aligned}$$

With luck, you will be able to look at the inside of the last integral and tell whether the desired limit is zero or whether the limit does not exist.

Of course, you may ask, why not just compute the limit by bringing the limit inside the integral,

$$\lim_{R\to\infty}\int_\alpha^\beta \left|f\left(Re^{i\theta}\right)\right| R\,d\theta \;=\; \int_\alpha^\beta \lim_{R\to\infty}\left|f\left(Re^{i\theta}\right)\right| R\,d\theta \quad ?$$

Because this last equation is not always valid. There are choices for f such that

$$\lim_{R\to\infty}\int_\alpha^\beta \left|f\left(Re^{i\theta}\right)\right| R\,d\theta \;\neq\; \int_\alpha^\beta \lim_{R\to\infty}\left|f\left(Re^{i\theta}\right)\right| R\,d\theta \quad !$$

For example, $f(z) = e^{-z}$ with $(\alpha,\beta) = (0, \pi/2)$.

Appendices to this Section

Two issues are addressed here. One is how to rigorously verify that certain integrals over arcs of radii R vanish as R goes to infinity. The other is how to evaluate $\int_0^\infty e^{-s^2}\,ds$. These appendices are included for the sake of completeness. You should be acquainted with the general results, but don't worry about reproducing the sort of analysis given here.

The Vanishing of Certain Integrals as $R \to \infty$

Often, when using residues to compute integrals with exponentials, it is necessary to verify that certain integrals over arcs of radii R vanish as R goes to infinity. Sometimes this is easy to show; sometimes it is not. To illustrate how we might verify the cases involving complex exponentials, we will rigorously verify the following lemma. It is the lemma needed to rigorously verify equation (17.7) on page 17–8.

Lemma 17.2
Let $\alpha > 0$, and assume g is any "reasonably smooth" function on the complex plane[2] satisfying

$$g(z) \;\to\; 0 \qquad \text{as} \quad |z| \to \infty \quad .$$

Then

$$\lim_{R\to\infty}\int_{C_R^+} g(z)e^{i\alpha z}\,dz \;=\; 0$$

where C_R^+ is the upper half of the circle of radius R about the origin.

PROOF: Keep in mind that C_R^+ is getting longer as R gets larger. So the simple fact that the integrand gets smaller as R gets larger is not enough to ensure that the above limit is zero. To help take into account the increasing length of the curve and to help convert the integral to something a little more easily to deal with, we make use of the fact that this curve can be parameterized by

$$z \;=\; z(\theta) \;=\; Re^{i\theta} \;=\; R\,[\cos(\theta) + i\sin(\theta)] \qquad \text{where} \quad 0 \;\le\; \theta \;\le\; \pi \quad .$$

For convenience, let

$$M(R) \;=\; \text{maximum value of } g(z(\theta)) \qquad \text{when} \quad 0 \;\le\; \theta \;\le\; \pi \quad .$$

[2] it suffices to assume $g(z)$ is continuous on the region where $|z| > R_0$ for some finite real value R_0.

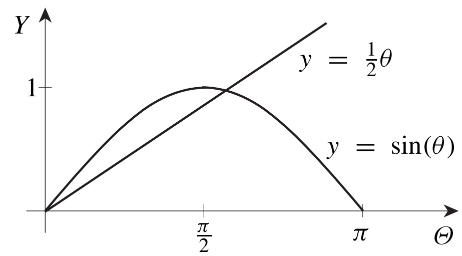

Figure 17.3: The graphs of $y = \sin(\theta)$ and $y = \theta/2$ on the interval $[0, \pi/2]$.

It isn't hard to show that the fact that $g(z) \to 0$ as $|z| \to \infty$ implies that

$$M(R) \to 0 \qquad \text{when} \quad R \to \infty \quad .$$

Also note that that, using the above parametrization,

$$dz = iRe^{i\theta}\, d\theta$$

and

$$\begin{aligned} \left|e^{i\alpha z(\theta)}\right| = \left|e^{i\alpha R[\cos(\theta)+i\sin(\theta)]}\right| &= \left|e^{i\alpha R\cos(\theta) - \alpha R\sin(\theta)}\right| \\ &= \left|e^{i\alpha R\cos(\theta)} e^{-\alpha R\sin(\theta)}\right| = e^{-\alpha R\sin(\theta)} \quad . \end{aligned}$$

So,

$$\begin{aligned} \left|\int_{C_R^+} g(z)e^{i\alpha z}\, dz\right| &= \left|\int_0^\pi g(z(\theta))e^{i\alpha z(\theta)} iRe^{i\theta}\, d\theta\right| \\ &\leq \int_0^\pi \left|g(z(\theta))e^{i\alpha z(\theta)} iRe^{i\theta}\right| d\theta \leq \int_0^\pi M(R)e^{-\alpha R\sin(\theta)} R\, d\theta \quad . \end{aligned}$$

Pulling out $M(R)$ and using the fact that $\sin(\theta)$ is symmetric about $\theta = \pi/2$, this becomes

$$\left|\int_{C_R^+} g(z)e^{i\alpha z}\, dz\right| \leq 2M(R)\int_0^{\pi/2} e^{-\alpha R\sin(\theta)} R\, d\theta \quad . \tag{17.13}$$

Now, it is easy to see and easy to confirm that

$$\sin(\theta) > \frac{\theta}{2} \qquad \text{for} \quad 0 \leq \theta \leq \frac{\pi}{2} \tag{17.14}$$

(see figure 17.3). It then follows that

$$e^{-\alpha R\sin(\theta)} < e^{-\alpha R\theta/2} \qquad \text{for} \quad 0 \leq \theta \leq \frac{\pi}{2} \quad . \tag{17.15}$$

Plugging this into inequality (17.13) and computing the resulting integral yields

$$\begin{aligned} \left|\int_{C_R^+} g(z)e^{i\alpha z}\, dz\right| &\leq 2M(R)\int_0^{\pi/2} e^{-\alpha R\theta/2} R\, d\theta \\ &= \frac{4M(R)}{\alpha}\left[1 - e^{-\alpha R\pi/4}\right] < \frac{4M(R)}{\alpha} \quad . \end{aligned}$$

Thus,

$$\lim_{R\to\infty}\left|\int_{C_R^+} g(z)e^{i\alpha z}\,dz\right| \;<\; \lim_{R\to\infty}\frac{4M(R)}{\alpha} \;=\; \frac{4\cdot 0}{\alpha} \;=\; 0 \quad ,$$

which, of course, means that

$$\lim_{R\to\infty}\int_{C_R^+} g(z)e^{i\alpha z}\,dz \;=\; 0 \quad ,$$

as claimed. ▌

?►Exercise 17.2: *Using the ideas behind inequalities (17.14) and (17.15), show that*

$$\int_0^{\pi/4} e^{-R^2\sin(2\theta)}R\,d\theta \;\le\; \int_0^{\pi/4} e^{-R^2\theta}R\,d\theta \;=\; \frac{1}{R} \;\to\; 0 \quad \text{as} \quad R\to\infty \quad .$$

Integral of the Basic Gaussian

While there is no simple formula for the indefinite integral of e^{-x^2}, the value of the definite integral

$$\int_0^\infty e^{-s^2}\,ds$$

is easily computed via a clever trick.

To begin, observe that, by symmetry,

$$\int_0^\infty e^{-s^2}\,ds \;=\; \frac{1}{2}I$$

where

$$I \;=\; \int_{-\infty}^\infty e^{-s^2}\,ds \;=\; \int_{-\infty}^\infty e^{-x^2}\,dx \;=\; \int_{-\infty}^\infty e^{-y^2}\,dy \quad .$$

The "clever trick" is based on the observation that I^2, the product of I with itself, can be expressed as a double integral over the entire XY–plane,

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)\left(\int_{-\infty}^\infty e^{-y^2}\,dy\right) \\ &= \int_{-\infty}^\infty\left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)e^{-y^2}\,dy \\ &= \int_{-\infty}^\infty\left(\int_{-\infty}^\infty e^{-x^2}e^{-y^2}\,dx\right)dy \;=\; \int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(x^2+y^2)}\,dx\,dy \quad . \end{aligned}$$

This double integral is easily computed using polar coordinates (r,θ) where

$$x \;=\; r\cos(\theta) \qquad \text{and} \qquad y \;=\; r\sin(\theta) \quad .$$

Recall that

$$x^2+y^2 \;=\; r^2 \qquad \text{and} \qquad dx\,dy \;=\; r\,dr\,d\theta \quad .$$

So, converting to polar coordinates and using elementary integration techniques,

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \, dr \, d\theta = \int_0^{2\pi} \frac{1}{2} \, d\theta = \pi \quad .$$

Taking the square root gives

$$I = \pm\sqrt{\pi} \quad .$$

Because e^{-s^2} is a positive function, its integral must be positive. Thus,

$$\int_{-\infty}^{\infty} e^{-s^2} \, ds = I = \sqrt{\pi}$$

and

$$\int_0^{\infty} e^{-s^2} \, ds = \frac{1}{2} I = \frac{1}{2}\sqrt{\pi} \quad .$$

17.3 Integrals Over Branch Cuts of Multi-Valued Functions

If you need to compute

$$\int_{\alpha}^{\beta} f(x) \, dx$$

when $f(z)$ is a multi-valued function, then a clever choice for the closed curve may include using the interval (α, β) as a branch cut for f, and letting it serve as two parts of the curve enclosing the residues of f. Just what I mean will be a lot clearer if we do one example.

But first, go back and re-read the discussion of multi-valued functions starting on page 14–5. Also, re-read the bit about *Square Roots and Such* starting on page 14–8.

!►*Example 17.5:* *Consider evaluating*

$$\int_0^{\infty} \frac{\sqrt{x}}{1 + x^2} \, dx \quad .$$

Naturally, we will let

$$f(z) = \frac{z^{1/2}}{1 + z^2} \quad .$$

This function has singularities at $z = \pm i$ *(where the denominator is zero). Also, because of the* $z^{1/2}$ *factor,* $f(z)$ *is multi-valued with a branch point at* $z = 0$ *. We will cleverly take the cut line to be the positive* X*–axis, and define* $z^{1/2}$ *to be given by*

$$z^{1/2} = \sqrt{|z|} \, e^{i\theta/2}$$

where θ *is the polar angle (argument) of* z *satisfying*

$$0 < \theta < 2\pi \quad .$$

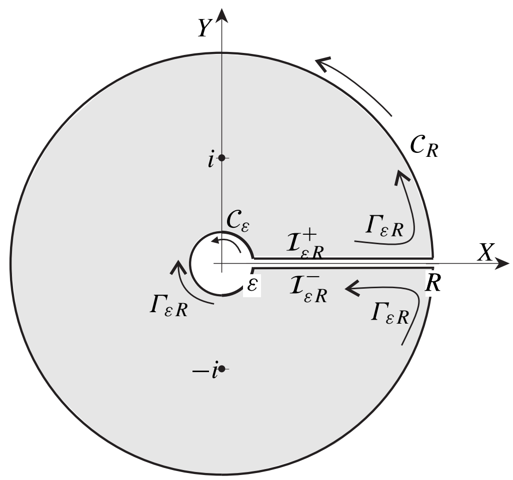

Figure 17.4: The curves for example 17.5. The region enclosed by $\Gamma_{\varepsilon R}$ is shaded, and the arrows indicate the direction of travel along $\Gamma_{\varepsilon R}$.

This defines the branch of f that we will use. This also means that we will have to be careful about the value of this function on the positive X–axis since, for each point x_0 on this interval,

$$\lim_{\substack{z\to x_0\\ \operatorname{Im} z>0}} z^{1/2} = \lim_{\theta\to 0^+} \sqrt{x_0}\, e^{i\theta/2} = \sqrt{x_0}\, e^{i\cdot 0} = \sqrt{x_0}$$

while

$$\lim_{\substack{z\to x_0\\ \operatorname{Im} z<0}} z^{1/2} = \lim_{\theta\to 2\pi^-} \sqrt{x_0}\, e^{i\theta/2} = \sqrt{x_0}\, e^{i\pi} = -\sqrt{x_0} \quad .$$

Thus,

$$\lim_{\substack{z\to x_0\\ \operatorname{Im} z>0}} f(z) = \lim_{\substack{z\to x_0\\ \operatorname{Im} z>0}} \frac{z^{1/2}}{1+z^2} = \frac{\sqrt{x_0}}{1+{x_0}^2}$$

while

$$\lim_{\substack{z\to x_0\\ \operatorname{Im} z<0}} f(z) = \lim_{\substack{z\to x_0\\ \operatorname{Im} z>0}} \frac{z^{1/2}}{1+z^2} = -\frac{\sqrt{x_0}}{1+{x_0}^2} \quad .$$

The closed curve we will use with the residue theorem is constructed from the curves in figure 17.4. For each pair of positive values ε and R satisfying $\varepsilon < 1 < R$, we let $\Gamma_{\varepsilon R}$ be the closed curve given by

$$\Gamma_{\varepsilon R} = \mathcal{I}^+_{\varepsilon R} + C_R + \mathcal{I}^-_{\varepsilon R} - C_\varepsilon$$

where, as indicated in figure 17.4,

C_R = *circle of radius R about 0, oriented counterclockwise* ,

C_ε = *circle of radius ε about 0, oriented counterclockwise* ,

$\mathcal{I}^+_{\varepsilon R}$ = *straight line on the X–axis from $x = \varepsilon$ to $x = R$* ,

and

$\mathcal{I}^-_{\varepsilon R}$ = *straight line on the X–axis from $x = R$ to $x = \varepsilon$* .

The circle C_ε isolates the branch point from the region encircled by $\Gamma_{\varepsilon R}$. This is a good idea because "bad things" can happen near branch points. Later, we will check that we can (or cannot) let $\varepsilon \to 0$.

The curves $\mathcal{I}^+_{\varepsilon R}$ and $\mathcal{I}^-_{\varepsilon R}$ are each, in fact, simply the subinterval (ε, R) on the X–axis oriented in opposing directions. They are treated, however, as two distinct pieces of $\Gamma_{\varepsilon R}$, with $\mathcal{I}^+_{\varepsilon R}$ viewed as a lower boundary to the region above it, and $\mathcal{I}^-_{\varepsilon R}$ viewed as the upper boundary to the region below it. Indeed, it may be best to first view them as laying a small distance above and below the X–axis, exactly as sketched in figure 17.4, with this distance shrunk to zero by the end of the computations. Consequently, the region encircled by $\Gamma_{\varepsilon R}$ is the shaded region in figure 17.4.

Applying the residue theorem, we have

$$\begin{aligned} &i2\pi \times [\text{sum of the resides of } f \text{ in the shaded region}] \\ &\qquad = \oint_\Gamma f(z)\,dz \\ &\qquad = \int_{\mathcal{I}^+_{\varepsilon R}} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{\mathcal{I}^-_{\varepsilon R}} f(z)\,dz - \int_{C_\varepsilon} f(z)\,dz \end{aligned}$$

where

$$\int_{\mathcal{I}^+_{\varepsilon R}} f(z)\,dz = \int_\varepsilon^R \left[\lim_{\substack{z\to x \\ \operatorname{Im} z>0}} f(z)\right] dx = \int_\varepsilon^R \frac{\sqrt{x}}{1+x^2}\,dx$$

and

$$\int_{\mathcal{I}^-_{\varepsilon R}} f(z)\,dz = \int_R^\varepsilon \left[\lim_{\substack{z\to x \\ \operatorname{Im} z<0}} f(z)\right] dx = -\int_\varepsilon^R \left[-\frac{\sqrt{x}}{1+x^2}\right] dx = \int_\varepsilon^R \frac{\sqrt{x}}{1+x^2}\,dx \quad .$$

Thus,

$$\begin{aligned} &i2\pi \times [\text{sum of the resides of } f \text{ in the shaded region}] \\ &\qquad = 2\int_\varepsilon^R \frac{\sqrt{x}}{1+x^2}\,dx + \int_{C_R} f(z)\,dz - \int_{C_\varepsilon} f(z)\,dz \quad , \end{aligned}$$

and, so,

$$\begin{aligned} \int_\varepsilon^R \frac{\sqrt{x}}{1+x^2}\,dx &= i\pi \times [\text{sum of the resides of } f \text{ in the shaded region}] \\ &\qquad + \frac{1}{2}\int_{C_\varepsilon} f(z)\,dz - \frac{1}{2}\int_{C_R} f(z)\,dz \quad . \end{aligned} \tag{17.16}$$

(Notice that, because of the multi-valueness of f,

$$\int_{\mathcal{I}^+_{\varepsilon R}} f(z)\,dz \qquad \text{and} \qquad \int_{\mathcal{I}^-_{\varepsilon R}} f(z)\,dz$$

did not cancel each other out even though $\mathcal{I}^+_{\varepsilon R}$ and $\mathcal{I}^-_{\varepsilon R}$ are the same curve oriented in opposite directions. That is what will make these computations work.)

Naturally, we want $\varepsilon \to 0$ and $R \to \infty$. Now, if $|z| = \varepsilon$ and $\varepsilon < 1$, then

$$|f(z)| = \left|\frac{z^{1/2}}{1+z^2}\right| \le \frac{\varepsilon^{1/2}}{1-\varepsilon^2} \quad .$$

So, using the parametrization $z = z(\theta) = \varepsilon e^{i\theta}$,

$$\begin{aligned}\left|\int_{C_\varepsilon} f(z)\,dz\right| &= \left|\int_0^{2\pi} f(z(\theta))\, i\varepsilon e^{i\theta}\,d\theta\right| \\ &\leq \int_0^{2\pi} |f(z(\theta))|\;\varepsilon\,d\theta \\ &\leq \int_0^{2\pi} \frac{\varepsilon^{1/2}}{1-\varepsilon^2}\,\varepsilon\,d\theta \;=\; \frac{\varepsilon^{3/2}}{1-\varepsilon^2}2\pi \;\rightarrow\; 0 \qquad \text{as} \quad \varepsilon \rightarrow 0 \quad .\end{aligned}$$

By a very similar computations, we get (since $R > 1$ *),*

$$\left|\int_{C_R} f(z)\,dz\right| \;\leq\; \int_0^{2\pi} \frac{R^{1/2}}{R^2-1}\,R\,d\theta \;=\; \frac{R^{3/2}}{R^2-1}2\pi \;\rightarrow\; 0 \qquad \text{as} \quad R \rightarrow \infty \quad .$$

Thus, after letting $\varepsilon \rightarrow 0$ *and* $R \rightarrow \infty$ *, equation (17.16) reduces to*

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx \;=\; i\pi \times [\text{sum of the resides of } f \text{ in the shaded region}] \tag{17.17}$$

Clearly, the only singularities of

$$f(z) \;=\; \frac{z^{1/2}}{1+z^2}$$

are at $z = \pm i$ *, and*

$$f(z) \;=\; \frac{g(z)}{z-i} \qquad \text{and} \qquad f(z) \;=\; \frac{h(z)}{z-(-i)} \quad .$$

where

$$g(z) \;=\; \frac{z^{1/2}}{z+i} \qquad \text{and} \qquad h(z) \;=\; \frac{z^{1/2}}{z-i} \quad .$$

So these singularities are simple poles, and

$$\begin{aligned}\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx &= i\pi \times [\text{sum of the resides of } f \text{ in the shaded region}] \\ &= i\pi \left[\operatorname{Res}_i(f) \;+\; \operatorname{Res}_{-i}(f)\right] \\ &= i\pi[g(i)+h(-i)] \\ &= i\pi\left[\frac{i^{1/2}}{i+i} \;+\; \frac{(-i)^{1/2}}{-i-i}\right] \\ &= \pi\left[\frac{e^{i\pi/4}}{2} \;-\; \frac{e^{i3\pi/4}}{2}\right] \;=\; \cdots \;=\; \frac{\pi}{\sqrt{2}} \quad .\end{aligned}$$

17.4 Integrating Through Poles and the Cauchy Principal Value

The Cauchy Principal Value of an Integral

The "Cauchy principal value" of an integral is actually a redefinition of the integral so that certain types of singularities are ignored through the process of taking "symmetric limits".

To be precise: Let f be a function on an interval (a, b) and assume f is "well behaved" (say, is analytic) at every point in the interval except one, x_0. We then define

$$\text{The Cauchy principal value of } \int_a^b f(x)\,dx \;=\; \text{CPV}\int_a^b f(x)\,dx \qquad \text{(my notation)}$$
$$= \lim_{\varepsilon\to 0^+}\left[\int_a^{x_0-\varepsilon} f(x)\,dx \;+\; \int_{x_0+\varepsilon}^b f(x)\,dx\right] \quad .$$

By the way, in Arfken, Weber and Harris you'll find

$$\text{P}\int_a^b f(x)\,dx \quad \text{and} \quad ⨍_a^b f(x)\,dx$$

denoting the above Cauchy principal value.

If f is continuous, or even just has a jump discontinuity, at x_0, then

$$\text{CPV}\int_a^b f(x)\,dx \;=\; \lim_{\varepsilon\to 0^+}\left[\int_a^{x_0-\varepsilon} f(x)\,dx \;+\; \int_{x_0+\varepsilon}^b f(x)\,dx\right]$$
$$= \int_a^{x_0} f(x)\,dx \;+\; \int_{x_0}^b f(x)\,dx$$
$$= \int_a^b f(x)\,dx \quad .$$

This simply means that the Cauchy principal value reduces to the integral when the integral is well defined. However, when the singularity is a simple pole then you can easily verify that the Cauchy principal value "removes" the effect of the singularity by "canceling out infinities".[3]

!►*Example 17.6:* *Consider*

$$\int_{-3}^4 \frac{1}{x}\,dx \qquad \textit{and} \qquad \textit{CPV}\int_{-3}^4 \frac{1}{x}\,dx \quad .$$

The function being integrated has a singularity at $x = 0$, *and*

$$\int_{-3}^4 \frac{1}{x}\,dx \;=\; \int_{-3}^0 \frac{1}{x}\,dx \;+\; \int_0^4 \frac{1}{x}\,dx$$
$$= \ln|x|\Big|_{-3}^0 \;+\; \ln|x|\Big|_0^4$$
$$= -\infty \;-\; \ln 3 \;+\; \ln 4 \;-\; (-\infty)$$

[3] In practice, make sure you can justify this "canceling out of infinities". Otherwise, your use of the Cauchy principal value is probably fraudulent.

$$= \infty - \infty + \ln \frac{4}{3} \quad ,$$

which, for very good reasons, is considered to be "undefined". On the other hand,

$$\begin{aligned} CPV \int_{-3}^{4} \frac{1}{x}\, dx &= \lim_{\varepsilon \to 0^+} \left[\int_{-3}^{-\varepsilon} \frac{1}{x}\, dx + \int_{\varepsilon}^{4} \frac{1}{x}\, dx \right] \\ &= \lim_{\varepsilon \to 0^+} [\ln \varepsilon - \ln 3 + \ln 4 - \ln \varepsilon] \\ &= \lim_{\varepsilon \to 0^+} [-\ln 3 + \ln 4] = \ln \frac{4}{3} \quad . \end{aligned}$$

The Cauchy principal part should not just be thought of as a way to get around "technical difficulties" with integrals that blow up, and its indiscriminate use can lead to dangerously misleading results. For example, if the force on some object at position x is given by $1/x$, then one could naively argue that the work needed to move that object from $x = -3$ to $x = 4$ is just

$$\text{work} = \text{CPV} \int_{-3}^{4} \frac{1}{x}\, dx = \ln \frac{4}{3} \quad ,$$

but just see if you really can push that object past $x = 0$.

The Cauchy principal value can be useful, but its use must be justified by something more than a desire to "cancel out infinities". Also, sometimes those infinities just don't cancel out.

?►*Exercise 17.3:* *Let n be any positive integer, and show that*

$$CPV \int_{-2}^{2} \frac{1}{x^n}\, dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \infty & \text{if } n \text{ is even} \end{cases} \quad .$$

By the way, if the integrand has several singularities on the interval (a, b), then the Cauchy principal value is computed by taking the above described symmetric limit at each singularity.

Also, if $(a, b) = (-\infty, \infty)$, then we have the "improper integral version of the Cauchy principal value" given by

$$\text{CPV} \int_{-\infty}^{\infty} f(x)\, dx = \lim_{R \to +\infty} \int_{R}^{R} f(x)\, dx \quad .$$

Again, its use must be justified.

Linearity of the Cauchy Principal Value

It should be briefly noted that, like the regular integral, the Cauchy principal value is linear. To be precise, you can easily show using the linearity of integrals and limits that, if at least two of the following exist as finite numbers,

$$\text{CPV} \int_{a}^{b} f(x)\, dx \quad , \quad \text{CPV} \int_{a}^{b} g(x)\, dx \quad \text{and} \quad \text{CPV} \int_{a}^{b} [f(x) + g(x)]\, dx \quad ,$$

then they all exist and, moreover,

$$\text{CPV} \int_{a}^{b} [\alpha f(x) + \beta g(x)]\, dx = \alpha\, \text{CPV} \int_{a}^{b} f(x)\, dx + \beta\, \text{CPV} \int_{a}^{b} g(x)\, dx \quad .$$

for any pair of numbers α and β. In particular, if the principal values exist, we have

$$\text{CPV}\int_a^b [u(x) + i v(x)]\,dx \;=\; \text{CPV}\int_a^b u(x)\,dx \;+\; i\,\text{CPV}\int_a^b v(x)\,dx \quad .$$

From this it almost immediately follows that

$$\text{Re}\left[\text{CPV}\int_a^b f(x)\,dx\right] \;=\; \text{CPV}\int_a^b \text{Re}[f(x)]\,dx$$

and

$$\text{Im}\left[\text{CPV}\int_a^b f(x)\,dx\right] \;=\; \text{CPV}\int_a^b \text{Im}[f(x)]\,dx \quad .$$

We will be using this later.

Integrating Through Singularities

Sometimes the function you are integrating has a singularity at a point on the curve over which you must integrate the function. When this happens, there are at least three things you can do to deal with this integral:

1. Use the Cauchy principal value.
2. Modify the function slightly to move the singularity off the curve by a distance of ε and then see what happens as $\varepsilon \to 0$.
3. Look very closely at your problem and decide if the fact that this integral doesn't really exist is telling you something about the physics of the problem.

If you do either the first or the second of the above, then make sure you can justify your choice by something other than "I don't like infinities." This means that you often should do the third thing in the above list to help justify the your choice.

Simple Poles and the Cauchy Principal Value

The discussion starting on page 17–6 concerning integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,dx \quad ,$$

can be easily adapted to take into account the possibility of f having simple poles on the X–axis. What we will discover is that we don't actually end up with the above integral, but with the Cauchy principal value of that integral.

So let's consider evaluating the above integral assuming that:

1. Except for a finite number of poles and/or essential singularities, f is a single-valued analytic function on the entire complex plane.
2. Only one of these singularities is on the X–axis, and that is a simple pole at x_0
3. One of the following holds:

   (a) $zf(z) \to 0$ as $|z| \to \infty$.

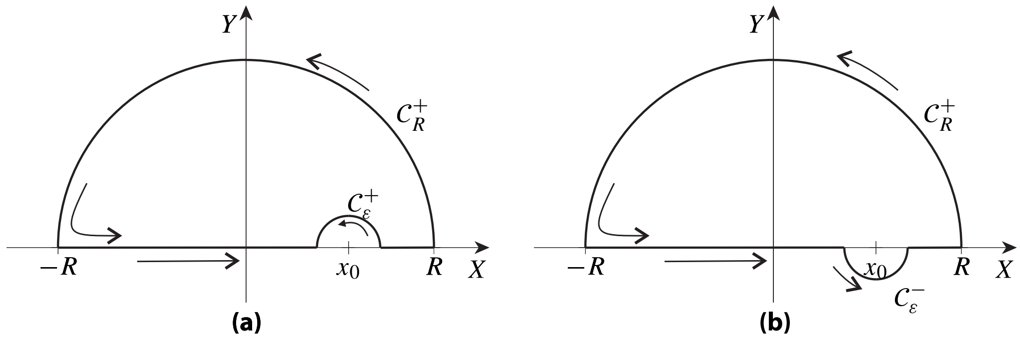

Figure 17.5: Isolating a pole on the X–axis. (Compare with figure 17.1a on page 17–6.)

(b) $f(z) = g(z)e^{i\alpha z}$ where $\alpha > 0$ and

$$g(z) \rightarrow 0 \qquad \text{as} \qquad |z| \rightarrow \infty \quad .$$

The computations here differ from those on page 17–6 only in that we initially isolate the pole at x_0 by a circle of radius ε centered at x_0 (oriented counterclockwise) with, unsurprisingly, C_ε^+ and C_ε^- denoting the upper and lower halves (see figures 17.5a and 17.5b). This also means that, instead of using the entire interval $(-R, R)$ in the closed curve for the residue theorem, we use the subintervals $(-R, x_0 - \varepsilon)$ and $(-R, x_0 - \varepsilon)$ along with either C_ε^+ or C_ε^- (possibly re-oriented). For these calculations, C_ε^+ will be used, as in figure 17.5a.

Take R large enough and ε small enough that the closed curve figure 17.5a encircles all the singularities of f in the upper half plane (UHP).

The residue theorem then tells us that

$$i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big]$$
$$= \int_{-R}^{x_0-\varepsilon} f(x)\,dx \;-\; \int_{C_\varepsilon^+} f(z)\,dz \;+\; \int_{x_0+\varepsilon}^{R} f(x)\,dx \;+\; \int_{C_R^+} f(z)\,dz \quad .$$

By the same arguments as given before, the integral over C_R^+ shrinks to 0 as $R \rightarrow \infty$. So, after letting $R \rightarrow \infty$, the last equation becomes

$$i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big]$$
$$= \int_{-\infty}^{x_0-\varepsilon} f(x)\,dx \;-\; \int_{C_\varepsilon^+} f(z)\,dz \;+\; \int_{x_0+\varepsilon}^{\infty} f(x)\,dx \quad .$$

Hence,

$$\int_{-\infty}^{x_0-\varepsilon} f(x)\,dx \;+\; \int_{x_0+\varepsilon}^{\infty} f(x)\,dx \;=\; i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big]$$
$$+ \int_{C_\varepsilon^+} f(z)\,dz \quad .$$

Letting $\varepsilon \rightarrow 0$ this becomes

$$\text{CPV}\int_{-\infty}^{\infty} f(x)\,dx \;=\; i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big]$$
$$+ \lim_{\varepsilon\rightarrow 0^+}\int_{C_\varepsilon^+} f(z)\,dz \quad . \tag{17.18}$$

On C_ε^+ we can use the Laurent series for f around x_0, along with the parametrization

$$z = z(\theta) = x_0 + \varepsilon e^{i\theta} \qquad \text{with} \quad 0 \leq \theta \leq \pi \quad .$$

Since f is assumed to have a simple pole at x_0, the Laurent series will be of the form

$$f(z(\theta)) = \sum_{k=-1}^{\infty} c_k z^k = \sum_{k=-1}^{\infty} c_k \varepsilon^k e^{ik\theta} = c_{-1}\varepsilon^{-1}e^{-i\theta} + \sum_{k=0}^{\infty} c_k \varepsilon^k e^{ik\theta}$$

Remember $c_{-1} = \text{Res}_{x_0}(f)$. So

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \int_{C_\varepsilon^+} f(z)\,dz &= \lim_{\varepsilon\to 0^+} \int_0^\pi \left[c_{-1}\varepsilon^{-1}e^{-i\theta} + \sum_{k=0}^{\infty} c_k \varepsilon^k e^{ik\theta} \right] \underbrace{i\varepsilon e^{i\theta}\,d\theta}_{dz} \\
&= \lim_{\varepsilon\to 0^+} \int_0^\pi \left[c_{-1}i + \sum_{k=0}^{\infty} c_k i \varepsilon^{k+1} e^{i(k+1)\theta} \right] d\theta \\
&= \lim_{\varepsilon\to 0^+} \left[ic_{-1} \int_0^\pi d\theta + \sum_{k=0}^{\infty} ic_k \varepsilon^{k+1} \int_0^\pi e^{i(k+1)\theta}\,d\theta \right] \\
&= \lim_{\varepsilon\to 0^+} \left[ic_{-1}\pi + \sum_{k=0}^{\infty} ic_k \varepsilon^{k+1} \int_0^\pi e^{i(k+1)\theta}\,d\theta \right] \\
&= \lim_{\varepsilon\to 0^+} \left[ic_{-1}\pi + \sum_{k=0}^{\infty} c_k \varepsilon^{k+1} \frac{1}{k+1} \left(e^{i(k+1)\pi} - 1\right) \right] \\
&= c_{-1}i\pi + \sum_{k=0}^{\infty} c_k \cdot 0^{k+1} \frac{1}{k+1} \left(e^{i(k+1)\pi} - 1\right) \\
&= i\pi\,\text{Res}_{x_0}(f) \quad .
\end{aligned}$$

Plugging this result back into equation 17.18 gives us

$$\begin{aligned}
&\text{CPV} \int_{-\infty}^{\infty} f(x)\,dx \\
&\qquad = i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big] + i\pi\,\text{Res}_{x_0}(f) \quad .
\end{aligned} \tag{17.19}$$

This last equation was derived using the curves in figure 17.5a. It turns out that you get exactly the same result using the curves in figure 17.5b. You should verify this yourself.

If there is more than one simple pole on the X–axis, then the residue of each contributes, and we have the following result:

Lemma 17.3
Let f be a function which, except for a finite number of poles and/or essential singularities, is a single-valued analytic function on the entire complex plane. Assume, further, that the singularities on the X–axis are all simple poles, and that either

1. $zf(z) \to 0$ *as* $|z| \to \infty$.

2. *or* $f(z) = g(z)e^{i\alpha z}$ *where* $\alpha > 0$ *and*

$$g(z) \ \rightarrow\ 0 \qquad \text{as} \quad |z| \rightarrow \infty \quad .$$

Then

$$\begin{aligned} \text{CPV}\int_{-\infty}^{\infty} f(x)\,dx \ &=\ i2\pi\big[\text{sum of the residues of } f \text{ in the UHP}\big] \\ &\quad + \ i\pi\big[\text{sum of the residues of } f \text{ in the } X\text{–axis}\big] \quad . \end{aligned} \tag{17.20}$$

Deriving the corresponding results when we use the residues in the lower half plane (LHP) will be left to you.

?►Exercise 17.4: *Assume the following concerning some function* f *on* $\mathbb{C}$ *:*

1. *Except for a finite number of poles and/or essential singularities,* f *is a single-valued analytic function on the entire complex plane.*
2. *All the singularities on the* X*–axis are simple poles.*

and

3. $f(z) = g(z)e^{-i\alpha z}$ *where* $\alpha > 0$ *and*

$$g(z) \ \rightarrow\ 0 \qquad \text{as} \quad |z| \rightarrow \infty \quad .$$

Then show that

$$\begin{aligned} \text{CPV}\int_{-\infty}^{\infty} f(x)\,dx \ &=\ -i2\pi\big[\text{sum of the residues of } f \text{ in the LHP}\big] \\ &\quad - \ i\pi\big[\text{sum of the residues of } f \text{ in the } X\text{–axis}\big] \quad . \end{aligned} \tag{17.21}$$

What if the singularities on the X–axis are not simple poles — say, poles of higher order or essential singularities? The computations done above can still be attempted, but, in computing

$$\lim_{\varepsilon \rightarrow 0^+} \int_{C_\varepsilon^+} f(z)\,dz$$

there will generally be terms involving $1/\varepsilon$ that blow up. So, in general, we cannot deal with singularities worse than simple poles on the X–axis, at least not using the Cauchy principal value.

Let's do an example where the Cauchy principal value is of value.

!►Example 17.7: *Consider*

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx \quad .$$

The only possible point where we may have a pole in the integrand is at $x = z = 0$ *. In fact, though, you can easily show that*

$$\lim_{z \rightarrow 0} \frac{\sin(z)}{z} \ =\ 1 \quad .$$

So, in fact, $\sin(z)/z$ *is analytic at* $z = 0$*, the integrand in the above integral is continuous at* $x = 0$*, and we have*

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx \;=\; CPV\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx \quad .$$

This becomes something other than a stupid, obvious observation after also noting that, for $x \in \mathbb{R}$*,*

$$\frac{\sin(x)}{x} \;=\; \operatorname{Im}\left[\frac{e^{ix}}{x}\right]$$

and that

$$f(z) \;=\; \frac{e^{iz}}{z}$$

satisfies the conditions for f *in lemma 17.3. Applying that lemma (and noting that the only residue of* f *is at* $z = 0$ *), we have*

$$CPV\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx \;=\; i\pi\operatorname{Res}_0\left(\frac{e^{ix}}{x}\right) \;=\; i\pi\operatorname{Res}_0\left(e^{i\cdot 0}\right) \;=\; i\pi\cdot 1 \;=\; i\pi \quad .$$

So,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx \;&=\; CPV\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx\\ &=\; CPV\int_{-\infty}^{\infty} \operatorname{Im}\left[\frac{e^{ix}}{x}\right]dx\\ &=\; \operatorname{Im}\left[CPV\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx\right] \;=\; \operatorname{Im}[i\pi] \;=\; \pi \quad .\end{aligned}$$

Moving Singularities

`To be written, someday.`

Basically, this is the idea of replacing

$$\int_a^b \frac{g(x)}{x - x_0}\,dx$$

with either

$$\lim_{y_0\to 0^+}\int_a^b \frac{g(x)}{x - (x_0 + y_0)}\,dx \qquad \text{or} \qquad \lim_{y_0\to 0^+}\int_a^b \frac{g(x)}{x - (x_0 - y_0)}\,dx \quad .$$

Again, there must be some justification for this. Be warned that, in general,

$$\begin{aligned}\text{CPV}\int_a^b \frac{g(x)}{x - x_0}\,dx \;&\neq\; \lim_{y_0\to 0^+}\int_a^b \frac{g(x)}{x - (x_0 + y_0)}\,dx\\ &\neq\; \lim_{y_0\to 0^+}\int_a^b \frac{g(x)}{x - (x_0 - y_0)}\,dx \;\neq\; \text{CPV}\int_a^b \frac{g(x)}{x - x_0}\,dx \quad .\end{aligned}$$