16

Complex Analysis III: Laurent Series and Singularities

A "Laurent series" for a function is a generalization of the Taylor series, and like a Taylor series, it is essentially a power series. With a Laurent series, however, the powers can be negative. A major advantage of the Laurent series over the Taylor series is that Laurent series can be expanded around singular points for a given function, and can then be used to analyze the function in the neighborhoods of its singularities. This analysis will be needed when we develop "residue theory" for computing all sorts of weird integrals that can arise in applications.

By the way, a *singular point* for a function f is a point on the complex plane where f might not be analytic.

16.1 Laurent Series

Derivation and Main Theorem

Let f be some single-valued function analytic in some region. To derive a Laurent series for f, we need to restrict our attention to an open "annular" subregion $\mathcal{A}$ on which f is analytic. Now, when we say $\mathcal{A}$ is an "annular region", we mean $\mathcal{A}$ is bounded by two concentric circles C_{inner} and C_{outer} about some point z_0, with radii r_{inner} and R_{outer}, respectively, satisfying

$$0 \leq r_{\text{inner}} < R_{\text{outer}} \leq \infty \quad .$$

If $r_{\text{inner}} = 0$, then C_{inner} is simply the point z_0, and $\mathcal{A}$ is a disk without its center z_0. This case will be important for analyzing the function near z_0. If $R_{\text{outer}} = \infty$, then there really isn't an outer circle, and $\mathcal{A}$ is the entire complex plane outside of the inner circle. This case will be important for analyzing the behavior of the function at z as $z \to \infty$. In practice, the radii of the inner and outer circles will typically depend on the distances between z_0 and the various singular points for f. Often, there will be many possible choices for the annular region $\mathcal{A}$, and, thus, many possible Laurent series for a function about a given point, with each valid on a different annular region.

Now let z be any point in $\mathcal{A}$, choose any two positive values r and R such that

$$r_{\text{inner}} < r < |z - z_0| < R < R_{\text{outer}} \quad ,$$

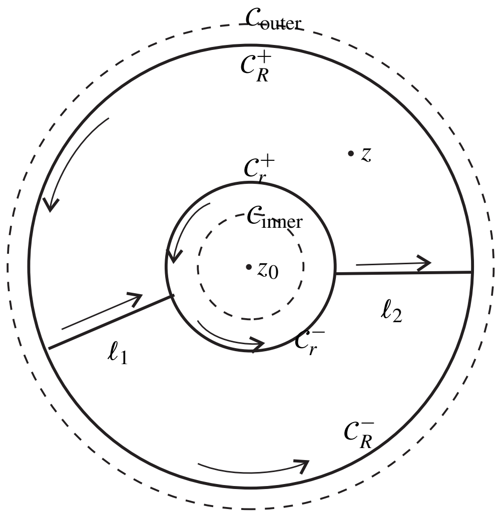

Figure 16.1: Figure for deriving the Laurent series. In this figure, $C_R = C_R^+ + C_R^-$ and the "small" circle around z_0 is $C_\varepsilon = C_\varepsilon^+ + C_\varepsilon^-$.

and let C_r and C_R be the two counterclockwise oriented circles centered at z_0 with radii r and R , respectively. Add two smooth oriented curves ℓ_1 and ℓ_2 between C_r and C_R as indicated in figure 16.1, with neither touching z . Using the endpoints of ℓ_1 and ℓ_2 , break C_r and C_R into two pieces each, C_r^+ and C_r^- , and C_R^+ and C_R^- , respectively, as also indicated in figure 16.1. Finally, keeping track of the orientations of the subcurves, let Γ_1 and Γ_2 be the closed curves given by

$$\Gamma^+ = C_R^+ + \ell_1 - C_r^+ + \ell_2 \qquad \text{and} \qquad \Gamma^- = C_R^- - \ell_2 - C_r^- - \ell_1 \quad ,$$

and consider the integrals

$$\oint_{\Gamma^+} \frac{f(\zeta)}{\zeta - z}\,d\zeta \qquad \text{and} \qquad \oint_{\Gamma^-} \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad .$$

By the Cauchy integral theorem, we know that the integral over Γ^- (the curve not encircling z) is 0 , while the basic Cauchy integral formula tells us that the other integral is $i2\pi f(z)$. Thus,

$$\begin{aligned}
i2\pi f(z) + 0 &= \oint_{\Gamma.^+} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \oint_{\Gamma^-} \frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \int_{C_R^+} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \int_{\ell_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int_{C_r^+} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \int_{\ell_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&\quad + \int_{C_R^-} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int_{\ell_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int_{C_r^-} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int_{\ell_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \oint_{C_R^+ + C_R^-} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \oint_{C_r^+ + C_r^-} \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad .
\end{aligned}$$

That is,

$$f(z) = \frac{1}{i2\pi} \oint_{C_R} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{i2\pi} \oint_{C_r} \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad . \tag{16.1}$$

Consider the first integral on the right side of equation (16.1). More precisely, let $\zeta \in C_R$. Then

$$|z - z_0| < R = |\zeta - z_0| \quad ,$$

which means that

$$\left|\frac{z - z_0}{\zeta - z_0}\right| < 1 \quad ,$$

and the formula for the sum of the geometric series can be applied as follows:

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0 - (z - z_0)} \\ &= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\ &= \frac{1}{\zeta - z_0} \cdot \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}} \quad . \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{1}{i2\pi} \oint_{C_R} \frac{f(\zeta)}{\zeta - z} \, d\zeta &= \frac{1}{i2\pi} \oint_{C_R} \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}} f(\zeta) \, d\zeta \\ &= \frac{1}{i2\pi} \sum_{k=0}^{\infty} \oint_{C_R} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}} f(\zeta) \, d\zeta \\ &= \sum_{k=0}^{\infty} (z - z_0)^k \cdot \frac{1}{i2\pi} \oint_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \quad . \end{aligned}$$

(I know what you are wondering; namely: *Do we have the necessary uniform convergence of the series in the integral to justify the above interchanging of the integral and the summation?* The answer is *yes*. This can be verified using the continuity of f on the region and straightforward extensions of theorem 12.17 on page 12–32.)

Note the following about the integrals in the last line:

1. They do not depend on z.
2. By a corollary of the Cauchy integral theorem (namely, theorem 15.5 on page 15–7) we can replace C_R with any simple, counterclockwise oriented loop in $\mathcal{A}$ about z_0.

This means the last sequence of equations reduces to

$$\frac{1}{i2\pi} \oint_{C_R} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{16.2a}$$

where

$$a_k = \frac{1}{i2\pi} \oint_{C} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \tag{16.2b}$$

and C is any simple, counterclockwise oriented loop in $\mathcal{A}$ about z_0.

An analogous derivation can be done for the other integral in equation (16.1). You do it.

?►Exercise 16.1: *Show that, if $\zeta \in C_r$, then*

$$\frac{1}{z-\zeta} = \sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z-z_0)^{n+1}} \quad ,$$

and, using that, show that

$$\frac{-1}{i2\pi} \oint_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \sum_{n=0}^{\infty} b_n \frac{1}{(z-z_0)^{n+1}}$$

where

$$b_n = \frac{1}{i2\pi} \oint_C (\zeta - z_0)^n f(\zeta) \, d\zeta$$

and C is any simple, counterclockwise oriented loop in $\mathcal{A}$ about z_0.

To combine the results from the exercise with the results given in equation set (16.2), reindex the last summation in the exercise using $k = -(n+1)$ and let $a_k = b_{k-1}$. The results from the exercise then become

$$-\frac{1}{i2\pi} \oint_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k \tag{16.3a}$$

where

$$a_k = \frac{1}{i2\pi} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \tag{16.3b}$$

and C is any simple, counterclockwise oriented loop in $\mathcal{A}$ about z_0

Combining equation sets (16.2) and (16.3) with equation (16.1) then gives us the following theorem.

Theorem 16.1 (Laurent Series)

Assume f is a single-valued function analytic in an open annular subregion $\mathcal{A}$ bounded by two concentric circles C_{inner} and C_{outer} about some point z_0, with radii r_{inner} and R_{outer}, respectively, satisfying

$$0 \le r_{\text{inner}} < R_{\text{outer}} \le \infty \quad .$$

Then, for every point z in $\mathcal{A}$,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \tag{16.4a}$$

where

$$a_k = \frac{1}{i2\pi} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \tag{16.4b}$$

and C is any simple, counterclockwise oriented loop in $\mathcal{A}$ about z_0.

The series in equation (16.4a) (with coefficients given by formula (16.4b)) is called the *Laurent series* for f about z_0 (in the annulus $\mathcal{A}$). It is worth noting that the summation is shorthand for

$$f(z) = \cdots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)^1}$$
$$+ a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots \quad .$$

The relation with the Taylor series is discussed in the next exercise.

?►Exercise 16.2: *Let z_0 be a point in the complex plane, and assume f is a single-valued analytic function on a disk of radius R_{outer} centered at z_0. (In particular, then, f is analytic at z_0.) Using the Cauchy integral theorem and the Cauchy integral formulas, verify that the Laurent series formula for f on this disk reduces to the Taylor series formula*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \quad .$$

It immediately follows from this exercise that saying "f is analytic as a complex function at a point z_0" is equivalent to saying that "f can be represented by a power series expansion about z_0". In other words, for complex functions, the definition of analytic given on page 14–10 (based on differentiability) is equivalent to the one given on page 12–33 in chapter 12 when we were discussing power series representations for functions.

16.2 Computing Laurent Series

In practice, we are rarely interested in explicitly computing all the coefficients of a Laurent series (though we will later have good reason for finding a_{-1}, the coefficient for the $(z-z_0)^{-1}$ term). Even when we do want to explicitly find a Laurent series for a particular function f, the integral formula (16.4b) is often too difficult to calculate to be of much use. What is often more helpful is the clever use of the geometric series formula,

$$\frac{1}{1-A} = \sum_{k=0}^{\infty} A^k \qquad \text{for} \quad |A| < 1 \quad ,$$

and/or clever use of a known Taylor or Laurent series.

Using Geometric Series

To illustrate both an honest-to-goodness Laurent series and how to use the geometric series, let us attempt to find all the Laurent series for

$$f(z) = \frac{1}{(z-1)(z-4)} \qquad \text{about} \qquad z_0 = 1 \quad .$$

The singular points for this function are $z = 1$ and $z = 4$, so our annular regions about $z_0 = 1$ must not contain these points. And since these regions are centered at $z_0 = 1$, this means that the bounding circle of these regions is the circle centered at $z_0 = 1$ of radius $R = |4-1| = 3$ (see figure 16.2). So we will have two Laurent series expansions of the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-1)^k \quad ,$$

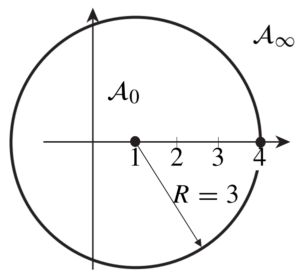

Figure 16.2: The two regions $\mathcal{A}_0$ (inside the circle) and $\mathcal{A}_\infty$ (outside the circle) for the Laurent series about $z_0 = 1$ of

$$f(z) = \frac{1}{(z-1)(z-4)} \quad .$$

one for the function on the region

$$\mathcal{A}_0 \qquad \text{where} \quad z \in \mathcal{A}_0 \quad \Longleftrightarrow \quad 0 < |z-1| < 3 \quad ,$$

and one for the function on the region

$$\mathcal{A}_\infty \qquad \text{where} \quad z \in \mathcal{A}_\infty \quad \Longleftrightarrow \quad 3 < |z-1| \quad .$$

We will find these two series in the following two examples.

!►*Example 16.1 (Laurent series in $\mathcal{A}_0$):* *Consider finding the Laurent series about $z_0 = 1$ for*

$$f(z) = \frac{1}{(z-1)(z-4)}$$

when

$$0 < |z-1| < 3 \quad .$$

Since the goal is to express $f(z)$ in terms of $z-1$ and

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{z-4} = (z-1)^{-1} \frac{1}{z-4} \quad ,$$

much of the work will be done once we've expressed $(z-4)^{-1}$ in terms of $z-1$. By the assumption on z, we have

$$\left| \frac{z-1}{3} \right| < 1 \quad .$$

Being mildly clever, then, we get

$$\begin{aligned}
\frac{1}{z-4} &= \frac{1}{z-1-3} \\
&= \frac{1}{3} \cdot \frac{1}{\frac{z-1}{3} - 1} \\
&= \frac{-1}{3} \cdot \frac{1}{1 - \frac{z-1}{3}} = \frac{-1}{3} \sum_{k=0}^{\infty} \left[\frac{z-1}{3} \right]^k \quad .
\end{aligned}$$

Thus, if $0 < |z-1| < 3$,

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{z-4}$$

$$= (z-1)^{-1} \cdot \frac{-1}{3} \sum_{k=0}^{\infty} \left[\frac{z-1}{3}\right]^k = \sum_{k=0}^{\infty} \frac{-1}{3^{k+1}} (z-1)^{k-1} \quad .$$

With a slight change of index ($n = k - 1$) we have the Laurent series representation

$$f(z) = \sum_{n=-1}^{\infty} \frac{-1}{3^{n+2}} (z-1)^n = \frac{-1}{3(z-1)} - \frac{1}{9} - \frac{1}{27}(z-1) - \frac{1}{81}(z-1)^2 - \cdots$$

valid when $0 < |z-1| < 3$.

!►Example 16.2 (Laurent series in $\mathcal{A}_\infty$): *Now consider finding the Laurent series about* $z_0 = 1$ *for*

$$f(z) = \frac{1}{(z-1)(z-4)}$$

when

$$3 < |z-1| \quad .$$

This time,

$$\left|\frac{3}{z-1}\right| < 1 \quad .$$

So,

$$\frac{1}{z-4} = \frac{1}{z-1-3}$$

$$= \frac{1}{z-1} \cdot \frac{1}{1 - \dfrac{3}{z-1}} = \frac{1}{z-1} \sum_{k=0}^{\infty} \left[\frac{3}{z-1}\right]^k \quad ,$$

and

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{z-4}$$

$$= \frac{1}{z-1} \cdot \frac{1}{z-1} \sum_{k=0}^{\infty} \left[\frac{3}{z-1}\right]^k = \sum_{k=0}^{\infty} \frac{3^k}{(z-1)^{k+2}} \quad .$$

With a slight change of index ($n = -(k+2)$) we have the Laurent series representation

$$f(z) = \sum_{n=-2}^{-\infty} \frac{1}{3^{n+2}} (z-1)^n$$

$$= \frac{1}{(z-1)^2} + \frac{3}{(z-1)^3} + \frac{9}{(z-1)^4} + \frac{27}{(z-1)^5} + \cdots$$

valid when $3 < |z-1|$.

The derivations of the above expansions were simplified by the fact that the expansions were about $z_0 = 1$ and one factor of the function of interest was already of the form $(z-1)^{-1}$. If that had not been the case, say $z_0 = i$, then we could have first expanded $f(z)$ by partial fractions,

$$f(z) = \frac{1}{(z-1)(z-4)} = \frac{-1/3}{z-1} + \frac{1/3}{z-4} \quad ,$$

and then expanded each term using a geometric series as illustrated in the examples.

Using Known Taylor or Laurent Series

Of course, you know of many power series other than the geometric series, and you are free to use your knowledge of these series to find Laurent series.

!►Example 16.3: *Find all Laurent series for*

$$f(z) = z^2 \exp\left(\frac{1}{z}\right)$$

about $z_0 = 0$ *.*

Since we know

$$\exp(\zeta) = e^\zeta = \sum_{k=0}^{\infty} \frac{1}{k!}\zeta^k \qquad \text{for all} \quad \zeta \in \mathbb{C} \quad ,$$

we immediately have

$$z^2 \exp\left(\frac{1}{z}\right) = z^2 \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{z}\right)^k = \sum_{k=0}^{\infty} \frac{1}{k!} z^{2-k}$$

whenever $^1/_z$ *is a finite complex number. This is the the Laurent series expansion for our function. Reindexing it and expanding it, we have*

$$\begin{aligned} z^2 \exp\left(\frac{1}{z}\right) &= \sum_{n=2}^{-\infty} \frac{1}{(2-n)!} z^n \\ &= z^2 + z + \frac{1}{2} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots \quad , \end{aligned}$$

and this expansion is valid for $|z| > 0$ *.*

?►Exercise 16.3: *Find the Laurent expansions about* $z_0 = 0$ *for the following using known power series (Taylor series) expansions. Also state the region in which the Laurent series expansion is valid.*

a: $\sin\left(\frac{1}{z}\right)$ **b:** $z \cos\left(\frac{3}{z}\right)$ **c:** $z^2 \left[1 - \exp\left(\frac{1}{z}\right)\right]$

d: $z^{-2}\left[1 - \exp\left(z^2\right)\right]$ **e:** $\ln\left(\frac{z-2}{z}\right)$

Products of Laurent Series

Suppose our function of interest f is the product of functions whose Laurent series around some point z_0 are known (or are reasonably easy to compute). That is, suppose we know

$$f(z) = g(z)h(z)$$

where, for each z in an annular region around z_0 ,

$$g(z) = \sum_{m=-\infty}^{\infty} g_m (z - z_0)^m \qquad \text{and} \qquad h(z) = \sum_{n=-\infty}^{\infty} h_n (z - z_0)^n \quad .$$

Of course, if these functions are simple enough, then it is relatively easy to find the coefficients in the corresponding Laurent series expansion for f ,

$$f(z) = \sum_{k=-\infty}^{\infty} f_k (z - z_0)^k \quad ,$$

in the region. Alternatively, we can multiply the series for g and h to get the series for f . Here is a relatively simple approach to computing this product of infinite series, using the basic formula for the coefficients in a Laurent series and one integral we should know by heart at this point. Remember, the basic formula for the Laurent series coefficients is

$$f_k = \frac{1}{i2\pi} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \quad .$$

where C is some simple loop about z_0 oriented counterclockwise. Replacing f with gh and then replacing g and h with their Laurent series, we get

$$\begin{aligned}
f_k &= \frac{1}{i2\pi} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \\
&= \frac{1}{i2\pi} \oint_C g(\zeta) h(\zeta) (\zeta - z_0)^{-k-1} \, d\zeta \\
&= \frac{1}{i2\pi} \oint_C \sum_{m=-\infty}^{\infty} g_m (\zeta - z_0)^m \sum_{n=-\infty}^{\infty} h_n (\zeta - z_0)^n (\zeta - z_0)^{-k-1} \, d\zeta \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_m h_n \frac{1}{i2\pi} \oint_C (\zeta - z_0)^{m+n-k-1} \, d\zeta \quad .
\end{aligned}$$

The last integral is one we've computed before. It is $i2\pi$ if $m + n - k - 1 = -1$ (i.e., n=k-m), and 0 otherwise. Thus,

$$f_k = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_m h_n \frac{1}{i2\pi} \left\{ \begin{matrix} i2\pi & \text{if} \quad n = k - m \\ 0 & \text{otherwise} \end{matrix} \right\} = \sum_{m=-\infty}^{\infty} g_m h_{k-m} \quad . \tag{16.5}$$

In particular,

$$f_{-1} = \sum_{m=-\infty}^{\infty} g_m h_{1-m} \quad . \tag{16.6}$$

(We mention this particular coefficient because, as we will see in the next chapter, it is often the only coefficient of real interest.)

In practice, rather than blindly applying the above two formulas, you may just want to "rederive" the above from basics for the problem at hand.

!►Example 16.4: *Suppose we want to find the Laurent series in the region just around* $z_0 = 0$,

$$f(z) = \sum_{k=-\infty}^{\infty} f_k z^k \quad ,$$

for the function

$$f(z) = \frac{\sin(1/z)}{z^2 + a^2} \qquad \textit{with} \quad 0 < |z| < a \quad .$$

Here,

$$f = gh \qquad \text{where} \quad g(z) = \frac{1}{z^2 + a^2} \quad \text{and} \quad h(z) = \sin\left(\frac{1}{z}\right) \quad .$$

Using the geometric series (and remembering that $|z| < a$),

$$g(z) = \frac{1}{a^2} \cdot \frac{1}{1 - \left[-\frac{z^2}{a^2}\right]} = \frac{1}{a^2} \sum_{m=0}^{\infty} \left[-\frac{z^2}{a^2}\right]^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{2m+2}} z^{2m} \quad .$$

And using the Taylor series for the sine function, we get the corresponding Laurent series expansion

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{2n+1}} \quad .$$

Applying "basics":

$$\begin{aligned}
f_k &= \frac{1}{i2\pi} \oint_C \frac{f(\zeta)}{(\zeta - 0)^{k+1}}\, d\zeta \\
&= \frac{1}{i2\pi} \oint_C g(\zeta) h(\zeta) \zeta^{-k-1}\, d\zeta \\
&= \frac{1}{i2\pi} \oint_C \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{2m+2}} \zeta^{2m} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{\zeta^{2n+1}} \zeta^{-k-1}\, d\zeta \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{a^{2m+2}(2n+1)!} \cdot \frac{1}{i2\pi} \oint_C \zeta^{2m-2n-1-k-1}\, d\zeta \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{a^{2m+2}(2n+1)!} \cdot \frac{1}{i2\pi} \begin{Bmatrix} i2\pi & \text{if} \quad 2m - 2n - 1 - k - 1 = -1 \\ 0 & \text{otherwise} \end{Bmatrix} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{a^{2m+2}(2m-k)!} \begin{Bmatrix} 1 & \text{if} \quad 2n = 2m - 1 - k \\ 0 & \text{otherwise} \end{Bmatrix} \quad .
\end{aligned}$$

In particular,

$$\begin{aligned}
f_{-1} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{a^{2m-2}(2m-k)!} \begin{Bmatrix} 1 & \text{if} \quad 2n = 2m - 1 - (-1) \\ 0 & \text{otherwise} \end{Bmatrix} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^{m+m}}{a^{2m+2}(2m+1)!} \\
&= \frac{1}{a} \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \left(\frac{1}{a}\right)^{2m+1} \quad .
\end{aligned}$$

After recalling that

$$\sinh(\zeta) = \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} (\zeta)^{2m+1} \quad ,$$

we see that the above reduces to

$$f_{-1} = \frac{1}{a} \sinh\left(\frac{1}{a}\right) \quad .$$

16.3 Singularities and the Laurent Series

Singularities at Finite Points

Let z_0 be a point in the complex plane and f a single-valued function analytic in some region about z_0, but, possibly, not at z_0. To be precise, we assume there is a $R > 0$ such that f is analytic on

$$\mathcal{R} = \{z : 0 < |z - z_0| < R\} \quad .$$

In this region, we can expand f in its Laurent series,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{16.7}$$

For our convenience, let's break this into the natural pieces,

$$\begin{aligned} f(z) &= \sum_{k=-\infty}^{-1} a_k (z - z_0)^k + \sum_{k=0}^{\infty} a_k (z - z_0)^k \\ &= \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{k=0}^{\infty} a_k (z - z_0)^k \\ &= f_S(z) + f_A(z) \quad , \end{aligned}$$

where, for even more convenience, we've let

$$f_S(z) = \text{the 'singular part' of } f(z) \text{ in this region} = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}$$

and

$$f_A(z) = \text{the 'analytic part' of } f(z) \text{ in this region} = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad .$$

Clearly, f_A will be analytic at every point z with $|z - z_0| < R$, including the point $z = z_0$. Moreover, by what we know about power series, we know the series for $f_A(z)$ will converge uniformly on any disk centered about z_0 of radius less than R.

The convergence properties of $f_S(z)$ can easily be deduced by considering the function

$$h(\zeta) = f_S\left(z_0 + \frac{1}{\zeta}\right) \quad .$$

I'll leave the details to you:

?►Exercise 16.4: *Let f and h be as above, and assume that f_S is not trivial (i.e., at least one of the a_{-n}'s is nonzero).*

a: *Find the power series for $h(\zeta)$ from the series for $f_S(z)$, and verify that the radius of convergence for the power series for $h(\zeta)$ is infinite. (Use facts learned a few weeks ago about power series in general, and the fact that the series for $f_S(z)$ converges at least on $0 < |z - z_0| < R$.)*

b: *From what you just derived about h, show that*

i: *The series for* $f_S(z)$ *converges at* every *complex point* z *except* $z = z_0$ *.*

ii: f_S *is analytic everywhere in the complex plane except at* z_0

iii: *If* ε *is any positive real number, then the series for* $f_S(z)$ *converges uniformly on the region* $\varepsilon \leq |z - z_0|$ *.*

Now, as we already know, f will be analytic at z_0 if and only if

$$f(z) \;=\; \sum_{k=0}^{\infty} a_k (z - z_0)^k \;=\; f_A(z)$$

on $\mathcal{R}$. This, of course, means that f_S is trivial. If f_S is not trivial, we say f has a singularity at z_0 and classify that singularity as being either a 'pole' or an 'essential' singularity according to the following criteria:

1. We say f has a *pole* at z_0 of *order* M (for some positive integer M) if and only if the following equivalent conditions hold:

- In formula (16.7), $a_{-M} \neq 0$ but $a_k = 0$ whenever $k < -M$.
- On $\mathcal{R}$,

$$\begin{aligned} f(z) \;&=\; \sum_{k=-M}^{\infty} a_k (z - z_0)^k \\ &=\; \frac{a_{-M}}{(z - z_0)^M} \;+\; \frac{a_{-M+1}}{(z - z_0)^{M-1}} \;+\; \cdots \;+\; \frac{a_{-1}}{z - z_0} \;+\; \sum_{k=0}^{\infty} a_k (z - z_0)^k \end{aligned}$$

  with $a_{-M} \neq 0$.
- $a_{-M} \neq 0$ and

$$f_S(z) \;=\; \frac{a_{-M}}{(z - z_0)^M} \;+\; \frac{a_{-M+1}}{(z - z_0)^{M-1}} \;+\; \cdots \;+\; \frac{a_{-1}}{z - z_0} \quad .$$

2. We say f has a *simple pole* at z_0 if and only if it has a pole of order 1 at z_0 .

3. We say f has an *essential singularity* (or *pole of infinite order*) at z_0 if and only if the following equivalent conditions hold:

- In formula (16.7), for any integer M , there is a $k \leq -M$ for which $a_k \neq 0$.
- On $\mathcal{R}$,

$$\begin{aligned} f(z) \;&=\; \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \\ &=\; \cdots \;+\; \frac{a_{-M+1}}{(z - z_0)^{M-1}} \;+\; \cdots \;+\; \frac{a_{-1}}{z - z_0} \;+\; \sum_{k=0}^{\infty} a_k (z - z_0)^k \end{aligned}$$

  and the series for the singular part has infinitely many nonzero terms.
- There are infinitely many nonzero terms in

$$f_S(z) \;=\; \cdots \;+\; \frac{a_{-M}}{(z - z_0)^M} \;+\; \frac{a_{-M+1}}{(z - z_0)^{M-1}} \;+\; \cdots \;+\; \frac{a_{-1}}{z - z_0} \;=\; \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} \quad .$$

Typically, a function behaves "very badly" near any essential singularity. In particular, "it can be shown" that, if f has an essential singularity at z_0, then we can pick any fixed value A and find a corresponding sequence of points $\{\zeta_1, \zeta_2, \zeta_3, \ldots\}$ such that

$$\zeta_k \to z_0 \quad \text{and} \quad f(\zeta_k) \to A \qquad \text{as } k \to \infty \quad .$$

Because of this,

$$\text{we do NOT have } |f(z)| \to \infty \qquad \text{as } k \to \infty$$

when f has an essential singularity at z_0. This is illustrated in the following exercise.

?►*Exercise 16.5:* *Let*

$$f(z) = \exp\left(\frac{1}{z}\right) = e^{1/z} \quad .$$

a: *Verify that f has an essential singularity at $z = 0$ and find its Laurent series about 0. (Hint: see example 16.3 on page 16–8.)*

b: *Pick any real or complex value A other than 0 (go ahead, really pick one this time) and let*

$$\zeta_k = \frac{1}{\ln(A) + i2\pi k} \qquad \textit{for} \quad k = 1, 2, 3, \ldots \quad .$$

i: *Show that $f(\zeta_k) = A$ for each positive integer k.*

ii: *Show that $\zeta_k \to 0$ as $k \to \infty$.*

iii: *By what you've just shown, you have*

$$\lim_{k\to\infty} f(\zeta_k) = \lim_{k\to\infty} A = A \qquad \textit{and} \qquad \lim_{k\to\infty} |f(\zeta_k)| = \lim_{k\to\infty} |A| = |A| \quad .$$

Why does this NOT mean that

$$\lim_{\zeta\to 0} f(\zeta) = A \qquad \textit{and} \qquad \lim_{\zeta\to 0} |f(\zeta_k)| = |A|$$

even though $\zeta_k \to 0$ as $k \to \infty$?

Behavior of a function near a pole is not so terrible. To see this, assume f has a pole at z_0 of finite order M, and let g be the function

$$g(z) = (z - z_0)^M f(z) \quad .$$

About z_0, f has the Laurent expansion

$$f(z) = \sum_{k=-M}^{\infty} a_k (z - z_0)^k$$

with the first coefficient, a_{-M} not being zero. For g we have

$$\begin{aligned} g(z) &= (z - z_0)^M f(z) \\ &= (z - z_0)^M \sum_{k=-M}^{\infty} a_k (z - z_0)^k \\ &= \sum_{k=-M}^{\infty} a_k (z - z_0)^{M+k} = \sum_{n=0}^{\infty} a_{n-M} (z - z_0)^n \quad . \end{aligned}$$

That is,

$$g(z) \;=\; \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

with

$$c_n \;=\; a_{n-M} \qquad \text{(and, hence, } a_k \;=\; c_{k+M}\text{)} \quad .$$

From this, all the following follows:

1. g is analytic at z_0 with $g(z_0) \;=\; c_0 \;=\; a_{-M} \;\neq\; 0$.

2. $f(z)$ can be written as
$$f(z) \;=\; \frac{g(z)}{(z - z_0)^M}$$
for some function g which is analytic and nonzero at z_0 .

3. If $z \approx z_0$, then
$$f(z) \;\approx\; \frac{A}{(z - z_0)^M}$$
for some nonzero value A ($A = g(z_0) = a_{-M}$). (But integrals of f may still depend more on higher-order terms.)

4. For each integer $k \geq -M$,
$$\begin{aligned} a_k \;=\; c_{M+k} \;&=\; \text{the } (M + K)^{\text{th}} \text{ term in the Taylor series formula for } g \text{ about } z_0 \\ &=\; \frac{g^{(M+k)}(z_0)}{(M + k)!} \\ &=\; \frac{1}{(M + k)!} \frac{d^{M+k}}{dz^{M+k}} \big[(z - z_0)^M f(z) \big] \Big|_{z=z_0} \quad . \end{aligned}$$
In particular,
$$a_{-1} \;=\; \frac{1}{(M - 1)!} \frac{d^{M-1}}{dz^{M-1}} \big[(z - z_0)^M f(z) \big] \Big|_{z=z_0} \quad , \tag{16.8}$$
and if the pole is simple (i.e., $M = 1$, then
$$a_{-1} \;=\; (z - z_0) f(z)\big|_{z=z_0} \quad . \tag{16.9}$$
We will find uses for these formulas soon. (Note: The evaluation of the above formulas at z_0 may require computing the limit of the formulas as z_0 , possibly using L'Hôpital's rule.)

Identifying Singularities in Practice

In practice, it is usually easy to identify where a function has singularities. This is because singularities typically arise due to a division by zero, and it is usually easy to spot where a division by zero occurs. (But if, on closer inspection, you really have something like " $f(z_0) = {}^0\!/_0$ " then try to compute

$$\lim_{z \to z_0} f(z) \quad ,$$

possibly using L'Hôpital's rule. If you get a finite number, the function is actually analytic at z_0.)

To determine whether a singularity is a finite or infinite order pole, you can simply use the methods discussed earlier (using known Taylor series or the geometric series) to first find the Laurent series for the function f about each singularity, and examine the series found to see how many terms are in the singular part, $f_S(z)$.

!► *Example 16.5:* *Consider*

$$f(z) = \exp\left(\frac{1}{z}\right) \quad .$$

Clearly, the only 'trouble spot' is $z = 0$, and, in exercise 16.5 on page 16–13 you saw that the corresponding Laurent series,

$$\exp\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{z}\right)^k = 1 + \frac{1}{z} + \frac{1}{2} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + \cdots \quad ,$$

has infinitely many nonzero terms in the singular part. Hence, $\exp(1/z)$ has an essential singularity about 0.

Alternatively, you can use the observations noted above regarding the behavior of a function f near a singularity. The simplest case is where f can clearly be written as

$$f(z) = \frac{g(z)}{(z - z_0)^M}$$

for some positive integer M and a function g that is analytic and nonzero at z_0. If this is the case, then f has a pole of order M. (If, however, $g(z_0) = 0$, then either f has a pole of lower order, or is analytic at z_0.)

If it is not clear that $f(z)$ can be written as just described, try computing

$$\lim_{z \to z_0} (z - z_0)^N f(z)$$

for a "suitably chosen" positive integer N (use your judgement). If f does have a pole of order M at z_0, then $f(z)$ can be written as

$$f(z) = \frac{g(z)}{(z - z_0)^M}$$

for some function g that is analytic and nonzero at z_0, and, thus,

$$\begin{aligned}
\lim_{z \to z_0} \left|(z - z_0)^N f(z)\right| &= \lim_{z \to z_0} \left|(z - z_0)^N \frac{g(z)}{(z - z_0)^M}\right| \\
&= \lim_{z \to z_0} |z - z_0|^{N-M} |g(z)| = \begin{cases} 0 & \text{if} \quad N > M \\ g(z_0) & \text{if} \quad N = M \\ \infty & \text{if} \quad N < M \end{cases} \quad .
\end{aligned}$$

From this and what we know about the behavior of functions near essential singularities, we then get the following:

1. If
$$\lim_{z\to z_0} (z - z_0)^N f(z) = 0 \quad ,$$
then f either has a pole of order less than N at z_0 or is analytic at z_0.

2. If
$$\lim_{z\to z_0} (z - z_0)^N f(z)$$
is a finite number other than 0, then f has a pole of order N at z_0.

3. If
$$\lim_{z\to z_0} (z - z_0)^N f(z) = \infty \quad ,$$
(more precisely, if
$$\lim_{z\to z_0} \left|(z - z_0)^N f(z)\right| = +\infty \quad ,)$$
then f has a pole of order greater than N at z_0.

4. Finally, if
$$\lim_{z\to z_0} (z - z_0)^N f(z)$$
is not well defined, then f has an essential singularity at z_0.

Zeroes at Finite Points

A discussion somewhat analogous to that for poles can be had for "zeroes" of functions. Remember, given a function f and a point z_0,
$$z_0 \text{ is a } \textit{zero} \text{ of } f \quad \Longleftrightarrow \quad f(z_0) = 0 \quad .$$
(And, perhaps once again, we'll note how stupid this standard terminology is since, in fact, most zeroes are nonzero.)

So assume z_0, a point in $\mathbb{C}$, is a zero for some single-valued function f analytic in some region "right around" z_0. Since $f(z_0) = 0$, f clearly cannot have a singularity at z_0. Hence f is analytic at z_0, and its Laurent series "right around" z_0 will only contain an analytic part. That is, for some positive radius R and all z with $|z - z_0| < R$,
$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} a_k (z - z_0)^k \\ &= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots \quad . \end{aligned}$$
Of course, since z_0 is a zero of f, we have, upon plugging $z = z_0$ into the above,
$$0 = f(z_0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots \quad .$$
So, in fact, $a_0 = 0$ and, for $|z - z_0| < R$,
$$\begin{aligned} f(z) &= a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots \\ &= (z - z_0)\left[a_1) + a_2(z - z_0) + a_3(z - z_0)^2 + \cdots\right] \\ &= (z - z_0) \sum_{k=1}^{\infty} a_k (z - z_0)^{k-1} \quad . \end{aligned}$$

And if a_1 also happens to be 0, then we can factor out another $z - z_0$, obtaining, for $|z - z_0| < R$,

$$\begin{aligned} f(z) &= (z - z_0)^2 \sum_{k=2}^{\infty} a_k (z - z_0)^{k-2} \\ &= (z - z_0)^2 \left[a_2 + a_3(z - z_0) + a_4(z - z_0)^2 + \cdots \right] \quad . \end{aligned}$$

Continuing, we eventually factor out $z - z_0$ as many times as possible without introducing any singularities in the remaining series, obtaining, for some positive integer M and all z with $|z - z_0| < R$,

$$\begin{aligned} f(z) &= (z - z_0)^M \sum_{k=M}^{\infty} a_k (z - z_0)^{k-2} \\ &= (z - z_0)^M \left[a_M + a_{M+1}(z - z_0) + a_{M+2}(z - z_0)^2 + \cdots \right] \end{aligned}$$

with

$$a_M \neq 0 \quad .$$

This integer M is called the *order* of the zero z_0 (of f). If $M = 1$, we say the zero is *simple*; if $m = 2$ we may say we have a *double zero*, and so on.

Do note some completely equivalent definitions of "z_0 being a zero of order M for f":

1. In some region containing z_0,

$$f(z) = (z - z_0)^M g(z)$$

for some function g analytic in the region with $g(z_0) \neq 0$.

2. The function

$$g(z) = \frac{f(z)}{(z - z_0)^M}$$

is analytic in a region containing z_0 and is nonzero at z_0.

In a sense, zeroes are "anti-poles" of functions. As you can easily confirm, poles and zeroes can cancel each other out, and division by a pole or zero can generate a zero or pole, respectively.

?►*Exercise 16.6:* *Let g have a pole of order M at z_0, and let h have a zero of order N at z_0. What can be said about the product $f = gh$ at z_0 when*

a: $M < N$? **b:** $M = N$? **c:** $M > N$?

?►*Exercise 16.7:* *Let g have a pole of order M at z_0, and let h have a zero of order N at z_0. What can be said about the quotients $^1/_g$ and $^1/_h$ at z_0 ?*

Singularities and Zeroes at Infinity (A Sidenote)

Let f be a function, and suppose, for some z_0 in $\mathbb{C}$ and some finite positive value R, that f can be expressed as a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

in the region where $R < |z - z_0|$. Instead of considering the behavior of this function 'near z_0', let us consider its behavior 'near ∞'. In fact, just consider what happens to

$$(z - z_0)^k \qquad \text{and} \qquad \frac{1}{(z - z_0)^k}$$

when k is any positive integer and $|z - z_0| \to \infty$.

For reasons that should now be obvious, we say

1. "f is *analytic at* ∞" if and only if, in this region,

$$f(z) = \sum_{k=-\infty}^{0} a_k (z - z_0)^k = a_0 + \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots \quad .$$

In this case, we also say "$f(\infty) = a_0$".

Moreover, we say "∞ is a zero of order M" (for some positive integer M) if and only if, in this region,

$$\begin{aligned} f(z) &= \sum_{k=-\infty}^{-M} a_k (z - z_0)^k \\ &= \frac{a_{-M}}{(z - z_0)^M} + \frac{a_{-M-1}}{(z - z_0)^{M+1}} + \frac{a_{-M-2}}{(z - z_0)^{M+2}} + \cdots \\ &= \frac{1}{(z - z_0)^M} \left[a_{-M} + \frac{a_{-M-1}}{(z - z_0)} + \frac{a_{-M-2}}{(z - z_0)^2} + \cdots \right] \end{aligned}$$

with

$$a_{-M} \neq 0 \quad .$$

2. "f has *pole* at ∞" of order M (for some positive integer M) if and only if, in this region,

$$\begin{aligned} f(z) &= \sum_{k=-\infty}^{M} a_k (z - z_0)^k \\ &= a_M (z - z_0)^M + a_{M-1} (z - z_0)^{M-1} + \cdots + a_0 + \frac{a_{-1}}{z - z_0} + \cdots \\ &= (z - z_0)^M \left[a_M + \frac{a_{M-1}}{(z - z_0)} + \cdots + \frac{a_0}{(z - z_0)^M} + \frac{a_{-1}}{(z - z_0)^{M+1}} + \cdots \right] \quad . \end{aligned}$$

with $a_M \neq 0$. (And the pole is *simple* if $M = 1$.)

3. "f has an *essential singularity* (or a *pole of infinite order*) at ∞" if and only if, in this region,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \quad .$$

with an infinite number of a_k's with positive indices being nonzero.

If you think about it just a little, you will realize that, basically, the "singularity of $f(z)$ at ∞" is the same as the "singularity of $h(\zeta)$ at $\zeta = 0$" where

$$h(\zeta) = f\left(z_0 + \frac{1}{\zeta}\right) \quad .$$

Also, if you think about it, you will also realize that

1. If f is analytic at ∞, then there is a constant C such that

$$f(z) \approx C \quad \text{when } |z| \text{ is large.}$$

2. If f has a pole of order M at ∞, then there is a nonzero constant C such that

$$f(z) \approx Cz^M \quad \text{when } |z| \text{ is large.}$$

3. If ∞ is a zero of order M for f, then there is a nonzero constant C such that

$$f(z) \approx \frac{C}{z^M} \quad \text{when } |z| \text{ is large.}$$