16

Complex Analysis III: Laurent Series and Singularities

A “Laurent series” for a function is a generalization of the Taylor series, and like a Taylor series, it is essentially a power series. With a Laurent series, however, the powers can be negative. A major advantage of the Laurent series over the Taylor series is that Laurent series can be expanded around singular points for a given function, and can then be used to analyze the function in the neighborhoods of its singularities. This analysis will be needed when we develop “residue theory” for computing all sorts of weird integrals that can arise in applications.

By the way, a singular point for a function $f$ is a point on the complex plane where $f$ might not be analytic.

16.1 Laurent Series Derivation and Main Theorem

Let $f$ be some single-valued function analytic in some region. To derive a Laurent series for $f$, we need to restrict our attention to an open “annular” subregion $A$ on which $f$ is analytic. Now, when we say $A$ is an “annular region”, we mean $A$ is bounded by two concentric circles $C_{\text{inner}}$ and $C_{\text{outer}}$ about some point $z_0$, with radii $r_{\text{inner}}$ and $R_{\text{outer}}$, respectively, satisfying

$$0 \leq r_{\text{inner}} < R_{\text{outer}} \leq \infty.$$ 

If $r_{\text{inner}} = 0$, then $C_{\text{inner}}$ is simply the point $z_0$, and $A$ is a disk without its center $z_0$. This case will be important for analyzing the function near $z_0$. If $R_{\text{outer}} = \infty$, then there really isn’t an outer circle, and $A$ is the entire complex plane outside of the inner circle. This case will be important for analyzing the behavior of the function at $z$ as $z \to \infty$. In practice, the radii of the inner and outer circles will typically depend on the distances between $z_0$ and the various singular points for $f$. Often, there will be many possible choices for the annular region $A$, and, thus, many possible Laurent series for a function about a given point, with each valid on a different annular region.

Now let $z$ be any point in $A$, choose any two positive values $r$ and $R$ such that

$$r_{\text{inner}} < r < |z - z_0| < R < R_{\text{outer}},$$

2/26/2014
and let $C_r$ and $C_R$ be the two counterclockwise oriented circles centered at $z_0$ with radii $r$ and $R$, respectively. Add two smooth oriented curves $l_1$ and $l_2$ between $C_r$ and $C_R$ as indicated in figure 16.1, with neither touching $z$. Using the endpoints of $l_1$ and $l_2$, break $C_r$ and $C_R$ into two pieces each, $C_r^+$ and $C_r^-$, and $C_R^+$ and $C_R^-$, respectively, as also indicated in figure 16.1. Finally, keeping track of the orientations of the subcurves, let $\Gamma_1$ and $\Gamma_2$ be the closed curves given by

$$\Gamma_1 = C_R^+ + l_1 - C_r^+ + l_2$$
$$\Gamma_2 = C_R^- - l_2 - C_r^- - l_1$$

and consider the integrals

$$\oint_{\Gamma_1} f(z) \frac{d\zeta}{\zeta - z}$$
$$\oint_{\Gamma_2} f(z) \frac{d\zeta}{\zeta - z}$$

By the Cauchy integral theorem, we know that the integral over $\Gamma_2$ (the curve not encircling $z$) is 0, while the basic Cauchy integral formula tells us that the other integral is $i2\pi f(z)$. Thus,

$$i2\pi f(z) + 0 = \oint_{\Gamma_1} f(z) \frac{d\zeta}{\zeta - z} + \oint_{\Gamma_2} f(z) \frac{d\zeta}{\zeta - z}$$

That is,

$$f(z) = \frac{1}{i2\pi} \oint_{C_r} f(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{i2\pi} \oint_{C_R} f(\zeta) \frac{d\zeta}{\zeta - z}$$

(16.1)
Consider the first integral on the right side of equation (16.1). More precisely, let $\zeta \in C_R$. Then
\[ |z - z_0| < R = |\zeta - z_0|, \]
which means that
\[ \left| \frac{z - z_0}{\zeta - z_0} \right| < 1, \]
and the formula for the sum of the geometric series can be applied as follows:
\[
\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \cdot \sum_{k=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^k = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}.
\]
Consequently,
\[
\frac{1}{i2\pi} \oint_{C_R} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{i2\pi} \oint_{C_R} \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}} f(\zeta) \, d\zeta = \frac{1}{i2\pi} \sum_{k=0}^{\infty} \oint_{C_R} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}} f(\zeta) \, d\zeta = \sum_{k=0}^{\infty} (z - z_0)^k \cdot \frac{1}{i2\pi} \oint_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta.
\]
(I know what you are wondering: Do we have the necessary uniform convergence of the series in the integral to justify the above interchanging of the integral and the summation? The answer is yes. This can be verified using the continuity of $f$ on the region and straightforward extensions of theorem 12.17 on page 12–32.)

Note the following about the integrals in the last line:

1. They do not depend on $z$.
2. By a corollary of the Cauchy integral theorem (namely, theorem 15.5 on page 15–7) we can replace $C_R$ with any simple, counterclockwise oriented loop in $A$ about $z_0$.

This means the last sequence of equations reduces to
\[
\frac{1}{i2\pi} \oint_{C_R} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad (16.2a)
\]
where
\[
a_k = \frac{1}{i2\pi} \oint_{C} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta \quad (16.2b)
\]
and $C$ is any simple, counterclockwise oriented loop in $A$ about $z_0$.

An analogous derivation can be done for the other integral in equation (16.1). You do it.
Exercise 16.1: Show that, if $\zeta \in C_r$, then
\[
\frac{1}{z - \zeta} = \sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}},
\]
and, using that, show that
\[
-\frac{1}{i2\pi} \oint_{C_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \sum_{n=0}^{\infty} b_n \frac{1}{(z - z_0)^{n+1}}
\]
where
\[
b_n = \frac{1}{i2\pi} \oint_{C_r} (\zeta - z_0)^n f(\zeta) \, d\zeta
\]
and $C$ is any simple, counterclockwise oriented loop in $A$ about $z_0$.

To combine the results from the exercise with the results given in equation set (16.2), reindex the last summation in the exercise using $k = -(n + 1)$ and let $a_k = b_{k-1}$. The results from the exercise then become
\[
-\frac{1}{i2\pi} \oint_{C} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k
\]
where
\[
a_k = \frac{1}{i2\pi} \oint_{C} (\zeta - z_0)^{k+1} \, d\zeta
\]
and $C$ is any simple, counterclockwise oriented loop in $A$ about $z_0$.

Combining equation sets (16.2) and (16.3) with equation (16.1) then gives us the following theorem.

**Theorem 16.1 (Laurent Series)**

Assume $f$ is a single-valued function analytic in an open annular subregion $A$ bounded by two concentric circles $C_{\text{inner}}$ and $C_{\text{outer}}$ about some point $z_0$, with radii $r_{\text{inner}}$ and $R_{\text{outer}}$, respectively, satisfying
\[
0 \leq r_{\text{inner}} < R_{\text{outer}} \leq \infty.
\]

Then, for every point $z$ in $A$,
\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k
\]
where
\[
a_k = \frac{1}{i2\pi} \oint_{C} (\zeta - z_0)^{k+1} \, d\zeta
\]
and $C$ is any simple, counterclockwise oriented loop in $A$ about $z_0$.

The series in equation (16.4a) (with coefficients given by formula (16.4b)) is called the **Laurent series** for $f$ about $z_0$ (in the annulus $A$). It is worth noting that the summation is shorthand for
\[ f(z) = \cdots + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots. \]

The relation with the Taylor series is discussed in the next exercise.

**Exercise 16.2:** Let \( z_0 \) be a point in the complex plane, and assume \( f \) is a single-valued analytic function on a disk of radius \( R_{\text{outer}} \) centered at \( z_0 \). (In particular, then, \( f \) is analytic at \( z_0 \).) Using the Cauchy integral theorem and the Cauchy integral formulas, verify that the Laurent series formula for \( f \) on this disk reduces to the Taylor series formula

\[ f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k. \]

It immediately follows from this exercise that saying “\( f \) is analytic as a complex function at a point \( z_0 \)” is equivalent to saying that “\( f \) can be represented by a power series expansion about \( z_0 \).” In other words, for complex functions, the definition of analytic given on page 14–10 (based on differentiability) is equivalent to the one given on page 12–33 in chapter 12 when we were discussing power series representations for functions.

### 16.2 Computing Laurent Series

In practice, we are rarely interested in explicitly computing all the coefficients of a Laurent series (though we will later have good reason for finding \( a_{-1} \), the coefficient for the \((z-z_0)^{-1}\) term). Even when we do want to explicitly find a Laurent series for a particular function \( f \), the integral formula (16.4b) is often too difficult to calculate to be of much use. What is often more helpful is the clever use of the geometric series formula,

\[ \frac{1}{1-A} = \sum_{k=0}^{\infty} A^k \quad \text{for} \quad |A| < 1, \]

and/or clever use of a known Taylor or Laurent series.

#### Using Geometric Series

To illustrate both an honest-to-goodness Laurent series and how to use the geometric series, let us attempt to find all the Laurent series for

\[ f(z) = \frac{1}{(z-1)(z-4)} \quad \text{about} \quad z_0 = 1. \]

The singular points for this function are \( z = 1 \) and \( z = 4 \), so our annular regions about \( z_0 = 1 \) must not contain these points. And since these regions are centered at \( z_0 = 1 \), this means that the bounding circle of these regions is the circle centered at \( z_0 = 1 \) of radius \( R = |4-1| = 3 \) (see figure 16.2). So we will have two Laurent series expansions of the form

\[ f(z) = \sum_{k=-\infty}^{\infty} a_k(z-1)^k, \]
Figure 16.2: The two regions $A_0$ (inside the circle) and $A_\infty$ (outside the circle) for the Laurent series about $z_0 = 1$ of

$$f(z) = \frac{1}{(z-1)(z-4)}.$$ 

one for the function on the region

$A_0 \quad$ where $z \in A_0 \iff 0 < |z - 1| < 3$ ,

and one for the function on the region

$A_\infty \quad$ where $z \in A_\infty \iff 3 < |z - 1|$ .

We will find these two series in the following two examples.

Example 16.1 (Laurent series in $A_0$): Consider finding the Laurent series about $z_0 = 1$

for

$$f(z) = \frac{1}{(z-1)(z-4)}$$ 

when

$$0 < |z - 1| < 3 .$$

Since the goal is to express $f(z)$ in terms of $z - 1$ and

$$f(z) = \frac{1}{z-1} \cdot \frac{1}{z-4} = (z-1)^{-1} \frac{1}{z-4} ,$$

much of the work will be done once we’ve expressed $(z - 4)^{-1}$ in terms of $z - 1$. By the assumption on $z$, we have

$$\left| \frac{z - 1}{3} \right| < 1 .$$

Being mildly clever, then, we get

$$\frac{1}{z-4} = \frac{1}{z-1-3} = \frac{1}{\frac{z-1}{3} - 1} = \frac{-1}{3} \cdot \frac{1}{1 - \frac{z-1}{3}} = \frac{-1}{3} \sum_{k=0}^{\infty} \left[ \frac{z-1}{3} \right]^k .$$
Thus, if \( 0 < |z - 1| < 3 \),
\[
f(z) = \frac{1}{z - 1} \cdot \frac{1}{z - 4} = (z - 1)^{-1} \cdot \frac{-1}{3} \sum_{k=0}^{\infty} \left[ \frac{z - 1}{3} \right] = \sum_{k=0}^{\infty} \frac{-1}{3k+1} (z - 1)^{k-1}.
\]
With a slight change of index \((n = k - 1)\) we have the Laurent series representation
\[
f(z) = \sum_{n=-1}^{\infty} \frac{-1}{3^{n+2}} (z - 1)^n = \frac{-1}{3(z - 1)} - \frac{1}{9} - \frac{1}{27} (z - 1) - \frac{1}{81} (z - 1)^2 - \cdots
\]
valid when \( 0 < |z - 1| < 3 \).

Example 16.2 (Laurent series in \( A_\infty \)): Now consider finding the Laurent series about \( z_0 = 1 \) for
\[
f(z) = \frac{1}{(z - 1)(z - 4)}
\]
when \( 3 < |z - 1| \).
This time,
\[
|\frac{3}{z - 1}| < 1.
\]
So,
\[
\frac{1}{z - 4} = \frac{1}{z - 1} - \frac{1}{3}
\]
\[
= \frac{1}{z - 1} \cdot \frac{1}{z - 1} = \frac{1}{z - 1} \sum_{k=0}^{\infty} \left[ \frac{3}{z - 1} \right]^{k-1},
\]
and
\[
f(z) = \frac{1}{z - 1} \cdot \frac{1}{z - 4}
\]
\[
= \frac{1}{z - 1} \cdot \frac{1}{z - 1} \sum_{k=0}^{\infty} \left[ \frac{3}{z - 1} \right] = \sum_{k=0}^{\infty} \frac{3^k}{(z - 1)^{k+2}}.
\]
With a slight change of index \((n = -(k + 2))\) we have the Laurent series representation
\[
f(z) = \sum_{n=-2}^{\infty} \frac{1}{3^{n+2}} (z - 1)^n
\]
\[
= \frac{1}{(z - 1)^2} + \frac{3}{(z - 1)^3} + \frac{9}{(z - 1)^4} + \frac{27}{(z - 1)^5} + \cdots
\]
valid when \( 3 < |z - 1| \).

The derivations of the above expansions were simplified by the fact that the expansions were about \( z_0 = 1 \) and one factor of the function of interest was already of the form \((z - 1)^{-1}\). If that had not been the case, say \( z_0 = i \), then we could have first expanded \( f(z) \) by partial fractions,
\[
f(z) = \frac{1}{(z - 1)(z - 4)} = \frac{-\frac{1}{3}}{z - 1} + \frac{\frac{1}{3}}{z - 4},
\]
and then expanded each term using a geometric series as illustrated in the examples.
Using Known Taylor or Laurent Series

Of course, you know of many power series other than the geometric series, and you are free to use your knowledge of these series to find Laurent series.

Example 16.3: Find all Laurent series for

\[ f(z) = z^2 \exp\left(\frac{1}{z}\right) \]

about \( z_0 = 0 \).

Since we know

\[ \exp(\zeta) = e^\zeta = \sum_{k=0}^{\infty} \frac{1}{k!} \zeta^k \quad \text{for all} \quad \zeta \in \mathbb{C} , \]

we immediately have

\[ z^2 \exp\left(\frac{1}{z}\right) = z^2 \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{z}\right)^k = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-2k} \]

whenever \( \frac{1}{z} \) is a finite complex number. This is the Laurent series expansion for our function. Reindexing it and expanding it, we have

\[ z^2 \exp\left(\frac{1}{z}\right) = \sum_{n=2}^{2\infty} \frac{1}{(2-n)!} z^n \]

\[ = z^2 + z + \frac{1}{2} + \frac{1}{3!} \frac{1}{z} + \frac{1}{4!} \frac{1}{z^2} + \cdots , \]

and this expansion is valid for \(|z| > 0\).

Exercise 16.3: Find the Laurent expansions about \( z_0 = 0 \) for the following using known power series (Taylor series) expansions. Also state the region in which the Laurent series expansion is valid.

- **a:** \( \sin\left(\frac{1}{z}\right) \)
- **b:** \( z \cos\left(\frac{3}{z}\right) \)
- **c:** \( z^2 \left[ 1 - \exp\left(\frac{1}{z}\right) \right] \)
- **d:** \( z^{-2} \left[ 1 - \exp(z^2) \right] \)
- **e:** \( \ln\left(\frac{z-2}{z}\right) \)

Products of Laurent Series

Suppose our function of interest \( f \) is the product of functions whose Laurent series around some point \( z_0 \) are known (or are reasonably easy to compute). That is, suppose we know

\[ f(z) = g(z)h(z) \]

where, for each \( z \) in an annular region around \( z_0 \),

\[ g(z) = \sum_{m=-\infty}^{\infty} g_m (z - z_0)^m \quad \text{and} \quad h(z) = \sum_{n=-\infty}^{\infty} h_n (z - z_0)^n . \]
Of course, if these functions are simple enough, then it is relatively easy to find the coefficients in the corresponding Laurent series expansion for \( f \),

\[
f(z) = \sum_{k=-\infty}^{\infty} f_k (z - z_0)^k,
\]

in the region. Alternatively, we can multiply the series for \( g \) and \( h \) to get the series for \( f \).

Here is a relatively simple approach to computing this product of infinite series, using the basic formula for the coefficients in a Laurent series and one integral we should know by heart at this point. Remember, the basic formula for the Laurent series coefficients is

\[
f_k = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta.
\]

where \( C \) is some simple loop about \( z_0 \) oriented counterclockwise. Replacing \( f \) with \( gh \) and then replacing \( g \) and \( h \) with their Laurent series, we get

\[
f_k = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta = \frac{1}{2\pi i} \oint_C \sum_{m=-\infty}^{\infty} g_m (\zeta - z_0)^m \sum_{n=-\infty}^{\infty} h_n (\zeta - z_0)^n d\zeta = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_m h_n \frac{1}{2\pi i} \oint_C (\zeta - z_0)^{m+n-k-1} d\zeta.
\]

The last integral is one we’ve computed before. It is \( i2\pi \) if \( m + n - k - 1 = -1 \) (i.e., \( n = k - m \)), and 0 otherwise. Thus,

\[
f_k = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_m h_n \frac{1}{2\pi i} \oint_C (\zeta - z_0)^{m+n-k-1} d\zeta = \sum_{m=-\infty}^{\infty} g_m h_{k-m}. \quad (16.5)
\]

In particular,

\[
f_{-1} = \sum_{m=-\infty}^{\infty} g_m h_{1-m}. \quad (16.6)
\]

(We mention this particular coefficient because, as we will see in the next chapter, it is often the only coefficient of real interest.)

In practice, rather than blindly applying the above two formulas, you may just want to “rederive” the above from basics for the problem at hand.

**Example 16.4:** Suppose we want to find the Laurent series in the region just around \( z_0 = 0 \),

\[
f(z) = \sum_{k=-\infty}^{\infty} f_k z^k,
\]

for the function

\[
f(z) = \frac{\sin(1/z)}{z^2 + a^2} \quad \text{with} \quad 0 < |z| < a.
\]
Here,

\[ f = gh \text{ where } g(z) = \frac{1}{z^2 + a^2} \text{ and } h(z) = \sin\left(\frac{1}{z}\right). \]

Using the geometric series (and remembering that \(|z| < a\),

\[ g(z) = \frac{1}{a^2} \sum_{m=0}^{\infty} \left(\frac{-z^2}{a^2}\right)^m = \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{2m+2}} z^{2m}. \]

And using the Taylor series for the sine function, we get the corresponding Laurent series expansion

\[ \sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{2n+1}}. \]

Applying “basics”:

\[
\begin{align*}
    f_k &= \frac{1}{i2\pi} \oint_C f(\zeta) \frac{d\zeta}{(\zeta - 0)^{k+1}} \\
    &= \frac{1}{i2\pi} \oint_C g(\zeta) h(\zeta) \zeta^{-k-1} d\zeta \\
    &= \frac{1}{i2\pi} \oint_C \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{2m+2}} \zeta^{2m} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \zeta^{2n+1} \zeta^{-k-1} d\zeta \\
    &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{a^{2m+2}(2n+1)!} \cdot \frac{1}{i2\pi} \oint_C \zeta^{2m-2n-1-k-1} d\zeta \\
    &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{a^{2m+2}(2n+1)!} \left\{ \begin{array}{ll}
        \frac{i2\pi}{a} & \text{if } 2m - 2n - 1 - k - 1 = -1 \\
        0 & \text{otherwise}
    \end{array} \right\} \\
    &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{a^{2m+2}(2m - k)!} \left\{ \begin{array}{ll}
        1 & \text{if } 2n = 2m - 1 - k \\
        0 & \text{otherwise}
    \end{array} \right\}. 
\end{align*}
\]

In particular,

\[
\begin{align*}
    f_{-1} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{a^{2m+2}(2m - k)!} \left\{ \begin{array}{ll}
        1 & \text{if } 2n = 2m - 1 - (-1) \\
        0 & \text{otherwise}
    \end{array} \right\} \\
    &= \sum_{m=0}^{\infty} \frac{(-1)^m}{a^{2m+2}(2m + 1)!} \\
    &= \frac{1}{a} \sum_{m=0}^{\infty} \frac{1}{(2m + 1)!} \left( \frac{1}{a} \right)^{2m+1}. 
\end{align*}
\]

After recalling that

\[ \sinh(\zeta) = \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} (\zeta)^{2m+1}, \]

we see that the above reduces to

\[ f_{-1} = \frac{1}{a} \sinh\left(\frac{1}{a}\right). \]
16.3 Singularities and the Laurent Series

Singularities at Finite Points

Let $z_0$ be a point in the complex plane and $f$ a single-valued function analytic in some region about $z_0$, but, possibly, not at $z_0$. To be precise, we assume there is a $R > 0$ such that $f$ is analytic on

$$R = \{ z : 0 < |z - z_0| < R \}.$$

In this region, we can expand $f$ in its Laurent series,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k \quad (16.7)$$

For our convenience, let’s break this into the natural pieces,

$$f(z) = \sum_{k=-\infty}^{-1} a_k(z - z_0)^k + \sum_{k=0}^{\infty} a_k(z - z_0)^k$$

where, for even more convenience, we’ve let

$$f_S(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} \quad \text{and}$$

$$f_A(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k.$$

Clearly, $f_A$ will be analytic at every point $z$ with $|z - z_0| < R$, including the point $z = z_0$. Moreover, by what we know about power series, we know the series for $f_A(z)$ will converge uniformly on any disk centered about $z_0$ of radius less than $R$.

The convergence properties of $f_S(z)$ can easily be deduced by considering the function

$$h(\zeta) = f_S\left(z_0 + \frac{1}{\zeta}\right).$$

I’ll leave the details to you:

**Exercise 16.4:** Let $f$ and $h$ be as above, and assume that $f_S$ is not trivial (i.e., at least one of the $a_{-n}$’s is nonzero).

- **a:** Find the power series for $h(\zeta)$ from the series for $f_S(z)$, and verify that the radius of convergence for the power series for $h(\zeta)$ is infinite. (Use facts learned a few weeks ago about power series in general, and the fact that the series for $f_S(z)$ converges at least on $0 < |z - z_0| < R$.)

- **b:** From what you just derived about $h$, show that
i: The series for \( f_S(z) \) converges at every complex point \( z \) except \( z = z_0 \).

ii: \( f_S \) is analytic everywhere in the complex plane except at \( z_0 \)

iii: If \( \varepsilon \) is any positive real number, then the series for \( f_S(z) \) converges uniformly on the region \( \varepsilon \leq |z - z_0| \).

Now, as we already know, \( f \) will be analytic at \( z_0 \) if and only if
\[
f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = f_A(z)
\]
on \( \mathcal{R} \). This, of course, means that \( f_S \) is trivial. If \( f_S \) is not trivial, we say \( f \) has a singularity at \( z_0 \) and classify that singularity as being either a ‘pole’ or an ‘essential’ singularity according to the following criteria:

1. We say \( f \) has a pole at \( z_0 \) of order \( M \) (for some positive integer \( M \)) if and only if the following equivalent conditions hold:
   - In formula (16.7), \( a_{-M} \neq 0 \) but \( a_k = 0 \) whenever \( k < -M \).
   - On \( \mathcal{R} \),
     \[
f(z) = \sum_{k=-M}^{\infty} a_k (z - z_0)^k
      = \frac{a_{-M}}{(z - z_0)^M} + \frac{a_{-M+1}}{(z - z_0)^{M-1}} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{k=0}^{\infty} a_k (z - z_0)^k
     \]
   with \( a_{-M} \neq 0 \).
   - \( a_{-M} \neq 0 \) and
   \[
f_S(z) = \frac{a_{-M}}{(z - z_0)^M} + \frac{a_{-M+1}}{(z - z_0)^{M-1}} + \cdots + \frac{a_{-1}}{z - z_0}.
\]

2. We say \( f \) has a simple pole at \( z_0 \) if and only if it has a pole of order \( 1 \) at \( z_0 \).

3. We say \( f \) has an essential singularity (or pole of infinite order) at \( z_0 \) if and only if the following equivalent conditions hold:
   - In formula (16.7), for any integer \( M \), there is a \( k \leq -M \) for which \( a_k \neq 0 \).
   - On \( \mathcal{R} \),
     \[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k
     = \cdots + \frac{a_{-M+1}}{(z - z_0)^{M-1}} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{k=0}^{\infty} a_k (z - z_0)^k
     \]
   and the series for the singular part has infinitely many nonzero terms.
   - There are infinitely many nonzero terms in
   \[
f_S(z) = \cdots + \frac{a_{-M}}{(z - z_0)^M} + \frac{a_{-M+1}}{(z - z_0)^{M-1}} + \cdots + \frac{a_{-1}}{z - z_0} = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}.
\]
Typically, a function behaves “very badly” near any essential singularity. In particular, “it can be shown” that, if \( f \) has an essential singularity at \( z_0 \), then we can pick any fixed value \( A \) and find a corresponding sequence of points \( \{\zeta_1, \zeta_2, \zeta_3, \ldots\} \) such that
\[
\zeta_k \to z_0 \quad \text{and} \quad f(\zeta_k) \to A \quad \text{as} \quad k \to \infty.
\]
Because of this, we do NOT have \( |f(z)| \to \infty \) as \( k \to \infty \) when \( f \) has an essential singularity at \( z_0 \). This is illustrated in the following exercise.

\textbf{◮Exercise 16.5:} Let 
\[
f(z) = \exp\left(\frac{1}{z}\right) = e^{\frac{1}{z}}.
\]

\textbf{a:} Verify that \( f \) has an essential singularity at \( z = 0 \) and find its Laurent series about \( 0 \).
(Hint: see example 16.3 on page 16–8.)

\textbf{b:} Pick any real or complex value \( A \) other than \( 0 \) (go ahead, really pick one this time) and let
\[
\zeta_k = \frac{1}{\ln(A) + i2\pi k} \quad \text{for} \quad k = 1, 2, 3, \ldots.
\]

\textbf{i:} Show that \( f(\zeta_k) = A \) for each positive integer \( k \).

\textbf{ii:} Show that \( \zeta_k \to 0 \) as \( k \to \infty \).

\textbf{iii:} By what you’ve just shown, you have
\[
\lim_{k \to \infty} f(\zeta_k) = \lim_{k \to \infty} A = A \quad \text{and} \quad \lim_{k \to \infty} |f(\zeta_k)| = \lim_{k \to \infty} |A| = |A|.
\]
Why does this NOT mean that
\[
\lim_{\zeta \to 0} f(\zeta) = A \quad \text{and} \quad \lim_{\zeta \to 0} |f(\zeta)| = |A|
\]
even though \( \zeta_k \to 0 \) as \( k \to \infty \)?

Behavior of a function near a pole is not so terrible. To see this, assume \( f \) has a pole at \( z_0 \) of finite order \( M \), and let \( g \) be the function
\[
g(z) = (z - z_0)^M f(z).
\]
About \( z_0 \), \( f \) has the Laurent expansion
\[
f(z) = \sum_{k=-M}^{\infty} a_k (z - z_0)^k
\]
with the first coefficient, \( a_{-M} \) not being zero. For \( g \) we have
\[
g(z) = (z - z_0)^M f(z)
= (z - z_0)^M \sum_{k=-M}^{\infty} a_k (z - z_0)^k
= \sum_{k=-M}^{\infty} a_k (z - z_0)^{M+k}
= \sum_{n=0}^{\infty} a_{n-M} (z - z_0)^n.
\]
That is,
\[ g(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n \]
with
\[ c_n = a_{n-M} \quad (\text{and, hence, } a_k = c_{k+M}) \, . \]

From this, all the following follows:

1. \( g \) is analytic at \( z_0 \) with \( g(z_0) = c_0 = a_{-M} \neq 0 \).

2. \( f(z) \) can be written as
\[ f(z) = \frac{g(z)}{(z - z_0)^M} \]
for some function \( g \) which is analytic and nonzero at \( z_0 \).

3. If \( z \approx z_0 \), then
\[ f(z) \approx \frac{A}{(z - z_0)^M} \]
for some nonzero value \( A \) (\( A = g(z_0) = a_{-M} \)). (But integrals of \( f \) may still depend more on higher-order terms.)

4. For each integer \( k \geq -M \),
\[ a_k = c_{M+k} \quad \text{the } (M + K)^{th} \text{ term in the Taylor series formula for } g \text{ about } z_0 \]
\[ = \frac{g^{(M+k)}(z_0)}{(M + k)!} \]
\[ = \frac{1}{(M + k)!} \frac{d^{M+k}}{dz^{M+k}} \left[ (z - z_0)^M f(z) \right] |_{z=z_0} \, . \]

In particular,
\[ a_{-1} = \frac{1}{(M-1)!} \frac{d^{M-1}}{dz^{M-1}} \left[ (z - z_0)^M f(z) \right] |_{z=z_0} \, , \quad (16.8) \]
and if the pole is simple (i.e., \( M = 1 \), then
\[ a_{-1} = (z - z_0) f(z)|_{z=z_0} \, . \quad (16.9) \]

We will find uses for these formulas soon. (Note: The evaluation of the above formulas at \( z_0 \) may require computing the limit of the formulas as \( z_0 \), possibly using L'Hôpital's rule.)

**Identifying Singularities in Practice**

In practice, it is usually easy to identify where a function has singularities. This is because singularities typically arise due to a division by zero, and it is usually easy to spot where a division by zero occurs. (But if, on closer inspection, you really have something like \( "f(z_0) = 0/0" \) then try to compute
\[ \lim_{z \to z_0} f(z) \, . \]
possibly using L’Hôpital’s rule. If you get a finite number, the function is actually analytic at 
\( z_0 \).

To determine whether a singularity is a finite or infinite order pole, you can simply use the 
methods discussed earlier (using known Taylor series or the geometric series) to first find the 
Laurent series for the function \( f \) about each singularity, and examine the series found to see 
how many terms are in the singular part, \( f_S(z) \).

**Example 16.5:** Consider 
\[
f(z) = \exp\left(\frac{1}{z}\right) .
\]

Clearly, the only ‘trouble spot’ is \( z = 0 \), and, in exercise 16.5 on page 16–13 you saw that 
the corresponding Laurent series, 
\[
\exp\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{z}\right)^k = 1 + \frac{1}{z} + \frac{1}{2} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + \cdots ,
\]
has infinitely many nonzero terms in the singular part. Hence, \( \exp^{(1/z)} \) has an essential 
singularity about \( 0 \).

Alternatively, you can use the observations noted above regarding the behavior of a function 
\( f \) near a singularity. The simplest case is where \( f \) can clearly be written as
\[
f(z) = \frac{g(z)}{(z-z_0)^M}
\]
for some positive integer \( M \) and a function \( g \) that is analytic and nonzero at \( z_0 \). If this is the 
case, then \( f \) has a pole of order \( M \). (If, however, \( g(z_0) = 0 \), then either \( f \) has a pole of lower 
order, or is analytic at \( z_0 \).

If it is not clear that \( f(z) \) can be written as just described, try computing 
\[
\lim_{z \to z_0} (z - z_0)^N f(z)
\]
for a “suitably chosen” positive integer \( N \) (use your judgement). If \( f \) does have a pole of order 
\( M \) at \( z_0 \), then \( f(z) \) can be written as
\[
f(z) = \frac{g(z)}{(z-z_0)^M}
\]
for some function \( g \) that is analytic and nonzero at \( z_0 \), and, thus,
\[
\lim_{z \to z_0} \left| (z - z_0)^N f(z) \right| = \lim_{z \to z_0} \left| (z - z_0)^N \frac{g(z)}{(z-z_0)^M} \right| = \\
\quad = \lim_{z \to z_0} |z - z_0|^{N-M} |g(z)| = \begin{cases} 
0 & \text{if } N > M \\
g(z_0) & \text{if } N = M \\
\infty & \text{if } N < M
\end{cases}
\]

From this and what we know about the behavior of functions near essential singularities, we then 
get the following:
1. If
\[ \lim_{z \to z_0} (z - z_0)^N f(z) = 0 , \]
then \( f \) either has a pole of order less than \( N \) at \( z_0 \) or is analytic at \( z_0 \).

2. If
\[ \lim_{z \to z_0} (z - z_0)^N f(z) \]
is a finite number other than 0, then \( f \) has a pole of order \( N \) at \( z_0 \).

3. If
\[ \lim_{z \to z_0} (z - z_0)^N f(z) = \infty , \]
(more precisely, if
\[ \lim_{z \to z_0} \left| (z - z_0)^N f(z) \right| = +\infty , \]
then \( f \) has a pole of order greater than \( N \) at \( z_0 \).

4. Finally, if
\[ \lim_{z \to z_0} (z - z_0)^N f(z) \]
is not well defined, then \( f \) has an essential singularity at \( z_0 \).

**Zeroes at Finite Points**

A discussion somewhat analogous to that for poles can be had for “zeroes” of functions. Remember, given a function \( f \) and a point \( z_0 \),
\[ z_0 \text{ is a zero of } f \iff f(z_0) = 0 . \]
(And, perhaps once again, we’ll note how stupid this standard terminology is since, in fact, most zeroes are nonzero.)

So assume \( z_0 \), a point in \( \mathbb{C} \), is a zero for some single-valued function \( f \) analytic in some region “right around” \( z_0 \). Since \( f(z_0) = 0 \), \( f \) clearly cannot have a singularity at \( z_0 \). Hence \( f \) is analytic at \( z_0 \), and its Laurent series “right around” \( z_0 \) will only contain an analytic part. That is, for some positive radius \( R \) and all \( z \) with \( |z - z_0| < R \),
\[ f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \]
\[ = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots . \]

Of course, since \( z_0 \) is a zero of \( f \), we have, upon plugging \( z = z_0 \) into the above,
\[ 0 = f(z_0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots . \]
So, in fact, \( a_0 = 0 \) and, for \( |z - z_0| < R \),
\[ f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots \]
\[ = (z - z_0) \left[ a_1 + a_2(z - z_0) + a_3(z - z_0)^2 + \cdots \right] \]
\[ = (z - z_0) \sum_{k=1}^{\infty} a_k (z - z_0)^{k-1} . \]
And if $a_1$ also happens to be 0, then we can factor out another $z - z_0$, obtaining, for $|z - z_0| < R$,

$$f(z) = (z - z_0)^2 \sum_{k=2}^{\infty} a_k (z - z_0)^{k-2}$$

$$= (z - z_0)^2 \left[ a_2 + a_3 (z - z_0) + a_4 (z - z_0)^2 + \cdots \right].$$

Continuing, we eventually factor out $z - z_0$ as many times as possible without introducing any singularities in the remaining series, obtaining, for some positive integer $M$ and all $z$ with $|z - z_0| < R$,

$$f(z) = (z - z_0)^M \sum_{k=M}^{\infty} a_k (z - z_0)^{k-2}$$

$$= (z - z_0)^M \left[ a_M + a_{M+1} (z - z_0) + a_{M+2} (z - z_0)^2 + \cdots \right]$$

with

$$a_M \neq 0.$$

This integer $M$ is called the order of the zero $z_0$ (of $f$). If $M = 1$, we say the zero is simple; if $M = 2$ we may say we have a double zero, and so on.

Do note some completely equivalent definitions of “$z_0$ being a zero of order $M$ for $f$”:

1. In some region containing $z_0$,

$$f(z) = (z - z_0)^M g(z)$$

for some function $g$ analytic in the region with $g(z_0) \neq 0$.

2. The function

$$g(z) = \frac{f(z)}{(z - z_0)^M}$$

is analytic in a region containing $z_0$ and is nonzero at $z_0$.

In a sense, zeroes are “anti-poles” of functions. As you can easily confirm, poles and zeroes can cancel each other out, and division by a pole or zero can generate a zero or pole, respectively.

\textbf{Exercise 16.6:} Let $g$ have a pole of order $M$ at $z_0$, and let $h$ have a zero of order $N$ at $z_0$. What can be said about the product $f = gh$ at $z_0$ when

a: $M < N$  

b: $M = N$  

c: $M > N$?

\textbf{Exercise 16.7:} Let $g$ have a pole of order $M$ at $z_0$, and let $h$ have a zero of order $N$ at $z_0$. What can be said about the quotients $\frac{1}{g}$ and $\frac{1}{h}$ at $z_0$?

\textbf{Singularities and Zeroes at Infinity (A Sidenote)}

Let $f$ be a function, and suppose, for some $z_0$ in $\mathbb{C}$ and some finite positive value $R$, that $f$ can be expressed as a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

\textit{version: 2/26/2014}
in the region where \( R < |z - z_0| \). Instead of considering the behavior of this function ‘near \( z_0 \)’, let us consider its behavior ‘near \( \infty \)’. In fact, just consider what happens to
\[
(z - z_0)^k \quad \text{and} \quad \frac{1}{(z - z_0)^k}
\]
when \( k \) is any positive integer and \( |z - z_0| \to \infty \).

For reasons that should now be obvious, we say

1. “\( f \) is analytic at \( \infty \)” if and only if, in this region,
\[
f(z) = \sum_{k=-\infty}^{0} a_k(z - z_0)^k = a_0 + \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots .
\]
In this case, we also say “\( f(\infty) = a_0 \)”.

Moreover, we say “\( \infty \) is a zero of order \( M \)” (for some positive integer \( M \)) if and only if, in this region,
\[
f(z) = \sum_{k=-\infty}^{-M} a_k(z - z_0)^k
\]
\[
= \frac{a_{-M}}{(z - z_0)^M} + \frac{a_{-M-1}}{(z - z_0)^{M+1}} + \frac{a_{-M-2}}{(z - z_0)^{M+2}} + \cdots
\]
\[
= \frac{1}{(z - z_0)^M} \left[ a_{-M} + \frac{a_{-M-1}}{(z - z_0)} + \frac{a_{-M-2}}{(z - z_0)^2} + \cdots \right]
\]
with
\[
a_{-M} \neq 0 .
\]

2. “\( f \) has pole at \( \infty \)” of order \( M \) (for some positive integer \( M \)) if and only if, in this region,
\[
f(z) = \sum_{k=-\infty}^{M} a_k(z - z_0)^k
\]
\[
= a_M(z - z_0)^M + a_{M-1}(z - z_0)^{M-1} + \cdots + a_0 + \frac{a_{-1}}{z - z_0} + \cdots
\]
\[
= (z - z_0)^M \left[ a_M + \frac{a_{M-1}}{(z - z_0)} + \cdots + \frac{a_0}{(z - z_0)^M} + \frac{a_{-1}}{(z - z_0)^{M+1}} + \cdots \right].
\]
with \( a_M \neq 0 \). (And the pole is simple if \( M = 1 \).)

3. “\( f \) has an essential singularity (or a pole of infinite order) at \( \infty \)” if and only if, in this region,
\[
f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k
\]
with an infinite number of \( a_k \)'s with positive indices being nonzero.

If you think about it just a little, you will realize that, basically, the “singularity of \( f(z) \) at \( \infty \)” is the same as the “singularity of \( h(\zeta) \) at \( \zeta = 0 \)” where
\[
h(\zeta) = f\left( z_0 + \frac{1}{\zeta} \right).
\]
Also, if you think about it, you will also realize that
1. If $f$ is analytic at $\infty$, then there is a constant $C$ such that

$$f(z) \approx C \quad \text{when } |z| \text{ is large.}$$

2. If $f$ has a pole of order $M$ at $\infty$, then there is a nonzero constant $C$ such that

$$f(z) \approx Cz^M \quad \text{when } |z| \text{ is large.}$$

3. If $\infty$ is a zero of order $M$ for $f$, then there is a nonzero constant $C$ such that

$$f(z) \approx \frac{C}{z^M} \quad \text{when } |z| \text{ is large.}$$