15

Complex Analysis II: Cauchy Integral Theorems and Formulas

The main goals here are major results relating "differentiability" and "integrability". In a very real sense, it will be these results, along with the Cauchy-Riemann equations, that will make complex analysis so useful in many advanced applications.

By the way, we are taking a very simple notion of "a function being integrable". When we say that a function f is *integrable*, we simply mean there is another single-valued function F such that $F' = f$. This may differ from other definitions of integrability you have seen.

15.1 Relating Integrability with Differentiability

Recalling a Little Potential Theory

The issue of whether a given complex function f can be written as the derivative of some (single valued) function F is almost completely analogous to the issue discussed last term of whether a given vector field $\mathbf{F}$ could be written as the gradient of some scalar-valued function ϕ (i.e., whether $\mathbf{F}$ is "conservative" or not).[1]

To see the analogy, let f be any single-valued function on some region $\mathcal{R}$ of the complex plane, and observe the following:

1. If f is integrable in $\mathcal{R}$ and equals F' for some single-valued function F, and if C is any oriented curve in $\mathcal{R}$ starting at a point z_S and ending at a point z_E (both in $\mathcal{R}$), then we saw (theorem 14.3 on page 14–18) that

$$\int_C f(z)\,dz \;=\; \int_C F'(z)\,dz \;=\; F(z_E) \;-\; F(z_S) \quad .$$

That is, the integral is "path independent" — its value does not depend the path taken, only on the starting and end points — and we can safely write

$$\int_{z_S}^{z_E} f(z)\,dz$$

to denote the integral of f over any curve in $\mathcal{R}$ from z_S to z_E .

[1] See subsection 10.2, starting on page 10–6 of last term's notes.

2. On the other hand, suppose every integral of f over any curve in $\mathcal{R}$ is path independent. Define F on $\mathcal{R}$ by

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$$

where z_0 is some fixed point in $\mathcal{R}$. Then, for any z in $\mathcal{R}$, "it is easily verified that"

$$\begin{aligned} F'(z) &= \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(\zeta)\, d\zeta = f(z) \quad . \end{aligned}$$

3. If every integral of f over any curve in $\mathcal{R}$ is path independent, then we obviously must have

$$\oint_C f(z)\, dz = 0 \qquad \text{whenever } C \text{ is a closed curve in } \mathcal{R} ,$$

simply because that curve starts and ends at the same point

4. On the other hand, if

$$\oint_C f(z)\, dz = 0 \qquad \text{whenever } C \text{ is a closed curve in } \mathcal{R} ,$$

then every integral of f over any curve in $\mathcal{R}$ must be path independent (because any two curves between two given points form a loop).

In summary, we have the following lemma:

Lemma 15.1
If f is a single-valued function on a region $\mathcal{R}$ of the complex plane, then the following statements are equivalent:

1. $f = F'$ *on* $\mathcal{R}$ *for some single-valued, analytic function* F *on* $\mathcal{R}$.
2. *Every integral of* f *over a curve in* $\mathcal{R}$ *is path independent.*
3. *Every integral of* f *over a closed curve in* $\mathcal{R}$ *is zero.*

Cauchy's Theorem Relating Integrability with Differentiability

The last lemma relates integrability with path independence, but does not really give us an easy way to determine if a given f can be written as the derivative of some other function on a given region. If you recall our discussion of potential theory, however, you will recall that we could easily determine if path integrals of a vector field $\mathbf{F}$ were path independent by seeing if $\nabla \times \mathbf{F} = \mathbf{0}$. This result was derived using Stokes theorem and assumed the region in question was three-dimensional and "had no holes". A similar result was noted for vector fields on the plane (exercise 10.2 on page 10–11). The complex function analog can be derived from that, but we will just derive it directly from the classical Stokes theorem on the plane, better known as Green's theorem. To save you the trouble of looking it up, I've copied it from page 10–29:

Theorem 15.2 (Green's Theorem/Stokes' Theorem in the Plane)

Let $\mathcal{S}$ be a bounded region in a Euclidean plane with boundary curve $\mathcal{C}$ oriented in the standard way (i.e., counterclockwise), and let $\{(x, y)\}$ be Cartesian coordinates for the plane with corresponding orthonormal basis $\{\mathbf{i}, \mathbf{j}\}$. Assume, further, that $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is a sufficiently differentiable vector field on $\mathcal{S}$. Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\boldsymbol{r} = \iint_{\mathcal{S}} \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] dA \quad . \tag{15.1}$$

The derivation/proof of this theorem is in section 10.8 of our notes, and was pretty well based on the basic fundamental theorem of multidimensional calculus (theorem 10.4 on page 10–26). If you check, you will discover that "$\mathbf{F}$ is sufficiently differentiable" simply means that the component functions of $\mathbf{F}$ are continuous on $\mathcal{S}$ with its boundary $\mathcal{C}$, and are "piecewise partially differentiable" inside $\mathcal{S}$.[2]

So now let $\mathcal{C}$ be some simple closed, counterclockwise-oriented curve, and let $\mathcal{S}$ be the region enclosed by $\mathcal{C}$. Suppose $f = u + iv$ is any complex-valued function which is continuous on the union of $\mathcal{S}$ with its boundary $\mathcal{C}$, and whose real and imaginary parts, u and v, are at least piecewise partially differentiable on the enclosed region $\mathcal{S}$. Using first lemma 14.2 and then Green's theorem, we have

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\, dz &= \int_{\mathcal{C}} \mathbf{G} \cdot d\boldsymbol{r} + i \int_{\mathcal{C}} \mathbf{H} \cdot d\boldsymbol{r} \\ &= \iint_{\mathcal{S}} \left[\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right] dA + i \iint_{\mathcal{S}} \left[\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right] dA\end{aligned}$$

where $\mathbf{G}$ and $\mathbf{H}$ are the (real) vector fields given by

$$\mathbf{G} = u\mathbf{i} - v\mathbf{j} \qquad \text{and} \qquad \mathbf{H} = v\mathbf{i} + u\mathbf{j} \quad .$$

In terms of u and v, the above is

$$\oint_{\mathcal{C}} f(z)\, dz = -\iint_{\mathcal{S}} \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right] dA + i \iint_{\mathcal{S}} \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] dA \quad . \tag{15.2}$$

Let us now consider this equation under the added assumption that f is "piecewise analytic" in the region $\mathcal{S}$ — that is, we assume $\mathcal{S}$ can be partitioned into a few open regions on which f is analytic. We can then compute the integrals on the right assuming u and v satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad .$$

This, of course, means that

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \qquad \text{on} \quad \mathcal{S} \quad ,$$

[2] A real-valued function of two variables is *piecewise partially differentiable* in a region if its partial derivatives exist everywhere except, possibly, on a finite set of isolated points and piecewise smooth curves of finite lengths in that region. This also means that we can partition the region into open subregions on which the partial derivatives are well defined, with the points at which the partials may not exist being on the boundaries of the regions. The important thing here is that our derivation of Green's theorem only required that the components of the vector field be continuous and piecewise partially differentiable.

and, thus, equation (15.2) reduces to

$$\oint_C f(z)\,dz = -\iint_S 0\,dA + i\iint_S 0\,dA = 0 \quad .$$

Since every closed curve can be decomposed into a bunch of simple closed curves, the above yields:

Theorem 15.3 (Basic Cauchy Integral Theorem)
Let C be a closed curve in $\mathbb{C}$, and let S be the region enclosed by C. Assume f is a complex-valued function which is single valued and continuous on the union of S with its boundary, C, and which is at least piecewise analytic on S. Then

$$\oint_C f(z)\,dz = 0 \quad .$$

The above is not quite the version normally found in texts. A more traditional version is

Theorem 15.4 (Traditional Cauchy Integral Theorem)
Assume f is a single-valued, analytic function on a simply-connected region $\mathcal{R}$ in the complex plane. Then

$$\oint_C f(z)\,dz = 0$$

whenever C is a simple closed curve in $\mathcal{R}$.

It is trivial to show that the traditional version follows from the basic version of the Cauchy Theorem. Moreover, with a little work, you can derive the basic version from the traditional version; so, technically, the two versions are equivalent. I've just decided that the "more basic" version will be easier to use.

A few comments regarding the Cauchy Integral Theorem:

1. We cannot simply drop the continuity requirement on f in the basic version or the requirement in the traditional version that $\mathcal{R}$ be simply connected. The fact (from example 14.2) that

$$\oint_C \frac{1}{z}\,dz = i2\pi \neq 0$$

(where C is the unit circle oriented counterclockwise) demonstrates this — $1/z$ is not continuous in the region bounded by the unit circle, and the region $\mathcal{R}$ here cannot be simply connected because it cannot include the origin.

2. Remember that the vanishing of the integrals of a given (single-valued) function over arbitrary closed loops in some region occurs if and only if that function is integrable in that region. So the Cauchy Integral Theorem basically tells us that, at least in a simply connected region,

$$\text{analyticity} \quad \Longrightarrow \quad \text{integrability} \quad .$$

3. A few words must be said about "continuity of derivatives" simply because some authors (including Arfken, Weber & Harris) make an issue of it. The claim is sometimes made that using Green's Theorem to prove/derive the Cauchy Integral Theorem is somehow deficient because it supposedly requires the partial derivatives of the functions involved be continuous on the region in question. This is only the case if you have a particularly whimpy derivation of Green's theorem. Our derivation did not require the derivatives be continuous — it didn't even really require that the function's partial exist everywhere. Our version allows our function to be merely piecewise partially differentiable. So, our basic Cauchy Integral Theorem is actually better in some ways than the traditional version.

There is another basic approach to proving the Cauchy Integral Theorem which totally bypasses Green's theorem. You can find versions of this proof, originally developed by Goursat, in most decent texts on complex variables.[3] It is an especially useful approach in these texts since Green's Theorem is not normally part of a course in complex variables. It is also a pretty neat way to prove the theorem.

A Converse to the Cauchy Integral Theorem?

We've observed that analyticity implies integrability, at least for functions on simply-connected regions. Now let's briefly consider what equation (15.2) says about any continuous single-valued function $f = u + iv$ that is integrable on a simply-connected region $\mathcal{R}$. Remember, the integral of such a function over each closed curve in $\mathcal{R}$ vanishes. If, in addition, f is piecewise partially differentiable on $\mathcal{R}$, then equation (15.2) tells us that

$$0 = \oint_C f(z)\,dz = -\iint_S \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dA + i\iint_S \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dA$$

whenever $\mathcal{S}$ is a subregion of $\mathcal{R}$ with boundary C in $\mathcal{R}$. It then follows that, for every subregion $\mathcal{S}$ of $\mathcal{R}$,

$$\iint_S \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] dA = 0 \qquad \text{and} \qquad \iint_S \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dA = 0 \quad .$$

If we also know these partial derivatives are piecewise continuous, then it is easy to show that u and v satisfy the Cauchy-Riemann equations at every point in $\mathcal{R}$ at which the partial derivatives are continuous, and theorem 14.1 on page 14–12 assures us that f is at least piecewise analytic on $\mathcal{R}$. That is,

$$\text{integrability} \quad \Longrightarrow \quad \text{(piecewise) analyticity}$$

provided the partial derivatives are at least piecewise continuous.

Again, the continuity requirements are mainly so that we can use results we already have. However, by the end of this chapter, we will have developed better results. After developing those results, we will revisit the issue of the relation between analyticity and integrability, and greatly reduce the *a priori* requirements on the partial derivatives.

[3] If you are interested, a discussion of Goursat's proof is in an appendix to this chapter, starting on page 15–14.

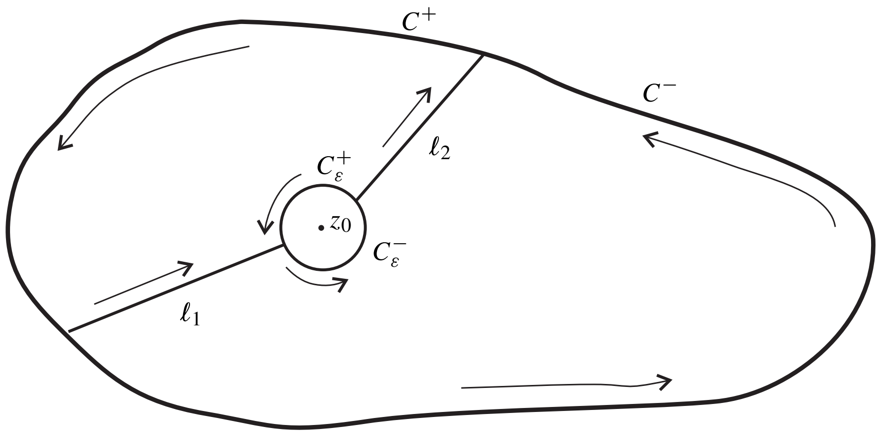

Figure 15.1: Isolating a bad point: The curves for deriving equation (15.3). The outer loop is $C = C^+ + C^-$ and the "small" circle around z_0 is $C_\varepsilon = C_\varepsilon^+ + C_\varepsilon^-$.

15.2 Line Integrals Around "Bad Points"

What if we have a few "bad points" for a function?

Well, let's start by letting z_0 be a point in a simply-connected region $\mathcal{R}$, and letting g be a single-valued, piecewise analytic function on $\mathcal{R}$ which is continuous at every point except, possibly, z_0. Now consider

$$\oint_C g(z)\,dx$$

where C is any simple closed curve in $\mathcal{R}$ encircling z_0 and oriented counterclockwise (as in figure 15.1).

Pick a positive value ε small enough that

$$\varepsilon \;<\; \text{smallest distance from } z_0 \text{ to } C$$

and let

$$C_\varepsilon \;=\; \text{the circle of radius } \varepsilon \text{ about } z_0\text{, oriented counterclockwise} \quad .$$

Next, draw two smooth oriented curves ℓ_1 and ℓ_2 between C and C_ε as indicated in figure 15.1. Using the endpoints of ℓ_1 and ℓ_2, break C and C_ε into two pieces each, C^+ and C^-, and C_ε^+ and C_ε^-, respectively, as also indicated in figure 15.1. Finally, keeping track of the orientations of the subcurves, let Γ_1 and Γ_2 be the closed curves given by

$$\Gamma^+ \;=\; C^+ \;+\; \ell_1 \;-\; C_\varepsilon^+ \;+\; \ell_2 \qquad \text{and} \qquad \Gamma^- \;=\; C^- \;-\; \ell_2 \;-\; C_\varepsilon^- \;-\; \ell_1 \quad .$$

Since z_0 is not on or enclosed by either Γ^+ or Γ^-, the Basic Cauchy Integral Theorem applies and tells us that

$$\oint_{\Gamma^+} g(z)\,dz \;=\; 0 \qquad \text{and} \qquad \oint_{\Gamma^-} g(z)\,dz \;=\; 0 \quad .$$

Adding these zeroes together and then decomposing Γ_1 and Γ_2 (keeping track of the orientations), we get

$$\begin{aligned}
0 &= \oint_{\Gamma^+} g(z)\,dz \;+\; \oint_{\Gamma^-} g(z)\,dz \\
&= \int_{C^+} g(z)\,dz \;+\; \int_{\ell_1} g(z)\,dz \;-\; \int_{C_\varepsilon^+} g(z)\,dz \;+\; \int_{\ell_2} g(z)\,dz \\
&\qquad + \int_{C^-} g(z)\,dz \;-\; \int_{\ell_2} g(z)\,dz \;-\; \int_{C_\varepsilon^-} g(z)\,dz \;-\; \int_{\ell_2} g(z)\,dz \\
&= \int_{C^+ + C^-} g(z)\,dz \;-\; \int_{C_\varepsilon^+ + C_\varepsilon^-} g(z)\,dz \\
&= \oint_C g(z)\,dz \;-\; \oint_{C_\varepsilon} g(z)\,dz \quad .
\end{aligned}$$

Thus,

$$\oint_C g(z)\,dz \;=\; \oint_{C_\varepsilon} g(z)\,dz \quad . \tag{15.3}$$

If, instead of just having just one point at which g is not continuous, we have a finite collection of points $\{z_1, z_2, \ldots, z_N\}$, then a straightforward extension of the above arguments yields

$$\oint_C g(z)\,dz \;=\; \sum_{k=1}^{N} \oint_{C_{k,\varepsilon}} g(z)\,dz$$

where, for each k,

$$C_{k,\varepsilon} \;=\; \text{the circle of radius } \varepsilon \text{ about } z_k \text{, oriented counterclockwise}$$

and with $\varepsilon > 0$ chosen small enough that each closed disk $\{z : |z - z_k| \leq \varepsilon\}$ does not contain any of the z_m's except z_k and does not intersect C.

Now since this holds for any simple closed curve C in $\mathcal{R}$ encircling the set $\{z_1, z_2, \ldots, z_N\}$, we clearly have that, if C and C' are any two simple, counter-clockwise oriented loops in $\mathcal{R}$ that encircle this set, then, by choosing ϵ small enough, we have

$$\oint_C g(z)\,dz \;=\; \sum_{k=1}^{N} \oint_{C_{k,\varepsilon}} g(z)\,dz \;=\; \oint_{C'} g(z)\,dz \quad ,$$

giving us:

Theorem 15.5

Let $\{z_1, z_2, \ldots, z_N\}$ be a finite set of points in some simply-connected region $\mathcal{R}$, and assume g is a single-valued piecewise analytic function that is continuous everywhere in $\mathcal{R}$ except, possibly, at the z_k's. Let C and C' be two simple closed curves in $\mathcal{R}$, each oriented counter-clockwise and each encircling the entire set $\{z_1, z_2, \ldots, z_N\}$. Then

$$\oint_C g(z)\,dz \;=\; \oint_{C'} g(z)\,dz \quad .$$

?►Exercise 15.1: *Let C be any simple loop enclosing a point z_0 and oriented counter-clockwise. Show that*

$$\int_C \frac{1}{z - z_0}\, dz = i2\pi \quad .$$

(Hint: Use the above theorem to convert the integral to one over a convenient circle about z_0. Then recall exercise 14.14 on page 14–17.)

Also compute this integral assuming the point z_0 is outside the loop C

What if a "bad point" for a function is on the curve over which the function is being integrated? We will discuss this later after getting a better feel for just how "bad" a "bad point" can be.

15.3 Cauchy's Integral Formulas

The Basic Cauchy Integral Formula

To derive some really nifty formulas, let us start with a single-valued function f which is continuous and piecewise analytic in some simply-connected region $\mathcal{R}$. Now choose any point z_0 in that region at which f is analytic, and consider the function

$$\frac{f(z) - f(z_0)}{z - z_0} \quad .$$

Clearly, the denominator is analytic and nonzero everywhere in $\mathcal{R}$ other than $z = z_0$. So this function is at least piecewise analytic on $\mathcal{R}$ and continuous everywhere in $\mathcal{R}$ except, possibly, at the point $z = z_0$. But

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \quad .$$

So

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if} \quad z \neq z_0 \\ f'(z_0) & \text{if} \quad z = z_0 \end{cases} \quad ,$$

is continuous at every point in $\mathcal{R}$ (including z_0). The Basic Cauchy Integral Theorem then tells us that, if C is any simple loop in $\mathcal{R}$ enclosing z_0, then

$$\int_C g(z)\, dz = 0 \quad .$$

But then,

$$\begin{aligned} 0 = \int_C g(z)\, dz &= \int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz \\ &= \int_C \frac{f(z)}{z - z_0}\, dz - \int_C \frac{f(z_0)}{z - z_0}\, dz \\ &= \int_C \frac{f(z)}{z - z_0}\, dz - f(z_0) \int_C \frac{1}{z - z_0}\, dz \quad . \end{aligned}$$

Thus,

$$\int_C \frac{f(z)}{z - z_0}\, dz = f(z_0) \int_C \frac{1}{z - z_0}\, dz \quad .$$

Now recall the results from exercise 15.1, namely that the integral on the right equals $i2\pi$ if C is oriented counter clockwise. Thus, assuming C is so oriented,

$$\int_C \frac{f(z)}{z - z_0}\,dz \;=\; f(z_0)i2\pi \quad .$$

Solving this for $f(z_0)$ then gives us the *(basic) Cauchy integral formula*,

$$f(z_0) \;=\; \frac{1}{i2\pi}\oint_C \frac{f(z)}{z - z_0}\,dz \tag{15.4}$$

provided f is analytic at z_0. But if f is not analytic at z_0, the other assumptions on f and $\mathcal{R}$ allow us to pick a sequence of points — $z_1,\ z_2,\ z_3,\ \dots$ — at which f is analytic and which converge to z_0 and are enclosed by C. By the continuity of f and the obvious continuity of the integral, we have

$$f(z_0) \;=\; \lim_{k\to\infty} f(z_k) \;=\; \lim_{k\to\infty}\frac{1}{i2\pi}\oint_C \frac{f(z)}{z - z_k}\,dz \;=\; \frac{1}{i2\pi}\oint_C \frac{f(z)}{z - z_0}\,dz$$

showing that the Cauchy integral formula holds at every z_0 in $\mathcal{R}$.

All the Cauchy Integral Formulas

For psychological reasons, let's replace the symbols z and z_0 in equation (15.4) with ζ and z, respectively. This gives us

$$f(z) \;=\; \frac{1}{i2\pi}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad .$$

Remember, C is any counter-clockwise oriented simple closed curve that encircles z and is in a simply-connected region $\mathcal{R}$ on which f is continuous and piecewise analytic.

Notice how simply the above integral depends on the variable z — it only appears in the denominator of the integrand. "Clearly", this integral is a differentiable function of z with

$$\begin{aligned}
\frac{d}{dz}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta \;&=\; \oint_C \frac{\partial}{\partial z}\left[\frac{f(\zeta)}{\zeta - z}\right] d\zeta \\
&=\; \oint_C f(\zeta)\frac{\partial}{\partial z}\left[\frac{1}{\zeta - z}\right] d\zeta \\
&=\; \oint_C f(\zeta)\left[\frac{1}{(\zeta - z)^2}\right] d\zeta \;=\; \oint_C \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta
\end{aligned}$$

Thus,

$$f'(z) \;=\; \frac{d}{dz}\left[\frac{1}{i2\pi}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta\right] \;=\; \frac{1}{i2\pi}\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta \quad .$$

And, just as clearly, this last integral is also a differentiable function of z. But before we compute that derivative, let's observe that the above tells us that f is differentiable at every point in the open region $\mathcal{R}$; that is, f is actually analytic on $\mathcal{R}$, not merely "piecewise analytic" as we assumed. This is worth remembering:

Theorem 15.6

If f is a continuous and piecewise analytic on a simply-connected region, then f is actually analytic on that region.

This fact will be especially relevant if we get to discuss "analytic continuation" of functions. It also means that we might as well stop saying "continuous and piecewise analytic" and just say "analytic".

But back to computing derivatives. Using the above integral formula for $f'(z)$, we have

$$\begin{aligned} f''(z) &= \frac{d}{dz}\left[\frac{1}{i2\pi}\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta\right] \\ &= \frac{1}{i2\pi}\oint_C \frac{\partial}{\partial z}\left[\frac{f(\zeta)}{(\zeta - z)^2}\right] d\zeta \\ &= \frac{1}{i2\pi}\oint_C f(\zeta)\frac{\partial}{\partial z}\left[\frac{1}{(\zeta - z)^2}\right] d\zeta \\ &= \frac{1}{i2\pi}\oint_C f(\zeta)\left[\frac{2}{(\zeta - z)^3}\right] d\zeta \;=\; \frac{2}{i2\pi}\oint_C \frac{f(\zeta)}{(\zeta - z)^3}\,d\zeta \quad . \end{aligned}$$

Doing this one more time just to be sure we have the hang of it:

$$\begin{aligned} f'''(z) &= \frac{d}{dz}\left[\frac{2}{i2\pi}\oint_C \frac{f(\zeta)}{(\zeta - z)^3}\,d\zeta\right] \\ &= \frac{2}{i2\pi}\oint_C \frac{\partial}{\partial z}\left[\frac{f(\zeta)}{(\zeta - z)^3}\right] d\zeta \\ &= \frac{2}{i2\pi}\oint_C f(\zeta)\frac{\partial}{\partial z}\left[\frac{1}{(\zeta - z)^3}\right] d\zeta \\ &= \frac{2}{i2\pi}\oint_C f(\zeta)\left[\frac{3}{(\zeta - z)^4}\right] d\zeta \;=\; \frac{3\cdot 2}{i2\pi}\oint_C \frac{f(\zeta)}{(\zeta - z)^4}\,d\zeta \quad . \end{aligned}$$

Clearly, we could do this forever, obtaining (after a slight change of notation "for purely psychological reasons"):

Theorem 15.7 (Cauchy Integral Formulas)

Suppose f is single valued and analytic on some region $\mathcal{R}$, and z_0 is any point in $\mathcal{R}$. Then f is infinitely differentiable at z_0. Moreover, if C is a counter-clockwise oriented simple closed curve encircling z_0, and f is analytic at each point enclosed by C, then

$$f(z_0) = \frac{1}{i2\pi}\oint_C \frac{f(z)}{z - z_0}\,dz \tag{15.5}$$

and, for $n = 1, 2, 3, \ldots$,

$$f^{(n)}(z_0) = \frac{n!}{i2\pi}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \quad . \tag{15.6}$$

Equations (15.5) and (15.6) are the Cauchy integral formulas for f and its derivatives. We will mainly use them to compute certain integrals (see example below) and to derive the Laurent series (see the next chapter of notes). But there are a lot of other results that can be derived from these formulas. Some of the more notable ones are briefly discussed in the next subsection (after the technical note).

!►Example 15.1: *Consider computing*

$$\int_C \frac{z^2+3z}{z-i}\,dz \qquad \text{and} \qquad \int_C \frac{z^2+3z}{(z-i)^2}\,dz$$

where C is the circle of radius 2 about the origin, oriented counterclockwise.

In this case, $z_0 = i$, and C does go around this point, and we can use equations (15.5) and (15.6) with

$$f(z) = z^2 + 3z \qquad \text{and} \qquad f'(z) = \frac{d}{dz}\left[z^2 + 3z\right] = 2z + 3 \quad ,$$

as follows:

$$\int_C \frac{z^2+3z}{z-i}\,dz = \int_C \frac{f(z)}{z-i}\,dz$$
$$= i2\pi f(i) = i2\pi\left[i^2+3i\right] = -6\pi - i2\pi \quad .$$

and

$$\int_C \frac{z^2+3z}{(z-i)^2}\,dz = \int_C \frac{f(z)}{(z-i)^2}\,dz$$
$$= i2\pi f'(i) = i2\pi\,[2i+3] = -4\pi + i6\pi \quad .$$

A Technical Math Note

The derivation of the Cauchy integral formulas for the derivatives were based on the "well-known" fact that

$$\frac{d}{dz}\int_C G(z,\zeta)\,d\zeta = \int_C \frac{\partial}{\partial z}G(z,\zeta)\,d\zeta \quad ,$$

at least whenever $G(z,\zeta)$ is a reasonable function of z. At some point, however, the more careful student will insist on a proof of this fact, instead of vague assurances that it is true. Well, here is a brief outline of the easy part of that proof:

$$\frac{d}{dz}\int_C G(z,\zeta)\,d\zeta = \lim_{\Delta z\to 0}\frac{\int_C G(z+\Delta z,\zeta)\,d\zeta - \int_C G(z,\zeta)\,d\zeta}{\Delta z}$$
$$= \lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_C \left[G(z+\Delta z,\zeta) - G(z,\zeta)\right]\,d\zeta$$
$$= \lim_{\Delta z\to 0}\int_C \frac{G(z+\Delta z,\zeta) - G(z,\zeta)}{\Delta z}\,d\zeta$$

Now, if we can show that

$$\lim_{\Delta z\to 0}\int_C \frac{G(z+\Delta z,\zeta) - G(z,\zeta)}{\Delta z}\,d\zeta = \int_C \lim_{\Delta z\to 0}\frac{G(z+\Delta z,\zeta) - G(z,\zeta)}{\Delta z}\,d\zeta \quad , \tag{15.7}$$

then the previous sequence of limits reduces to

$$\frac{d}{dz}\int_C G(z,\zeta)\,d\zeta = \lim_{\Delta z\to 0}\int_C \frac{G(z+\Delta z,\zeta) - G(z,\zeta)}{\Delta z}\,d\zeta$$
$$= \int_C \lim_{\Delta z\to 0}\frac{G(z+\Delta z,\zeta) - G(z,\zeta)}{\Delta z}\,d\zeta = \int_C \frac{\partial}{\partial z}G(z,\zeta)\,d\zeta \quad ,$$

proving the claim. The only trick is in confirming the interchanging of integration and the limit process indicated in equation (15.7). To properly discuss this, however, requires that some development of the concept of "uniform continuity", which is probably going deeper into theory than we have time for. So trust me that, at least as long as $\mathcal{C}$ is a closed (or finite curve) and $G(z, \zeta)$ is a "reasonably smooth" function of z and ζ, equation (15.7) holds. And trust me that our use of this in deriving the Cauchy integral formulas is valid.[4]

Some Other Consequences of the Cauchy Integral Formulas

One conclusion we can immediately draw from the Cauchy integral formulas is that any function analytic on a region is, in fact, infinitely analytic on that region. That is, the n^{th} derivative of the function exists at every point in $\mathcal{R}$ for every positive integer n. Thus, it also follows that every derivative of an analytic function is also an analytic function on the region where the original function is analytic.

Now let's go back to considering whether the converse of the Cauchy integral theorem holds. Let f be a single-valued, continuous function on a simply-connected region $\mathcal{R}$, and assume

$$\oint_{\mathcal{C}} f(\zeta)\, d\zeta \;=\; 0$$

for every simple closed curve in $\mathcal{R}$. Then, as noted earlier, we can pick any single point z_0 and define a new function F on $\mathcal{R}$ by

$$F(z) \;=\; \int_{z_0}^{z} f(\zeta)\, d\zeta \quad .$$

It is "easily verified" that F is analytic on $\mathcal{R}$ with $F' = f$. But we've just discovered that the derivative of analytic function is also analytic. So f, the derivative of the analytic function F on $\mathcal{R}$, is also analytic on $\mathcal{R}$. Thus, for a continuous single-valued function f on a simply-connected region $\mathcal{R}$,

$$f \text{ is integrable} \quad \Longrightarrow \quad f \text{ is analytic} \quad .$$

The statement of this fact is called *Morera's Theorem*. Combined with the Cauchy Integral Theorem, we now have, for any continuous single-valued function f on a simply-connected region $\mathcal{R}$,

$$f \text{ is integrable} \quad \Longleftrightarrow \quad f \text{ is analytic} \quad .$$

Other results are more easily seen if we use the Cauchy integral formulas with the curve $\mathcal{C}$ being a circle of radius R centered at z_0. This curve can be parameterized by

$$\zeta \;=\; z_0 + Re^{i\theta} \qquad \text{for} \quad 0 \;<\; \theta \;<\; 2\pi \quad .$$

Using this parametrization,

$$\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \;=\; \frac{f\big(z_0 + Re^{i\theta}\big)}{(Re^{i\theta})^{n+1}}\, d\big[Re^{i\theta}\big] \;=\; i R^{-n} f\big(z_0 + Re^{i\theta}\big)\, e^{-in\theta}\, d\theta \quad ,$$

[4] The interchanging of limits and integration may seem "obviously valid", but, in fact, a good mathematician can come up with examples for which "$\lim \int \neq \int \lim$". If you really want to see an example, look up problem 7.8 on page 91 of *Principles of Fourier Analysis* by Howell.

and the Cauchy integral formulas (equations (15.5) and (15.6)) become

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + Re^{i\theta}\right) d\theta \tag{15.8}$$

and, for $n = 1, 2, 3, \ldots$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi R^n} \int_0^{2\pi} f\left(z_0 + Re^{i\theta}\right) e^{-in\theta} d\theta \quad . \tag{15.9}$$

Taking absolute values then leads to the inequalities

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(z_0 + Re^{i\theta}\right)\right| d\theta \tag{15.10}$$

and, for $n = 1, 2, 3, \ldots$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi R^n} \int_0^{2\pi} \left|f\left(z_0 + Re^{i\theta}\right)\right| d\theta \quad . \tag{15.11}$$

From these equalities and inequalities, we can derive the following facts:

1. The value of $f(z_0)$ is the average of the values of $f(z)$ around circles centered at z_0. That, essentially, is exactly what equation (15.8) is telling us. This is sometimes called the *mean-value theorem of Gauss*.

2. The value of $|f(z_0)|$ is less than or equal to the average of the values of $|f(z)|$ around circles centered at z_0. That, essentially, is exactly what equation (15.8) is telling us.

3. (The *maximum principle*) The maximum value of $|f(z)|$ cannot occur at a point z_0 inside the region $\mathcal{R}$ on which it is analytic unless f is a constant on $\mathcal{R}$. If you think about the previously mentioned fact, and what it means to be less than or equal to the average of the values of $|f(z)|$ around circles, you may suspect that this fact can be proven using equation (15.8). You would be right (but a complete and rigorous proof does take a little cleverness).[5]

4. (Liouville's Theorem) If f is a bounded analytic function on the entire complex plane, then it must be a constant. This follows very easily from equation (15.11).

?►Exercise 15.2: *Assume f is analytic and bounded on the entire complex plane; that is, f is analytic at every point in $\mathbb{C}$, and there is some finite value M*

$$|f(z)| \leq M \qquad \text{for each} \quad z \in \mathbb{C} \quad .$$

Using equation (15.11), show that $f'(z) = 0$ for every $z \in \mathbb{C}$. (Hint, what happens if $R \to \infty$?) How does this essentially prove Liouville's theorem?

By the way, using Liouville's theorem you can prove that any nonconstant polynomial

$$p(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_2 z^2 + A_1 z + A_0$$

[5] A few versions of this principle — also called the *maximum modulus principle* — are noted in the appendix which follows.

has a root somewhere on the complex plane. Basically, you start by assuming $p(z)$ does not have a root, let $f(z) = {}^{1}/_{p(z)}$, and note that $f(z)$ then must be a bounded analytic function on $\mathbb{C}$. Liouville's theorem then tells you that $f(z)$ — and hence $p(z)$ — must be constant. Thus, if $p(z)$ is a *non*constant polynomial, it must have a root somewhere on the complex plane. From this you can build the proof that every n^{th} order polynomial can be written in completely factored form,

$$p(z) = A(z - z_1)(z - z_2)(z - z_3) \cdots (z - z_n) \quad .$$

15.4 APPENDIX

The following was taken from handouts prepared several years ago when I taught MA 656, Complex Variables I.:

Goursat's Proof of the Cauchy Integral Theorem

We can rephrase the classic Cauchy integral theorem[6] as follows:

> *Let C be a simple closed piecewise-smooth curve in $\mathbb{C}$, and let f be any function that is analytic at every point on and enclosed by C. Then*
>
> $$\int_C f(z)\, dz = 0$$

What follows is Goursat's proof of this theorem.[7] There are three parts. The first involves an iterative process to derive a bound on

$$\left| \int_{\mathcal{C}} f(z)\, dz \right|$$

in terms of an similar integral, but over a much smaller curve. That "similar integral" is further analyzed in the second part. The third part will be an exercise in which you use the results of the first two part to finish the proof.

PROOF (First part — the iterative process):

THE STARTING POINT: We start by enclosing the curve C in a rectangle whose sides are parallel to the X and Y axes. Using this rectangle, we set

$$Rect^0 = \text{the rectangular (open) region enclosed by the rectangle}$$

$$\mathcal{R}^0 = Rect^0 \cap \text{open region enclosed by } \mathcal{C} \quad (= \text{ open region enclosed by } \mathcal{C})$$

$$\mathcal{C}^0 = \text{boundary of } \mathcal{R}^0 \quad (= \mathcal{C})$$

$$B_0 = \text{base length of } Rect^0$$

$$H_0 = \text{height of } Rect^0$$

[6] also called the Cauchy-Goursat Theorem in some texts

[7] Actually, it's my version of someone else's version of … of Goursat's proof.

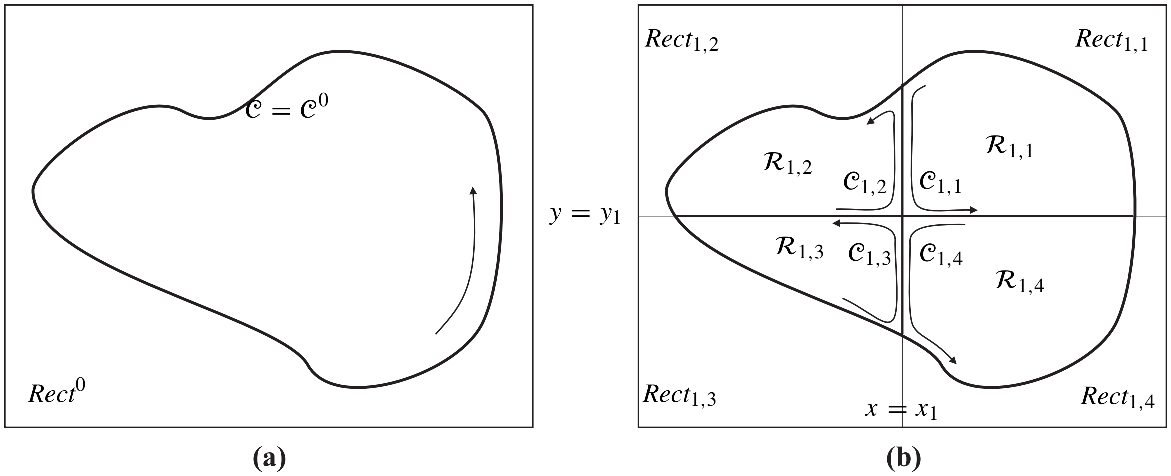

Figure 15.2: The curve $\mathcal{C}$ and the rectangles for (a) the "starting point" and (b) the "first iterative step".

$$x_0 = x \text{ coordinate of the midpoint on the horizontal sides of } Rect^0$$
$$y_0 = y \text{ coordinate of the midpoint on the vertical sides of } Rect^0$$

(See figure 15.2.)

THE FIRST ITERATIVE STEP: Partition the rectangular region $Rect^0$ into four congruent open rectangular regions

$$Rect_{1,1} \quad , \quad Rect_{1,2} \quad , \quad Rect_{1,3} \quad \text{and} \quad Rect_{1,4}$$

using horizontal and vertical lines through the midpoints of the sides of $Rect^0$. Set

$$\mathcal{R}_{1,k} = Rect_{1,k} \cap \mathcal{R}^0 \qquad \text{for} \quad k = 1,\ 2,\ 3,\ 4$$

and

$$\mathcal{C}_{1,k} = \text{boundary of } \mathcal{R}_{1,k} \qquad \text{for} \quad k = 1,\ 2,\ 3,\ 4 \quad .$$

Observe that any portion of each $\mathcal{C}_{1,k}$ that is not part of the original curve $\mathcal{C}$ is also a portion of another $\mathcal{C}_{1,k}$, but with opposite orientation. Consequently,

$$\sum_{k=1}^{4} \int_{\mathcal{C}_{1,k}} f(z)\, dz = \int_{\mathcal{C}} f(z)\, dz \quad .$$

(Note: If $\mathcal{R}_{1,k}$ is empty, then so is $\mathcal{C}_{1,k}$. In this case, we automatically define the integral of f over $\mathcal{C}_{1,k}$ to be zero.) Clearly, there must be an integer $\ell \in \{1, 2, 3, 4\}$ such that both of the following hold:

1. $\mathcal{R}_{1,\ell}$ is not empty.

2. $\left|\int_{\mathcal{C}_{1,\ell}} f(z)\, dz\right| = \max\left\{\left|\int_{\mathcal{C}_{1,k}} f(z)\, dz\right| : k = 1,\ 2,\ 3,\ 4\right\}$

Using that ℓ , set

$$\begin{aligned}
Rect^1 &= Rect_{1,\ell} \\
\mathcal{R}^1 &= Rect^1 \cap \mathcal{R}^0 \\
\mathcal{C}^1 &= \text{boundary of } \mathcal{R}^1 \\
B_1 &= \text{base length of } Rect^1 \\
H_1 &= \text{height of } Rect^1 \\
x_1 &= x \text{ coordinate of the midpoint on the horizontal sides of } Rect^1 \\
y_1 &= y \text{ coordinate of the midpoint on the vertical sides of } Rect^1
\end{aligned}$$

(Again, see figure 15.2.) Do note that

$$B_1 = \frac{1}{2}B_0 \qquad \text{and} \qquad H_1 = \frac{1}{2}H_0 \quad .$$

Also,

$$\left|\int_{\mathcal{C}} f(z)\,dz\right| = \left|\sum_{k=1}^{4}\int_{\mathcal{C}_{1,k}} f(z)\,dz\right| \leq \sum_{k=1}^{4}\left|\int_{\mathcal{C}_{1,k}} f(z)\,dz\right| \leq \sum_{k=1}^{4}\left|\int_{\mathcal{C}^1} f(z)\,dz\right| \quad ,$$

which, by basic algebra, simplifies to

$$\left|\int_{\mathcal{C}} f(z)\,dz\right| \leq 4\left|\int_{\mathcal{C}^1} f(z)\,dz\right| \quad . \tag{15.12}$$

ITERATIVE STEP #m: Simply repeat the first iterative step, replacing $Rect^0$ with $Rect^{m-1}$.

RESULTS OF THE ITERATION: We have a nested sequence of rectangular regions

$$Rect^0 \supset Rect^1 \supset Rect^2 \supset \cdots \supset Rect^{m-1} \supset Rect^m \supset Rect^{m+1} \supset \cdots$$

with each $Rect^m$ having base length and height

$$B_m = \frac{1}{2}B_{m-1} = \cdots = \frac{1}{2^m}B_0$$

and

$$H_m = \frac{1}{2}H_{m-1} = \cdots = \frac{1}{2^m}H_0 \quad ,$$

respectively. In addition, for each positive integer m , we have (see figure 15.3)

$$\begin{aligned}
x_m &= x \text{ coordinate of the midpoint of the horizontal sides of } Rect^m \\
y_m &= y \text{ coordinate of the midpoint of the vertical sides of } Rect^m \\
\mathcal{R}^m &= Rect^m \cap \mathcal{R}^0 \neq \text{ the empty set} \\
\mathcal{C}^m &= \text{boundary of } \mathcal{R}^m \neq \text{ the empty set}
\end{aligned}$$

and

$$\left|\int_{\mathcal{C}} f(z)\,dz\right| \leq 4^m\left|\int_{\mathcal{C}^m} f(z)\,dz\right| \quad . \tag{15.13}$$

You can also easily confirm the following:

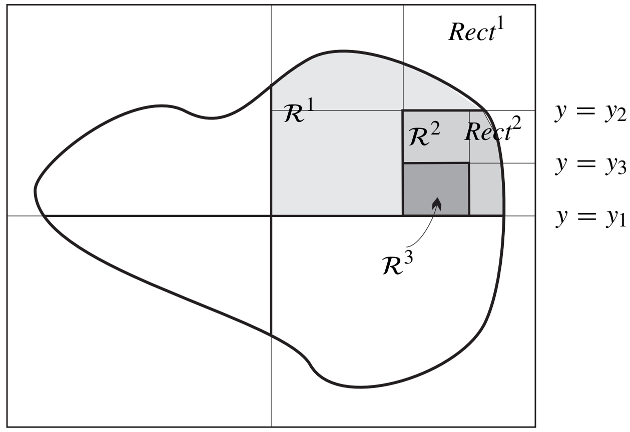

Figure 15.3: A crude attempt to indicate the nested sequences of rectangular regions and R^m s resulting from the iterative process.

1. The sequences $\{x_m : m \in \mathbb{N}\}$ and $\{y_m : m \in \mathbb{N}\}$ are convergent, allowing us to define

$$z_\infty = \lim_{m\to\infty} x_m + i y_m \quad .$$

2. z_∞ is in the closure of every $\mathcal{R}^m$.[8]

3. f is analytic at z_∞ .

4. If $z \in \mathcal{C}^m$, then $|z - z_\infty| \leq \frac{1}{2^m}[B_0 + H_0]$.

?►*Exercise 15.3:* *Verify the above statements.*

PROOF (Second Part): Here, we "compute" $\int_{\mathcal{C}^m} f(z)\,dz$ using the fact that f is analytic at z_∞ . For $z \in \mathcal{C}^m$ (and letting $\Delta z = z - z_\infty$), this analyticity tells us that

$$\begin{aligned} f(z) &= f(z_\infty + \Delta z) \\ &= f(z_\infty) + f'(z_\infty)\Delta z + \Delta z\, \varepsilon(z_\infty, \Delta z) \\ &= f(z_\infty) + f'(z_\infty)[z - z_\infty] + \Delta z\, \varepsilon(z_\infty, \Delta z) \\ &= A + Bz + [z - z_\infty]E(z) \end{aligned}$$

where

$$A = f(z_\infty) - f'(z_\infty)z_\infty \qquad \text{and} \qquad B = f'(z_\infty)$$

are constants, and

$$E(z) = \varepsilon(z_\infty, z - z_\infty) \to 0 \quad \text{as} \quad z \to z_\infty \quad .$$

From homework, we already know that, since $\mathcal{C}^m$ is closed,

$$\int_{\mathcal{C}^m} A\,dz = 0 \qquad \text{and} \qquad \int_{\mathcal{C}^m} Bz\,dz = 0 \quad .$$

[8] In fact, $z_\infty = \bigcap_{m=1}^{\infty}$ closure of $Rect^m = \bigcap_{m=1}^{\infty}$ closure of $\mathcal{R}^m$.

So

$$\int_{\mathcal{C}^m} f(z)\,dz \;=\; \int_{\mathcal{C}^m} A + Bz + [z - z_\infty]E(z)\,dz \;=\; \int_{\mathcal{C}^m} [z - z_\infty]E(z)\,dz \quad . \tag{15.14}$$

?►*Exercise 15.4:* *Finish the proof by using the results of the first two parts and results developed in class, to show that*

$$\left|\int_{\mathcal{C}} f(z)\,dz\right| \quad ,$$

which does not depend on m *, is bounded by something that shrinks to zero as* $m \to \infty$ *.*

Maximum Modulus Theorems

We should note *three* "Maximum Modulus Theorems":

Weak Local Maximum Modulus Theorem
Let f *be analytic on some domain* $\mathcal{D}$ *. If* $|f(z)|$ *has a local maximum at a point* $z_0 \in \mathcal{D}$ *, then* f *is constant in a neighborhood of* z_0 *(and that neighborhood is the same as the neighborhood on which* $|f(z_0)|$ *is a maximum).*[9]

Global Maximum Modulus Theorem
Let f *be analytic on some domain* $\mathcal{D}$ *. If* $|f|$ *has a global maximum in* $\mathcal{D}$ *, then* f *is constant on* $\mathcal{D}$ *.*

Strong Local Maximum Modulus Theorem
Let f *be analytic on some domain* $\mathcal{D}$ *. If* $|f|$ *has a local maximum in* $\mathcal{D}$ *, then* f *is constant on* $\mathcal{D}$ *.*

[9] Remember: We've defined a neighborhood about a point to be an open disk centered at the point. That this theorem also holds for the more general notions of "neighborhood" will follow from the same arguments used to prove the global maximum modulus theorem.