2

Traditional Vector Theory

The earliest definition of a "vector" usually encountered is that a vector is a thing possessing "length" and "direction". This is the "arrow in space" view with "length" naturally being the length of the arrow and "direction" being the direction the arrow is pointing. We will denote vectors by writing their name in boldface (e.g., $\mathbf{v}$ and $\mathbf{a}$) or by drawing a little arrow above their name (e.g., $\vec{v}$ and $\vec{a}$).

Some examples of such vectors include the velocity of an object at a particular time and the acceleration of an object at a particular time. (For now, we are *not* considering vector fields; that is, our vectors will not be functions of time or position.)

Perhaps the most fundamental of vectors are those describing "relative position" or "displacement": If A and B are two points in space (i.e., positions in space), then *the vector from* A *to* B, denoted $\overrightarrow{AB}$, is simply the arrow[1] starting at point A and ending at point B. $\overrightarrow{AB}$ gives you the direction and distance to move from position A to position B (hence the term "relative position").

We will use "displacement vectors" as the basic model for traditional vectors. Almost all other traditional vectors in physics — velocities, accelerations, forces, etc. — are derived from displacement vectors. For example, the velocity $\mathbf{v}$ of an object at a given position p is the limit

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\overrightarrow{pq}}{\Delta t}$$

where q is the position of the object Δt time units after being at p. It is through these derivations that the properties we derive for displacement (relative position) vectors can be shown to hold for all the traditional vectors in physics.

By the way, throughout this discussion, we are assuming that the points in space are points in a "Euclidean" space. We will discuss exactly what this means later (in another chapter). For now, "assume high school geometry". In particular:

1. Any two points can be connected by a straight line segment (totally contained in the space) whose length is the distance between the two points.

2. The angles of each triangle add up to π (or 180 degrees).

3. The laws of similar triangles hold.

4. Parallelograms are well defined.

[1] officially, $\overrightarrow{AB}$ is a "directed line segment".

Keep in mind that there are different Euclidean spaces. Two different "flat" planes, for example, are two different Euclidean spaces.

?►*Exercise 2.1:* *Give several reasons why a sphere is not a Euclidean space.*[2]

2.1 Fundamental Defining Concepts

Fundamental Geometrically Defined Concepts

The Two Fundamental Measurable Quantities

Our goal is to develop the fundamental theory of vectors as things that are completely defined by "length" and "direction", using the set of vectors describing relative position in some Euclidean space as a basic model. Keep in mind the requirement that "length" and "direction" completely defines a vector here. So if we have two "arrows", both pointing in the same direction and of the same length, then they are the same vector.

?►*Exercise 2.2:* *Let A, B, C and D be points in space. How does the concept that "Parallelograms are well defined" allow us to decide when $\overrightarrow{AB} = \overrightarrow{CD}$?*

With traditional vectors, there are two fundamental measurable quantities to work with:

Length: The *length* (also called the *magnitude* or *norm*) of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$ or $|\mathbf{v}|$ or v, depending on the whim of the author.

Angle Between Two Vectors: While it is intuitively clear what "direction" means, it is difficult to quantify. What can be measured (physically, in theory) is the smallest angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ visualized with their starting points touching. We will typically denote this angle by θ or $\theta(\mathbf{u}, \mathbf{v})$, and we will use radians or degrees to measure this angle as seems convenient. Do note that

$$\theta(\mathbf{u}, \mathbf{v}) = \theta(\mathbf{v}, \mathbf{u}) \quad .$$

It should be noted that "measurable" means that we can (conceptually, at least) find these quantities by directly measuring the length and the angle with fairly basic real-world tools. Consequently, we automatically have

$$\|\mathbf{v}\| \geq 0 \qquad \text{and} \qquad 0 \leq \theta \leq \pi \quad .$$

(We may later relax this restriction on θ.)

Units for Length and "Types" of Vectors

In many basic developments of traditional vector theory, the length of a vector, $\|\mathbf{v}\|$, is treated as a simple, unadorned real number. In fact, however, any vector length measurement is inherently

[2] Terminology: A sphere is just the surface of a ball. It does not contain the inside of the ball.

in terms of some “measurement unit” such as feet, meters, miles/hour, meters/second^2, newtons, etc. Even if we don’t explicitly state the “unit of length” for a vector we sketch on paper, we must have some notion of what “one unit of length” is, and that unnamed quantity is our basic unit of length — give it a name if you wish.

What’s more, we need to realize that there are different “types” of vectors according to what is being measured. For any vector describing displacement from one position to another, “length” really means distance in the normal sense, and can be measured in feet, meters, furlongs, and so forth. This is one type of vector. On the other hand, the “length” of a vector describing the velocity of an object at a given time is really the “speed” of that object, and can be measured in feet/second, meters/hour, furlongs/fortnight, and so on. The set of “velocity vectors” make up another type of vector. And we also have “acceleration vectors”, “force vectors”, “flux vectors”, ...

Subsequent Geometric Concepts

All of our traditional vector computations resulting in scalar values will be based on “length” and “angle” . Note that, right off, we can use the length to define one very special vector and a set of vectors that we may find useful:

1. The *zero vector*, denoted by $\mathbf{0}$ or $\vec{0}$, is the vector of length zero. One can argue that there are infinitely many zero vectors, each corresponding to a different direction, but that is being silly. The zero vector, in our model of relative position, corresponds to just “not moving from a point”. For that and other practical reasons, we take the position that there is only one zero vector in each set of vectors

2. A *unit vector* is any vector $\vec{v}$ of length one (i.e., $\|\vec{v}\| = 1$). If it is significant that this is a unit vector, then the vector may be denoted by $\hat{v}$ or $\hat{\mathbf{v}}$, instead of $\vec{v}$. We will find various uses for unit vectors.

And with the concept of “angles between vectors”, we can introduce terminology for when vectors point in special directions relative to each other. Most of this terminology is what we use everyday.

Parallelness: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be *parallel* if either they “point in the same direction” or they “point in opposite directions”. “Obviously”,

$$\text{“}\mathbf{u} \text{ and } \mathbf{v} \text{ point in the same direction”} \quad \Longleftrightarrow \quad \theta(\mathbf{u},\mathbf{v}) = 0$$

and

$$\text{“}\mathbf{u} \text{ and } \mathbf{v} \text{ point in opposite directions”} \quad \Longleftrightarrow \quad \theta(\mathbf{u},\mathbf{v}) = \pi \quad .$$

Orthogonality:

(a) A pair of vectors $\{\mathbf{u},\mathbf{v}\}$ is said to be *orthogonal*

$$\Longleftrightarrow \quad \text{Either } \theta(\mathbf{u},\mathbf{v}) = \frac{\pi}{2} \text{ or } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} .$$

(Thus, we’ve defined “orthogonality” so that $\mathbf{0}$ is automatically orthogonal to every vector. The reasons we include this in our definition is simply that it will simplify statements and formulas later.)

On occasion, we will indicate that $\{\mathbf{u},\mathbf{v}\}$ is orthogonal by writing $\mathbf{u} \perp \mathbf{v}$.

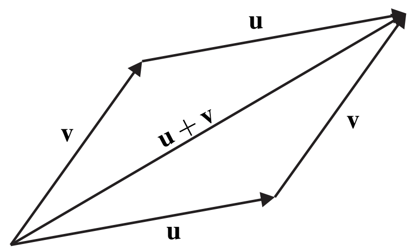

Figure 2.1: The vector sum, based on the well-defined parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as two sides.

(b) A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K\}$ is said to be *orthogonal*

$$\iff \quad \text{Every distinct pair in the set is orthogonal.}$$

With both "length" and "angle" we can define

Orthonormality: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K\}$ is said to be *orthonormal*

$$\iff \quad \text{It is an orthogonal set of unit vectors.}$$

Using orthogonal or orthonormal sets of vectors will greatly simplify life.

Fundamental Geometrically Defined Algebraic Operations

There are two (to begin with):

Scalar Multiplication of a Vector: Let α be a scalar[3] and let $\mathbf{v}$ be a vector. The product of α with $\mathbf{v}$, denoted $\alpha\mathbf{v}$ (or $\alpha \cdot \mathbf{v}$), is defined as follows:

If $\alpha > 0$, then $\alpha\mathbf{v}$ is the vector of length $\alpha \|\mathbf{v}\|$ pointing in the same direction as $\mathbf{v}$.

If $\alpha < 0$, then $\alpha\mathbf{v}$ is the vector of length $|\alpha| \|\mathbf{v}\|$ pointing in the direction opposite of $\mathbf{v}$.

If $\alpha = 0$, then $\alpha\mathbf{v} = \mathbf{0}$

Along these lines, we automatically have that the division of a vector $\mathbf{v}$ by a nonzero scalar α is given by

$$\frac{\mathbf{v}}{\alpha} = \frac{1}{\alpha} \cdot \mathbf{v} \quad .$$

Vector Addition: Within our model of vectors representing relative position, the addition of $\overrightarrow{AB}$ to $\overrightarrow{BC}$ naturally corresponds to "go from position A to position B" and then "go from position B to position C". The result, of course, is that you've gone from position A to position C. So we define the sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ to be $\overrightarrow{AC}$, and write

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad .$$

Geometrically, this is exactly the same as defining the sum of two vectors $\mathbf{u}$ and $\mathbf{v}$ using a parallelogram as in figure 2.1.

3 Until otherwise announced, assume all scalars are real numbers.

From the figure (and our assumptions about 'Euclideaness'), it should be obvious that

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad .$$

Other properties of addition and scalar multiplication should also be "obvious" or easily verified.

?►*Exercise 2.3:* *Using just the definitions, assumptions, pictures, and high school geometry, convince yourself that*

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v} \qquad \text{and} \qquad (\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$$

for arbitrary vectors $\mathbf{u}$ *and* $\mathbf{v}$ *, and arbitrary scalars* α *and* β *.*[4]

?►*Exercise 2.4:* *Convince yourself that vector addition is associative, i.e., that, for any three vectors* $\mathbf{u}$ *,* $\mathbf{v}$ *and* $\mathbf{w}$ *,*

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \quad .$$

at least when the vectors are defined in terms of relative position, as above.

Keep in mind that vector addition requires that the vectors be of the same "type". Adding a velocity vector to an acceleration vector makes no sense.

Decomposing Vectors, Part I (Length and Direction)

Let $\mathbf{v}$ be a nonzero (traditional) vector. Corresponding to $\mathbf{v}$ is the corresponding "unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$" is given by

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \quad .$$

Note that

$$\|\mathbf{u}\| = \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1 \quad ,$$

so, as the terminology suggests, it really is a unit vector. Also,

$$\mathbf{v} = 1\,\mathbf{v} = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|}\mathbf{v} = \|\mathbf{v}\| \frac{\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{v}\|\,\mathbf{u} \quad .$$

Thus, we can "decompose" (or "resolve") a nonzero vector $\mathbf{v}$ into the product of its length times the unit vector in the direction of $\mathbf{v}$. This will be useful in a number of situations.

2.2 Traditional Vector Spaces

Using the four geometric concepts discussed above (length, angle between vectors, scalar multiplication and vector addition), we can now develop the rest of the theory for traditional vectors. And as any good mathematician can tell you, half the work is coming up with the right definitions.

By the way, please note two things:

[4] Really convince yourself. Don't just say, "Oh yeah, I know it's true". Be able to convince skeptics.

1. At this point, our theory is completely "component free" — we are viewing traditional vectors the way God does. All those component formulas you've mucked about with for years have yet to be derived.

2. Much of the terminology and development that follows will apply to more general, non-traditional sets of "vectors". We will use this fact when we generalize our results.

I should also mention that K, N and M will denote arbitrary fixed positive integers throughout the rest of this chapter (unless otherwise indicated).

Linear Combinations and Related Notions

Linear Combinations and Span

A *linear combination* of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ... and $\mathbf{v}_N$ is any expression of the form

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_N\mathbf{v}_N$$

where α_1, α_2, ...and α_n are scalars. It is normally assumed that the number of terms in a linear combination (N in the above) is finite. However, when we get to more general vector spaces, especially the vector spaces that will be useful in solving partial differential equations, we will start allowing linear combinations with infinitely many terms (we will then also have worry about convergence issues).

Observe that the subtraction of one vector from another is just a particularly simple linear combination,

$$\mathbf{u} - \mathbf{v} = (1)\mathbf{u} + (-1)\mathbf{v} \quad .$$

The following should be obvious:

$$\mathbf{u} - \mathbf{u} = \mathbf{0}$$

and

$$\mathbf{w} = \mathbf{u} - \mathbf{v} \iff \mathbf{w} + \mathbf{v} = \mathbf{u}$$

?►*Exercise 2.5:* *Draw two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *with their "starting points" touching. (Make them of different lengths and pointing in different directions!) On this figure, sketch*

$$\mathbf{u} + \mathbf{v} \quad , \quad \mathbf{u} - \mathbf{v} \quad , \quad \mathbf{v} - \mathbf{u} \quad \textit{and} \quad 2\mathbf{u} + 4\mathbf{v} \quad .$$

Some more terminology: Given a set of vectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots\}$$

(not necessarily finite), the *span* of this set is the set of all (finite) linear combinations of these $\mathbf{v}_k$'s.

Linear (In)dependence

We will be interested in "minimal" sets of vectors for generating particular collections of linear combinations. To label when such a set

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_N\}$$

is or is not "minimal" in this sense, we have the following terminology:

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_N\}$ is *linearly dependent*

$\iff$ One of the $\mathbf{v}_k$'s can be expressed as a linear combination of the others.

and

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_N\}$ is *linearly **in**dependent*

$\iff$ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_N\}$ is **not** linearly dependent

$\iff$ None of the $\mathbf{v}_k$'s can be expressed as a linear combination of the others.

Our interest will often be in linearly *in*dependent sets. They will be the "minimal" sets for constructing linear combinations.

?►*Exercise 2.6:* *Why can the zero vector,* $\mathbf{0}$ *, never be in a linearly independent set of vectors.*

Alternative definitions for linear dependence and independence (which should really be considered as tests for linear dependence and independence) are described in the next exercises.

?►*Exercise 2.7:* *Show that*

a: $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ *is linearly dependent (as defined above) if and only if there are scalars* α_1 , α_2 , α_3 , ... *and* α_N *— not all zero — such that*

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_N\mathbf{v}_N = 0 \quad .$$

b: $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ *is linearly independent (as defined above) if and only if the only choice for scalars* α_1 , α_2 , α_3 , ... *and* α_N *such that*

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_N\mathbf{v}_N = 0$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$.

?►*Exercise 2.8:* *Come up with at least one alternative definition/test for the linear independence of* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ *based on the* span *of this set and subsets of this set.*

Vector Spaces

Let $\mathcal{V}$ be some (big) set of traditional vectors (all are the same "type" so they can be added together). Then this set is called a *vector space* (of traditional vectors) if and only if $\mathcal{V}$ is "closed" with respect to linear combinations. And what does "$\mathcal{V}$ is 'closed' with respect to linear combinations" mean? It means that every linear combination of vectors in $\mathcal{V}$ is another vector in $\mathcal{V}$. That is, a set $\mathcal{V}$ of vectors is a vector space if and only if, for each

$$N \in \mathbb{N} \quad , \quad \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N \in \mathcal{V} \quad \text{and} \quad \alpha_1, , \alpha_2, \ldots, \alpha_N \in \mathbb{R} \quad ,$$

we have

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_N\mathbf{v}_N \in \mathcal{V} \quad .$$

For example, all the displacement vectors generated from points in a single plane is a vector space.

Note that the span of a set of vectors is automatically a vector space. It is often helpful to find minimal sets of vectors that span a given vector space $\mathcal{V}$. Such a set is called a "basis" for $\mathcal{V}$. That is, a *basis* for a vector space $\mathcal{V}$ is any (finite) set of vectors

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N\}$$

(all from $\mathcal{V}$) that satisfies both of the following:

1. $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N\}$ is linearly independent .
2. Each vector in $\mathcal{V}$ is a linear combination of vectors from $\mathcal{B}$. That is, if $\mathbf{v}$ is any vector in $\mathcal{V}$, then there is a corresponding ordered set of N scalars $(v_1, v_2, \ldots, v_N)$ such that

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_N\mathbf{b}_N \quad .$$

The v_k's just described are called the *(scalar) components*[5] of $\mathbf{v}$ with respect to the basis $\mathcal{B}$.

The following three statements are all easily verified. In them

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N\}$$

is a basis for some vector space $\mathcal{V}$, $\mathbf{v}$ is some vector in $\mathcal{V}$, and $(v_1, v_2, \ldots, v_N)$ are the components of $\mathbf{v}$ with respect to $\mathcal{B}$.

1. The v_k's are unique. That is, given any set of N scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$, then

$$\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots + \alpha_N\mathbf{b}_N = \mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_N\mathbf{b}_N$$

   if and only if

$$\alpha_1 = v_1 \quad , \quad \alpha_2 = v_2 \quad , \quad \ldots \quad \text{and} \quad \alpha_N = v_N \quad .$$

2. Any set of more than N vectors in $\mathcal{V}$ can*not* be linearly independent.
3. Any set of fewer than N vectors in $\mathcal{V}$ can*not* span all of $\mathcal{V}$ (i.e., there will be vectors in $\mathcal{V}$ that cannot be expressd as linear combinations of vectors from that set).

The last two facts tell us that a set of vectors can be a basis for $\mathcal{V}$ only if it is a linearly independent set of exactly N vectors. On the other hand, if we have a linearly independent set of N vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$, and $\mathbf{w}$ is another vector in $\mathcal{V}$, then $\mathbf{w}$ must be a linear combination of the $\mathbf{v}_k$'s — otherwise $\{\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ would be a linearly independent set of more than N vectors, contrary to fact 2, above. Consequently, we must have

4. A set of vectors in $\mathcal{V}$ is a basis for $\mathcal{V}$ if and only if it is a linearly independent set of exactly N vectors.

The number N of vectors in any basis of a vector space $\mathcal{V}$ is called the *dimension* of that vector space. In applications with traditional vectors (relative positions, velocities, etc.), the dimension is finite, usually 3 (or 2 if we want to simplify things).

It is often convenient to be able to switch from one basis to another. We will discuss this in great detail later. For now:

[5] NOT the "coordinates" of $\mathbf{v}$! Vectors do NOT have "coordinates".

?▶*Exercise 2.9:* *Let $\mathcal{V}$ be a two-dimensional vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and let*

$$\mathbf{c}_1 = 2\mathbf{b}_1 + 3\mathbf{b}_2 \quad \text{and} \quad \mathbf{c}_2 = \mathbf{b}_1 - \mathbf{b}_2 \quad .$$

a: *Is $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ a basis for $\mathcal{V}$. (Give good reasons for your answer, based on the above facts.)*

b: *Suppose $\mathbf{v} = 5\mathbf{b}_1 + 4\mathbf{b}_2$.*

Find $\mathbf{v}$ in terms of $\mathbf{c}_1$ and $\mathbf{c}_2$. In other words, find scalars a and b so that

$$\mathbf{v} = a\mathbf{c}_1 + b\mathbf{c}_2 \quad .$$

What are the components of $\mathbf{v}$ with respect to $\mathcal{B}$?

What are the components of $\mathbf{v}$ with respect to C ?

c: *Find $\mathbf{b}_1$ and $\mathbf{b}_2$ in terms of $\mathbf{c}_1$ and $\mathbf{c}_2$.*

d: *Suppose $\mathbf{w} = 5\mathbf{c}_1 + 4\mathbf{c}_2$.*

Find the components of $\mathbf{w}$ with respect to $\mathcal{B}$.

Find the components of $\mathbf{w}$ with respect to C.

Let $(v_1, v_2, \ldots, v_N)$ be the N-tuple of components of a vector $\mathbf{v}$ with respect to some basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N\}$. In computations, we often substitute $(v_1, v_2, \ldots, v_N)$ for $\mathbf{v}$, sometimes even writing

$$\mathbf{v} = (v_1, v_2, \ldots, v_N) \quad .$$

This is sloppy. Strictly speaking, it is even wrong. $\mathbf{v}$ is a vector (relative position, velocity, etc.) while $(v_1, v_2, \ldots, v_N)$ is an element of $\mathbb{R}^N$. The two are not the same thing, and identifying the two is safe only under the simplest of circumstances. It is certainly not a good idea when there are two different bases at hand. So don't write things like

$$\mathbf{v} = (v_1, v_2, \ldots, v_N)$$

unless you keep reminding yourself that this is really shorthand for

$$\mathbf{v} = v_1\mathbf{b}_2 + v_2\mathbf{b}_2 + \cdots + v_N\mathbf{b}_N \quad .$$

In other words, $(v_1, v_2, \ldots, v_N)$ should be viewed as a convenient *description* of $\mathbf{v}$ with respect to a given basis $\mathcal{B}$. Change the basis, and the corresponding representation for $\mathbf{v}$ can change dramatically. (So $(v_1, v_2, \ldots, v_N)$ is a basis-dependent description of $\mathbf{v}$.)

Still, this basis-dependent description can be useful, and is used in basis-independent formulas. This is illustrated in the next exercise, in which you confirm your favorite formulas for vector addition and scalar multiplication.

?▶*Exercise 2.10:* *Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in a vector space having basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N\}$. Let α and β be any two scalars and confirm that the components of $\alpha\mathbf{v} + \beta\mathbf{w}$ are*

$$(\alpha v_1 + \beta w_1,\ \alpha v_2 + \beta w_2,\ \ldots,\ \alpha v_N + \beta w_N)$$

where

$$(v_1, v_2, \ldots, v_N) \quad \text{and} \quad (w_1, w_2, \ldots, w_N)$$

are, respectively, the components of $\mathbf{v}$ and $\mathbf{w}$ with respect to $\mathcal{B}$.

2.3 The Dot Product
Geometric Definition

Now we combine the fundamental geometric notions of the length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ and the angle $\theta(\mathbf{v},\mathbf{w})$ between vectors $\mathbf{v}$ and $\mathbf{w}$ to define the classic "dot product". Throughout this section, all vectors are assumed to be from a single traditional vector space $\mathcal{V}$.

The *dot* (or *scalar* or *inner*) product of two vectors $\mathbf{v}$ and $\mathbf{w}$, denoted by either

$$\mathbf{v}\cdot\mathbf{w} \qquad \text{or} \qquad \langle\, \mathbf{v} \mid \mathbf{w}\,\rangle \quad ,$$

is the scalar given by

$$\mathbf{v}\cdot\mathbf{w} \;=\; \langle\, \mathbf{v} \mid \mathbf{w}\,\rangle \;=\; \|\mathbf{v}\|\,\|\mathbf{w}\|\cos(\theta) \qquad \text{where} \quad \theta = \theta(\mathbf{v},\mathbf{w}) \quad .$$

The "dot" notation, $\mathbf{v}\cdot\mathbf{w}$ is more traditional, but the "bracket" notation $\langle\,\mathbf{v}\,|\,\mathbf{w}\,\rangle$ corresponds to notation we will later use for generalizations of the dot product.[6]

Of course, we must observe that

$$\mathbf{v}\cdot\mathbf{w} \;=\; \mathbf{w}\cdot\mathbf{v} \qquad , \qquad \mathbf{v}\cdot\mathbf{v} \;=\; \|\mathbf{v}\|^2 \qquad \text{and} \qquad \cos(\theta) \;=\; \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\;\|\mathbf{w}\|} \quad .$$

(Note: If you ever see $\mathbf{v}^2$, it usually means $\mathbf{v}\cdot\mathbf{v}$, i.e., $\|\mathbf{v}\|^2$.)

?►*Exercise 2.11:* *Let α be any scalar, and $\mathbf{v}$ and $\mathbf{w}$ any two vectors. Show that α "factors" out of the dot product, i.e., that*

$$(\alpha\mathbf{v})\cdot\mathbf{w} \;=\; \alpha(\mathbf{v}\cdot\mathbf{w}) \quad .$$

Be sure to consider all three cases: $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$ (a picture may help with the $\alpha < 0$ case). Do NOT use the component formula for the dot product!

?►*Exercise 2.12:* *Using the result of the last exercise, show that two scalars α and β must be the same if*

$$\alpha\mathbf{v} \;=\; \beta\mathbf{v}$$

for some nonzero vector $\mathbf{v}$.

We can now express our definitions for orthogonality and orthonormality in terms of the dot product:

Orthogonality:

(a) A pair of vectors $\{\mathbf{v},\mathbf{w}\}$ is said to be orthogonal

$$\iff \quad \text{either } \mathbf{v} = \mathbf{0} \text{ or } \mathbf{w} = \mathbf{0} \text{ or the angle between } \mathbf{v} \text{ and } \mathbf{w} \text{ is } {}^{\pi}/_{2}$$

$$\iff \quad \mathbf{v}\cdot\mathbf{w} \;=\; 0 \quad .$$

[6] In the language of tensor analysis, the dot product is also called "the metric" on the vector space $\mathcal{V}$. This confuses some mathematicians because, for them, "the metric" on the vector space $\mathcal{V}$ means something else; namely, the function ρ given by $\rho(\mathbf{v},\mathbf{u}) = \|\mathbf{v}-\mathbf{u}\|$.

(b) A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K\}$ is said to be orthogonal

$$\iff \quad \text{every distinct pair in the set is orthogonal}$$

$$\iff \quad \mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 0 & \text{if } i \neq j \\ \|\mathbf{v}_i\|^2 & \text{if } i = j \end{cases} \quad .$$

Orthonormality: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K\}$ is said to be orthonormal

$$\iff \quad \text{it is an orthogonal set of unit vectors}$$

$$\iff \quad \mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad .$$

Using orthogonal or orthonormal bases will greatly simplify life.

At this point, let me introduce a commonly used symbol in mathematical physics: For any two integers i and j, the corresponding *Kronecker delta*, δ_{ij}, is defined by

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad .$$

Using the Kronecker delta, we can define set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K\}$ to be orthonormal if and only if

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} \quad .$$

(Okay, inventing this notation just to simplify the definition of orthonormality was silly. But the Kronecker delta will simplify notation elsewhere, too.)

?►*Exercise 2.13:* *Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K\}$ be any finite set of nonzero vectors. Show that*

$$\mathcal{S} \text{ is orthogonal.} \quad \Longrightarrow \quad \mathcal{S} \text{ is linearly independent.}$$

Note that it then follows that

$$\mathcal{S} \text{ is linearly dependent.} \quad \Longrightarrow \quad \mathcal{S} \text{ is not orthogonal.}$$

Is it true that a linearly independent set must be orthogonal? If so, explain why; if not, give an example of a linearly independent set that is not orthogonal.

Projections and Linearity of the Dot Product

Decomposing Vectors, Part II (Projections)

Let $\mathbf{v}$ and $\mathbf{a}$ be two vectors, with $\mathbf{a}$, at least, being nonzero. By elementary geometry, we can write $\mathbf{v}$ as the sum of two vectors

$$\mathbf{v} = \mathbf{A} + \mathbf{B}$$

where $\mathbf{A}$ is parallel to $\mathbf{a}$, and $\mathbf{B}$ is orthogonal to $\mathbf{a}$ (see figure 2.2). The vector parallel to $\mathbf{a}$ will be denoted by $\overrightarrow{\mathrm{pr}}_{\mathbf{a}}(\mathbf{v})$ (instead of $\mathbf{A}$), and will be called either the *projection of* $\mathbf{v}$ *onto* $\mathbf{a}$ or the *(vector) component of* $\mathbf{v}$ *in the direction of* $\mathbf{a}$ (depending on the whim of the speaker).

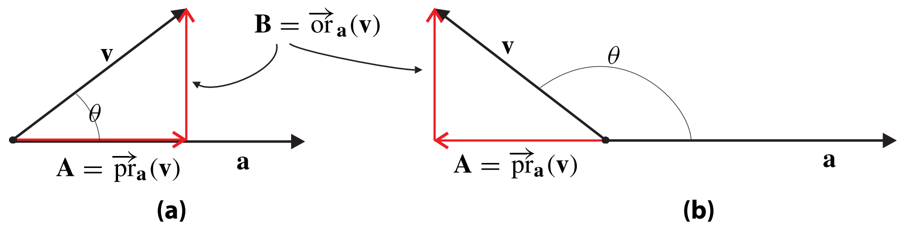

Figure 2.2: The projection of $\mathbf{v}$ onto $\mathbf{a}$, and the projection of $\mathbf{v}$ orthogonal to $\mathbf{a}$.

This is the "piece" of $\mathbf{v}$ we will usually be most interested in. The other vector component, $\mathbf{B}$, doesn't have a name or notation generally agreed upon. We will denote it by $\overrightarrow{\text{or}}_{\mathbf{a}}(\mathbf{v})$, and will call it either the *projection of* $\mathbf{v}$ *orthogonal to* $\mathbf{a}$ or the *(vector) component of* $\mathbf{v}$ *orthogonal to* $\mathbf{a}$ (depending on the whim of the speaker). So, using the direction given by $\mathbf{a}$ as a "base direction", we have now decomposed $\mathbf{v}$ into a pair of vectors

$$\mathbf{v} = \overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v}) + \overrightarrow{\text{or}}_{\mathbf{a}}(\mathbf{v})$$

where $\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v})$ is parallel to $\mathbf{a}$ and $\overrightarrow{\text{or}}_{\mathbf{a}}(\mathbf{v})$ is orthogonal to $\mathbf{a}$

To get a useful formula for $\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v})$, it helps to introduce the *scalar component of* $\mathbf{v}$ *in the direction of* $\mathbf{a}$, which, for convenience, we will briefly denote by $\sigma_{\mathbf{a}}(\mathbf{v})$ and define by

$$\sigma_{\mathbf{a}}(\mathbf{v}) = \begin{cases} \left\|\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v})\right\| & \text{if } \overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v}) \text{ points in the direction of } \mathbf{a} \\ -\left\|\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v})\right\| & \text{if } \overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v}) \text{ points in the direction opposite of } \mathbf{a} \end{cases} .$$

Thus,

$$\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v}) = \sigma_{\mathbf{a}}(\mathbf{v}) \times \text{the unit vector in the direction of } \mathbf{a} .$$

Using elementary trigonometry, you can easily derive a simple formula for this scalar component in terms of θ, and then rewrite that formula in terms $\mathbf{v} \cdot \mathbf{a}$ and $\|a\|$. Combining that with the last equation above then leads to the standard dot product formula for the vector projection. The details are left to you:

?►*Exercise 2.14:* *Let* $\mathbf{a}$ *and* $\mathbf{v}$ *be as above. Show, using trigonometry and the geometric definition of the dot product from page 2–10, that*

$$\sigma_{\mathbf{a}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{a}\|} ,$$

and then that

$$\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} .$$

(Be sure to consider both cases.)

Linearity and the Component Formulas for the Dot Product

To derive the standard component formula for the dot product, you start by verifying some simple "linearity" equations for projections and the dot product in two dimensions:

?►*Exercise 2.15:* *Let* $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{a}$ *be three vectors in a plane, with at least* $\mathbf{a}$ *being nonzero, and do the following (without using the well-known component formula for the dot product):*

a: *Draw a figure with* $\mathbf{a}$ *and* $\mathbf{v}$ *starting at the same point. Add* $\mathbf{w}$ *starting a the end of* $\mathbf{v}$ *and draw in the sum* $\mathbf{v}+\mathbf{w}$ *from the start of* $\mathbf{v}$ *to the end of* $\mathbf{w}$ *. Using this figure, convince yourself that*

$$\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v}+\mathbf{w}) \;=\; \overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v}) \;+\; \overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{w}) \quad .$$

b: *Now, using what you just found (along with the results from exercise 2.14), show that*

$$(\mathbf{v}+\mathbf{w})\cdot\mathbf{a} \;=\; \mathbf{v}\cdot\mathbf{a} \;+\; \mathbf{w}\cdot\mathbf{a} \quad .$$

With a little ability to visualize vectors in three dimensions, you should be able to see that the first part holds when $\mathbf{a}$, $\mathbf{v}$ and $\mathbf{w}$ are not all in one plane (no matter what dimension our space of traditional vectors is). Consequently, the result of the second part,

$$(\mathbf{v}+\mathbf{w})\cdot\mathbf{a} \;=\; \mathbf{v}\cdot\mathbf{a} \;+\; \mathbf{w}\cdot\mathbf{a} \quad ,$$

holds for any three vectors $\mathbf{a}$, $\mathbf{v}$ and $\mathbf{w}$ (in the same vector space, of course). By the commutativity of the dot product, we also then have

$$\mathbf{a}\cdot(\mathbf{v}+\mathbf{w}) \;=\; \mathbf{a}\cdot\mathbf{v} \;+\; \mathbf{a}\cdot\mathbf{w} \quad ,$$

from which it follows that, with four vectors,

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{v}+\mathbf{w}) \;&=\; \mathbf{a}\cdot(\mathbf{v}+\mathbf{w}) \;+\; \mathbf{b}\cdot(\mathbf{v}+\mathbf{w}) \\ &=\; \mathbf{a}\cdot\mathbf{v} \;+\; \mathbf{a}\cdot\mathbf{w} \;+\; \mathbf{b}\cdot\mathbf{v} \;+\; \mathbf{b}\cdot\mathbf{w} \quad .\end{aligned}$$

And, of course, similar expansions can be done with even more vectors. This now allows you to derive the various component formulas for the dot product, including the one most of you so love.

?►*Exercise 2.16:* *For the following, assume*

$$\mathcal{B} \;=\; \{\,\mathbf{b}_1,\,\mathbf{b}_2,\,\dots,\,\mathbf{b}_N\,\}$$

is a basis for a traditional vector space, and let

$$(\,v_1,\,v_2,\,\dots,\,v_N\,) \qquad \text{and} \qquad (\,w_1,\,w_2,\,\dots,\,w_N\,)$$

be the components with respect to $\mathcal{B}$ *of vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *, respectively. (So,*

$$\mathbf{v} \;=\; v_1\mathbf{b}_1 \;+\; v_2\mathbf{b}_2 \;+\; \dots \;+\; v_N\mathbf{b}_N \;=\; \sum_{j=1}^{N} v_j\mathbf{b}_j$$

and

$$\mathbf{w} \;=\; w_1\mathbf{b}_1 \;+\; w_2\mathbf{b}_2 \;+\; \dots \;+\; w_N\mathbf{b}_N \;=\; \sum_{k=1}^{N} w_k\mathbf{b}_k \qquad .)$$

a: *Show that*

$$\mathbf{v} \cdot \mathbf{w} = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk} v_j w_k$$

where each γ_{jk} *is the constant* $\mathbf{b}_j \cdot \mathbf{b}_k$ *.*

b: *To what does the formula for* $\mathbf{v} \cdot \mathbf{w}$ *reduce when the basis* $\mathcal{B}$ *is orthogonal?*

c: *To what does the formula for* $\mathbf{v} \cdot \mathbf{w}$ *reduce when the basis* $\mathcal{B}$ *is orthonormal?*

The answer to the very last part of the last exercise is, of course,

$$\mathbf{v} \cdot \mathbf{w} = \sum_{j=1}^{N} v_j w_j \quad .$$

Remember this only holds when the basis being used is orthonormal.

Because of the relation between the dot product and norms of vectors, you should be able to quickly do the corresponding exercise for obtaining the component formula(s) for $\|\mathbf{v}\|$:

?►*Exercise 2.17:* *For the following, assume*

$$\mathcal{B} = \{ \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N \}$$

is a basis for a traditional vector space, and let

$$(v_1, v_2, \ldots, v_N)$$

be the components with respect to $\mathcal{B}$ *of a vector* $\mathbf{v}$ *in this space.*

a: *Show that*

$$\|\mathbf{v}\| = \sqrt{\sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk} v_j v_k}$$

where the γ_{jk}*'s are constants computable from the basis vectors. Be sure to give the formula for computing these constants.*

b: *To what does the formula for* $\|\mathbf{v}\|$ *reduce when the basis* $\mathcal{B}$ *is orthogonal?*

c: *To what does the formula for* $\|\mathbf{v}\|$ *reduce when the basis* $\mathcal{B}$ *is orthonormal?*

And, of course, the answer to the last part of the last exercise is

$$\|\mathbf{v}\| = \sqrt{\sum_{j=1}^{N} {v_j}^2} \quad .$$

At this point, we can also verify a simple "dot product test for equality", which, oddly enough, will be useful when discussing the cross product in the next section.

?►*Exercise 2.18 (dot product test for equality):*

a: *Suppose* $\mathbf{D}$ *is a vector such that*

$$\mathbf{D} \cdot \mathbf{c} = 0 \qquad \text{for every} \quad \mathbf{c} \quad .$$

Show that $\mathbf{D} = \mathbf{0}$ *. (Hint: What if we choose* $\mathbf{c} = \mathbf{D}$ *?)*

b: *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are two vectors such that*

$$\mathbf{A} \cdot \mathbf{c} = \mathbf{B} \cdot \mathbf{c} \qquad \text{for every} \quad \mathbf{c} \quad .$$

Show that $\mathbf{A} = \mathbf{B}$ *. (Hint: What about* $\mathbf{D} = \mathbf{A} - \mathbf{B}$ *?)*

Components and the Dot Product

We need to be a little careful about using the word "components". If we have a vector $\mathbf{v}$ and a basis

$$\mathcal{B} = \{ \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N \} \quad ,$$

then $\mathbf{v}$ will have components $(v_1, v_2, \ldots, v_N)$ with respect to the basis. These are the scalars such that

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_N\mathbf{b}_N \quad .$$

Vector $\mathbf{v}$ will also have scalar and vector components with respect to each $\mathbf{b}_j$ given by

$$\sigma_{\mathbf{b}_j}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{b}_j}{\|\mathbf{b}_j\|} \qquad \text{and} \qquad \overrightarrow{\text{pr}}_{\mathbf{b_j}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{b}_j}{{\mathbf{b}_j}^2}\,\mathbf{b}_j \quad ,$$

respectively. As you can easily verify, these yield slightly different things when the basis is not orthonormal.

?►*Exercise 2.19:* *Let* $(v_1, v_2, \ldots, v_N)$ *be the components of a vector* $\mathbf{v}$ *with respect to some arbitrary basis*

$$\mathcal{B} = \{ \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N \} \quad .$$

a i: *Find, in terms of the* v_j *'s, the formulas for the vector and scalar components of* $\mathbf{v}$ *with respect to each* $\mathbf{b}_k$ *.*

ii: *How, then, would you find the* v_j *'s if you knew the vector and scalar components of* $\mathbf{v}$ *with respect to each* $\mathbf{b}_k$ *.*

b: *Show that, if* $\mathcal{B}$ *is orthogonal, then*

$$v_j = \frac{\mathbf{v} \cdot \mathbf{b}_j}{\|\mathbf{b}_j\|^2} \quad .$$

(A more general version of this — which will look almost the same — will be important in solving many partial differential equation problems later.)

c: *Show that, if* $\mathcal{B}$ *is orthonormal, then*

$$v_j = \mathbf{v} \cdot \mathbf{b}_j \quad .$$

2.4 The Cross Product

The dot product can be defined whatever the dimension of our traditional vector space is, and can (and will) be generalized to nontraditional vector spaces. On the other hand, the cross product is essentially an operation requiring that we have a three-dimensional space of traditional vectors. So, in the following, assume that the vector space is just such a space.

Geometric Definition

Given two vectors $\mathbf{v}$ and $\mathbf{w}$, the cross product $\mathbf{v}\times\mathbf{w}$ is defined to be the vector with the following length and direction:

Length: $\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\,\|\mathbf{w}\|\sin(\theta)$ where $\theta = \theta(\mathbf{v},\mathbf{w})$.

Direction: If $\|\mathbf{v}\times\mathbf{w}\| = 0$, then we automatically have $\mathbf{v}\times\mathbf{w} = \mathbf{0}$. Otherwise the direction of $\mathbf{v}\times\mathbf{w}$ is the one direction orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ given by the classic "right-hand rule" (which I will not attempt to sketch for you here — look in your old Intro to Physics or old Calculus text.)

There are a number of properties that are relatively easily verified from this definition:

1. The cross product is *anti*commutative. That is,

$$\mathbf{w}\times\mathbf{v} = -\mathbf{v}\times\mathbf{w} \quad .$$

2. Assuming $\mathbf{v}$ and $\mathbf{w}$ are nonzero, then

$$\mathbf{v}\times\mathbf{w} = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{v} \text{ and } \mathbf{w} \text{ are parallel} \quad .$$

In particular, we always have

$$\mathbf{v}\times\mathbf{v} = \mathbf{0} \quad .$$

3. If $\{\mathbf{v},\mathbf{w}\}$ is an orthonormal pair of vectors, then $\{\mathbf{v},\mathbf{w},\mathbf{v}\times\mathbf{w}\}$ is an orthnormal set.

4. Scalars "factor out". That is, for any scalar γ,

$$(\gamma\mathbf{v})\times\mathbf{w} = \gamma(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\times(\gamma\mathbf{w}) \quad .$$

5. The cross product is not associative. That is, in general,

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} \neq \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \quad .$$

6. The cross product is related to the dot product via

$$\|\mathbf{v}\times\mathbf{w}\|^2 = \|\mathbf{v}\|^2\,\|\mathbf{w}\|^2 - (\mathbf{v}\cdot\mathbf{w})^2 \quad .$$

(This follows from the definitions and an elementary trigonometric identity.)

?►*Exercise 2.20:* *Explain (to yourself, at least) why each of the previous statements is true.*

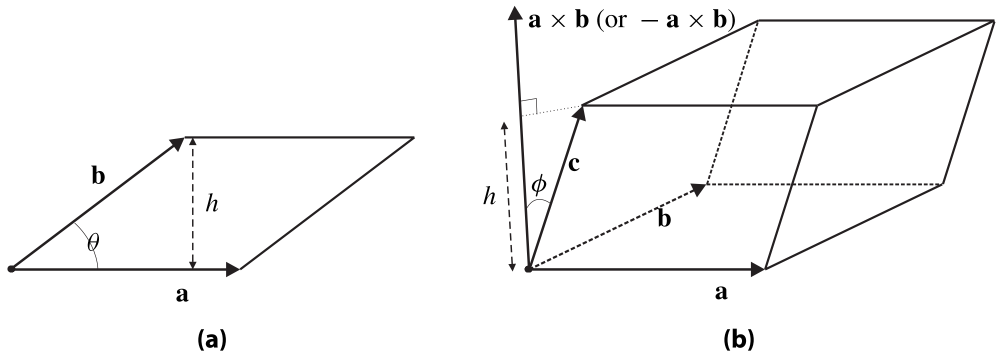

Figure 2.3: (a) The parallelogram generated by $\mathbf{a}$ and $\mathbf{b}$. (b) The parallelepiped generated by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Area and Volume and the Cross Product

When we talk about the parallelogram or parallelepiped generated by two or three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we are talking about the two- or three-dimensional objects with "flat/straight sides" sketched in figure 2.3.

Area of a Parallelogram

Look at the parallelogram generated by vectors $\mathbf{a}$ and $\mathbf{b}$ in figure 2.3a. Clearly, the length of the "base" of this rectangle is $\|\mathbf{a}\|$, and, from basic trigonometry, we know the height h of this parallelogram is

$$h = \|\mathbf{b}\| \sin(\theta) \quad .$$

So, applying a little elementary geometry, we have

$$\begin{aligned}
\text{area of the parallelogram} &= \big(\text{length of base}\big)\big(\text{height}\big) \\
&= \|\mathbf{a}\| \, \|\mathbf{b}\| \sin(\theta) \\
&= \|\mathbf{a} \times \mathbf{b}\| \quad .
\end{aligned}$$

(Aside from the obvious uses of this fact to find "areas of parallelograms floating in space", this fact can be used in finding the formula for the differential element of surface area using any coordinate system.)

By the way, the vector $\mathbf{a} \times \mathbf{b}$ is sometimes called the "vector area" of the parallelogram. (The concept of "vector area" can be relevant in computations of certain surface integrals.)

Volume of a Parallelepiped

Now look at the parallelepiped generated by vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in figure 2.3b (the squished box with every side being a parallelogram). Recall (from high school) that the volume of this box is the area of its base times its "height" h as measured from bottom to top along a line perpendicular to the bottom plane. From above we have that

$$\text{area of the base} = \text{area of parallelogram generated by } \mathbf{a} \text{ and } \mathbf{b} = \|\mathbf{a} \times \mathbf{b}\| \quad .$$

Since $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, it is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$, and, so, gives the direction for a line perpendicular to the bottom. Consequently, (see the figure) the height h is simply the magnitude of the projection of vector $\mathbf{c}$ onto $\mathbf{a} \times \mathbf{b}$. Using either the projection formulas from before or simple trigonometry, we have

$$h = \|\mathbf{c}\| \cos(\phi)$$

where ϕ is the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$. With this and the geometric definition of the dot product, we then get

$$\begin{aligned} \text{volume of parallelepiped} &= \big(\text{area of base}\big)\big(\text{height}\big) \\ &= \|\mathbf{a} \times \mathbf{b}\| \, \|\mathbf{c}\| \cos(\phi) \\ &= |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| \quad . \end{aligned}$$

The expression $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called a *triple product* because it has three factors (even though there are only two ‘multiplies’). Other triple products will be briefly discussed later. For now, however, observe that we have a certain symmetry in play here: It makes no difference which two of the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are used to generate the bottom. Consequently, we automatically have the identity

$$\text{volume of parallelepiped} = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}| \quad .$$

In the next subsection, we will expand upon this observation, and then use it to derive another fundamental identity which, in turn, will allow us to derive the component formula for the cross product.

(By the way, we may later use the formulas just derived for area and volume to develop the formulas for the integral elements of area and volume, dA and dV, using any coordinate system in any Euclidean or nonEuclidean space. With luck, we will even discuss finding “hyper-volumes” of arbitrary N-dimensional “hyper-parallelepipeds”, and use those formulas in multidimensional integration in arbitrary coordinate systems. All that, however, goes beyond our traditional vector theory.)

Linearity of the Cross Product

We start with an exercise:

?►*Exercise 2.21:* *We just saw that, given three vectors* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$, *then*

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \quad .$$

Thus, either

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \qquad \textit{or} \qquad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \quad . \tag{2.1}$$

a: *Now assume that* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are oriented so that the angle between each pair is no greater than* $\pi/2$ *(* $90°$ *), and that* $\mathbf{c}$ *is in the general direction of* $\mathbf{a} \times \mathbf{b}$ *(as in figure 2.3b). Convince yourself that, of the two equations in (2.1), it is*

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

which holds.

b: *Now try different orientations for* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *and convince yourself that, no matter how they are oriented with respect to each other,*

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \;=\; \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \quad .$$

(See if you can come up with every possible orientation! Then you have a proof of this identity!)

This last exercise demonstrates that, given any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (in a three-dimensional traditional vector space), then

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \;=\; \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \quad . \tag{2.2}$$

?►Exercise 2.22: *Let* $\mathbf{v}$, $\mathbf{w}$, $\mathbf{b}$ *and* $\mathbf{c}$ *be any four vectors in in three-dimensional space of traditional vectors. Using identity (2.2) and a result concerning the dot product (look at exercise 2.15), show that*

$$\big[(\mathbf{v}+\mathbf{w}) \times \mathbf{b}\big] \cdot \mathbf{c} \;=\; \big[(\mathbf{v} \times \mathbf{b}) \;+\; (\mathbf{w} \times \mathbf{b})\big] \cdot \mathbf{c} \quad .$$

(Do NOT assume $(\mathbf{v}+\mathbf{w}) \times \mathbf{b} = (\mathbf{v} \times \mathbf{b}) \;+\; (\mathbf{w} \times \mathbf{b})$ *; that's what we are working to derive!)*

Keep in mind that the equation derived in the last exercise holds for every possible $\mathbf{c}$. If you think about it (or recall exercise 2.18 on page 2–15), you will realize that could only mean that

$$(\mathbf{v}+\mathbf{w}) \times \mathbf{b} \;=\; (\mathbf{v} \times \mathbf{b}) \;+\; (\mathbf{w} \times \mathbf{b})$$

holds (this is why you did exercise 2.18). By the anticommutativity of the cross product, we also have

$$\mathbf{b} \times (\mathbf{v}+\mathbf{w}) \;=\; -(\mathbf{v}+\mathbf{w}) \times \mathbf{b} \;=\; -\big[(\mathbf{v} \times \mathbf{b}) \;+\; (\mathbf{w} \times \mathbf{b})\big] \;=\; (\mathbf{b} \times \mathbf{v}) \;+\; (\mathbf{b} \times \mathbf{w}) \quad .$$

And from these identities (the distributive properties of the cross product), along with a few more basic properties, you can derive a most general component formula for the cross product.

?►Exercise 2.23: *Let* $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ *be a basis for a three-dimensional space of traditional vectors, and let*

$$(\, v_1,\; v_2,\; v_3\,) \qquad \text{and} \qquad (\, w_1,\; w_2,\; w_3\,)$$

be the components with respect to $\mathcal{B}$ *of two vectors* $\mathbf{v}$ *and* $\mathbf{w}$. *Show that*

$$\mathbf{v} \times \mathbf{w} \;=\; A(\mathbf{b}_1 \times \mathbf{b}_2) \;+\; B(\mathbf{b}_1 \times \mathbf{b}_3) \;+\; C(\mathbf{b}_2 \times \mathbf{b}_3)$$

where A , B *and* C *are formulas of the* v_j*'s and* w_k*'s. Find those formulas.*

A clever choice of basis, of course, can lead to a "nicer" component formula for the cross product.

Right-Handed Bases and the Classical Formula for the Cross Product

A *right-handed basis* is an (ordered) orthogonal basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for a three-dimensional traditional vector space in which $\mathbf{b}_3$ points in the same direction as $\mathbf{b}_1 \times \mathbf{b}_2$. In practice, we typically restrict ourselves to orthonormal right-handed bases, in which case the requirement that "$\mathbf{b}_3$ points in the same direction as $\mathbf{b}_1 \times \mathbf{b}_2$" simplifies to

$$\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 \quad .$$

Typically, also, orthonormal right-handed bases are denoted by either

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \qquad \text{or} \qquad \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \qquad \text{or} \qquad \{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \quad ,$$

or

$$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \qquad \text{or} \qquad \{\hat{\imath}, \hat{\jmath}, \hat{k}\} \qquad \text{or} \qquad \{\hat{x}, \hat{y}, \hat{z}\} \quad ,$$

or even

$$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\} \qquad \text{or} \qquad \{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\} \qquad \text{or} \qquad \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}},\} \quad .$$

In the future, if a basis is referred to as a *standard* basis for a three-dimensional vector space of traditional vectors, assume it is a right-handed orthonormal basis.

So assume $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a standard basis. By the above definition, $\mathbf{k} = \mathbf{i} \times \mathbf{j}$. With a little thought and a few pictures, you can easily confirm that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \text{and} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j} \quad .$$

Combining this with the formula you derived for $\mathbf{v} \times \mathbf{w}$ (in exercise 2.23) yields the standard component formula

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\mathbf{i} - (v_1 w_3 - v_3 w_1)\mathbf{j} + (v_1 w_2 - v_2 w_1)\mathbf{k}$$

where, of course, (v_1, v_2, v_3) and (w_1, w_2, w_3) are the components of $\mathbf{v}$ and $\mathbf{w}$ with respect to the given standard basis. Fortunately for those who dislike memorizing formulas with indices in which order is important, this formula can also be written as the determinant of a more easily remembered matrix,

$$\mathbf{v} \times \mathbf{w} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \quad .$$

?►*Exercise 2.24:* *Let* $\mathbf{a}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be three vectors in a three-dimensional space of traditional vectors, and let* (a_1, a_2, a_3), (v_1, v_2, v_3) *and* (w_1, w_2, w_3) *be, respectively, their components with respect to the some standard basis. Verify that*

$$\mathbf{a} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \quad .$$

2.5 Triple Products

The Fundamental Products and the BAC-CAB Rule

Any product of three vectors is called a *triple product*. If you think about it, you will realize that the only legitimate triple products are of the form

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \quad , \quad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad \text{or} \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad .$$

The first one has already been discussed because of its relation to the volume of a parallelepiped and a distributive law for the cross product (and a component formula found in exercise 2.24). Since we already know that the cross product is not associative, we know that the other two triple products above are not equal. However, we do have

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \quad ,$$

so any identity for one can easily be converted to an identity for the other.

One famous identity is the "bac-cab rule",

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \quad . \tag{2.3}$$

This is significant for at least three reasons:

1. It explicitly shows that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a linear combination of $\mathbf{b}$ and $\mathbf{c}$. (In fact, a good derivation of the rule starts with the geometric derivation of this fact.)

2. It can be used to reduce the rather tedious computations of the triple cross product to a much simpler set of computations with two dot products.

3. It has a neat name.

A derivation of the bac-cab rule follows in the next subsection.

?►*Exercise 2.25:* *What would be the "cab-bac rule"?*

?►*Exercise 2.26:* *Three vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given in problem 1.5.10 on page 31 of our text. Using these vectors:*

a: *Compute $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ just using the standard formula for the cross product to compute both cross products.*

b: *Compute $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ using the bac-cab rule (or, in this case, the BAC-CAB rule).*

c: *Which way of computing $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ was easier?*

Deriving/Verifying the BAC-CAB Rule

Our goal is to verify identity (2.3), the bac-cab rule. We could verify this by just computing the two sides of the equation using standard components, and then verifying the two are the same — but that would be tedious and unenlightening.

For convenience, let $\mathbf{v}$ denote the triple product of interest,

$$\mathbf{v} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \quad .$$

Thus, to verify the bac-cab rule we need to show that

$$\mathbf{v} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \quad .$$

In our initial derivation, we will make two assumptions to avoid degeneracies:

$$\{\mathbf{b}, \mathbf{c}\} \text{ is linearly independent} \qquad \text{and} \qquad \mathbf{a} \cdot \mathbf{c} \neq 0 \quad .$$

Also keep in mind that we are dealing with a three-dimensional traditional vector space.

Now, by the definition of cross product, we know $\mathbf{v}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. Let's first consider the fact that $\mathbf{v}$ is perpendicular to $\mathbf{b} \times \mathbf{c}$. We also know $\mathbf{b}$ and $\mathbf{c}$ are perpendicular to $\mathbf{b} \times \mathbf{c}$. But, geometrically, the set of all vectors perpendicular to $\mathbf{b} \times \mathbf{c}$ forms a two-dimensional plane. Since this plane is a two-dimensional vector space, and $\{\mathbf{b}, \mathbf{c}\}$ is a linearly independent pair of vectors in it, the set $\{\mathbf{b}, \mathbf{c}\}$ is a basis for this space. Hence, $\mathbf{v}$, which is also in this plane, must be a linear combination of $\mathbf{b}$ and $\mathbf{c}$,

$$\mathbf{v} = \beta\mathbf{b} + \gamma\mathbf{c} \quad .$$

Combining this with the fact that $\mathbf{a}$ is perpendicular to $\mathbf{v}$ gives us

$$0 = \mathbf{a} \cdot \mathbf{v} = \mathbf{a} \cdot [\beta\mathbf{b} + \gamma\mathbf{c}] = \beta\mathbf{a} \cdot \mathbf{b} + \gamma\mathbf{a} \cdot \mathbf{c} \quad .$$

So

$$\gamma\mathbf{a} \cdot \mathbf{c} = -\beta\mathbf{a} \cdot \mathbf{b} \quad .$$

Thus,

$$\gamma = -\beta\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{c}}$$

and

$$\mathbf{v} = \beta\mathbf{b} + \gamma\mathbf{c} = \beta\mathbf{b} - \beta\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{c}}\mathbf{c} \quad .$$

Factoring out $w = {}^{\beta}/_{\mathbf{a} \cdot \mathbf{c}}$ then gives us

$$\mathbf{v} = w[(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] \quad ,$$

This is almost what we set out to derive. To finish, we need to show that $w = 1$.

Showing that $w = 1$ is the hardest part. To simplify our computation, we will be terribly clever and choose a standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with $\mathbf{i}$ in the same direction of $\mathbf{a}$. That way,

$$\mathbf{a} = a\mathbf{i} \qquad \text{with} \quad a = \|\mathbf{a}\| \quad ,$$

$$\mathbf{j} \times \mathbf{a} = \mathbf{j} \times a\mathbf{i} = -a\mathbf{k} \quad ,$$

$$\mathbf{a} \cdot \mathbf{b} = a\mathbf{i} \cdot \mathbf{b} = ab_1 \qquad \text{and} \qquad \mathbf{a} \cdot \mathbf{c} = a\mathbf{i} \cdot \mathbf{v} = ac_1$$

where (b_1, b_2, b_3) and (c_1, c_2, c_3) are the components of $\mathbf{b}$ and $\mathbf{c}$ with respect to the given basis. Taking the dot product of $\mathbf{v}$ and $\mathbf{j}$ using the last formula obtained for $\mathbf{v}$ yields

$$\begin{aligned} \mathbf{j} \cdot \mathbf{v} &= \mathbf{j} \cdot w[(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] \\ &= w[(ac_1)\mathbf{j} \cdot \mathbf{b} - (ab_1)\mathbf{j} \cdot \mathbf{c}] = w[ac_1b_2 - ab_1c_2] \quad . \end{aligned}$$

On the other hand, taking the dot product of $\mathbf{v}$ with $\mathbf{j}$ using the original triple product formula for $\mathbf{v}$, along with a clever use of equality (2.2) on page 2–19, yields

$$\begin{aligned}
\mathbf{j}\cdot\mathbf{v} &= \mathbf{j}\cdot[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})] \\
&= (\mathbf{j}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}) \\
&= -a\mathbf{k}\cdot\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3\end{bmatrix} \\
&= -a(b_1c_2 - b_2c_1) = ac_1b_2 - ab_1c_2 \quad .
\end{aligned}$$

Comparing the results of the last two sets of computations, we have

$$w[ac_1b_2 - ab_1c_2] = \mathbf{v}\cdot\mathbf{j} = ac_1b_2 - ab_1c_2$$

Solving for w then yields $w = 1$, finishing our derivation/confirmation of the bac-cab rule.

Remember though, we assumed a few things "to avoid degeneracies". In the following exercises, you will show that the bac-cab rule still holds when those degeneracies occur.

?►Exercise 2.27: *Assume $\{\mathbf{b},\mathbf{c}\}$ is not linearly independent. Show that the bac-cab rule still holds by showing that both sides of equation (2.3) are zero.*

?►Exercise 2.28: *In our derivation, we assumed $\mathbf{a}\cdot\mathbf{c}\neq 0$ and obtained*

$$\mathbf{v} = w[(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}] \quad .$$

Show that the above equation can also be obtained if $\mathbf{a}\cdot\mathbf{b}\neq 0$. Thus, the above derivation holds, with slight modification, if $\mathbf{a}\cdot\mathbf{c} = 0$, provided $\mathbf{a}\cdot\mathbf{b}\neq 0$.

?►Exercise 2.29: *Assume that both $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\cdot\mathbf{b}\neq 0$ are zero. Show that the bac-cab rule still holds by showing that both sides of equation (2.3) are zero.*

2.6 Reciprocal Bases*

Assume $\mathcal{V}$ is a traditional finite-dimensional vector space with basis

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_N\} \quad .$$

The corresponding *reciprocal basis* is the set of vectors

$$\mathcal{B}' = \{\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3, \ldots, \mathbf{b}'_N\}$$

such that

$$\mathbf{b}'_j\cdot\mathbf{b}_k = \delta_{jk} \qquad \text{for all} \quad j \text{ and } k \quad .$$

* In traditional vector theory, the idea of "reciprocal bases" is mainly of just academic interest. However, this idea will re-emerge when we develop "tensor theory". In fact, it is this idea that actually leads to the distinction between "covariance" and "contravariance" in tensor descriptions.

In other words, for each index j, $\mathbf{b}'_j$ is the vector orthogonal to all the $\mathbf{b}_k$'s except $\mathbf{b}_j$ that also satisfies

$$\mathbf{b}'_j \cdot \mathbf{b}_j = 1 \quad .$$

It is not hard to show that $\mathcal{B}'$, as defined above, exists, is unique, and really is a basis for $\mathcal{V}$. It is also easy to verify that the reciprocal basis of $\mathcal{B}'$ is $\mathcal{B}$. In fact, it would be more appropriate to refer to $\mathcal{B}$ and $\mathcal{B}'$ as a *reciprocal pair of bases*.

As the following exercises show, there isn't much to finding the reciprocal basis $\mathcal{B}'$ if the original basis $\mathcal{B}$ is orthogonal or even orthonormal.

?►*Exercise 2.30:* *Let $\mathcal{B}$ and $\mathcal{B}'$ be as above. Convince yourself of the following:*

a: *If $\mathcal{B}$ is orthonormal, then*

$$\mathbf{b}'_k = \mathbf{b}_k \qquad \text{for each} \quad k \quad .$$

b: *If $\mathcal{B}$ is orthogonal, then*

$$\mathbf{b}'_k = \frac{\mathbf{b}_k}{\|\mathbf{b}_k\|^2} \qquad \text{for each} \quad k \quad .$$

Finding the reciprocal basis corresponding to a nonorthogonal basis is a little more challenging.

?►*Exercise 2.31:* *Let $\mathcal{V}$ be a two-dimensional traditional vector space with standard basis $\{\mathbf{i}, \mathbf{j}\}$, and let*

$$\mathcal{B} = \{\mathbf{b}_1 , \mathbf{b}_2\}$$

be the basis with

$$\mathbf{b}_1 = 3\mathbf{i} \qquad \text{and} \qquad \mathbf{b}_2 = \mathbf{i} + 2\mathbf{j} \quad .$$

Find the reciprocal basis

$$\mathcal{B}' = \{\mathbf{b}'_1 , \mathbf{b}'_2\}$$

and sketch both the vectors in $\mathcal{B}$ and $\mathcal{B}'$. (You may want to do the sketching first.)

?►*Exercise 2.32:* *Let $\mathcal{V}$ be a three-dimensional traditional vector space with standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and let*

$$\mathcal{B} = \{\mathbf{b}_1 , \mathbf{b}_2 , \mathbf{b}_3\}$$

be the basis with

$$\mathbf{b}_1 = \mathbf{i} + \mathbf{j} \quad , \qquad \mathbf{b}_2 = 2\mathbf{j} \qquad \text{and} \qquad \mathbf{b}_3 = \mathbf{j} - 2\mathbf{k} \quad .$$

Find the reciprocal basis

$$\mathcal{B}' = \{\mathbf{b}'_1 , \mathbf{b}'_2 , \mathbf{b}'_3\}$$

and sketch both the vectors in $\mathcal{B}$ and $\mathcal{B}'$. (You may find the cross product useful in finding the $\mathbf{b}'_k$'s.)

So what? Well, let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in $\mathcal{V}$. These vectors will have components

$$(v_1, v_2, \ldots, v_N) \quad \text{and} \quad (w_1, w_2, \ldots, w_N)$$

with respect to $\mathcal{B}$, along with components

$$(v_1', v_2', \ldots, v_N') \quad \text{and} \quad (w_1', w_2', \ldots, w_N')$$

with respect to $\mathcal{B}'$. That is,

$$\mathbf{v} = \sum_{i=1}^{N} v_i \mathbf{b}_i = \sum_{j=1}^{N} v_j' \mathbf{b}_j'$$

and

$$\mathbf{w} = \sum_{m=1}^{N} w_m \mathbf{b}_m = \sum_{n=1}^{N} w_n' \mathbf{b}_n' \quad .$$

From this, you can easily show that

$$\mathbf{v} \cdot \mathbf{w} = \sum_{j=1}^{N} v_j' w_j \qquad \text{and} \qquad \|\mathbf{v}\| = \sqrt{\sum_{j=1}^{N} v_j' v_j}$$

even when $\mathcal{B}$ is not orthogonal.

?►Exercise 2.33: *Verify the above formulas for* $\mathbf{v} \cdot \mathbf{w}$ *and* $\|\mathbf{v}\|$.

That is the importance of the reciprocal basis. It let's us have a "simple" component formula for the inner product and the norm, provided we use the two different bases for the components.

Using the above, you can also discover that the reciprocal basis gives an "easy" way to find components of vectors with respect to an arbitrary basis (provided you've found that reciprocal basis).

?►Exercise 2.34: *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be a reciprocal pair of bases for an* N*-dimensional vector space, and let* $\mathbf{v}$ *be a vector in this space. Show that each component of* $\mathbf{v}$ *with respect to one of these bases can be expressed as an inner product of* $\mathbf{v}$ *with the corresponding element of the other basis. More precisely, show that*

$$\mathbf{v} = \sum_{k=1}^{N} v_k \mathbf{b}_k \qquad \textit{where} \quad v_k = \mathbf{b}_k' \cdot \mathbf{v}$$

and

$$\mathbf{v} = \sum_{k=1}^{N} v_k' \mathbf{b}_k' \qquad \textit{where} \quad v_k' = \mathbf{b}_k \cdot \mathbf{v} \quad .$$

(Later, compare this with the approach described in the next chapter.)

Now, use these results in the next exercises.

?► ***Exercise 2.35:*** *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be the reciprocal pair of basis from exercise 2.31, and find the components with respect to each of these bases for each of the following vectors:*

$$\mathbf{i} \quad , \quad \mathbf{j} \quad \textit{and} \quad 3\mathbf{i} - 5\mathbf{j} \quad .$$

?► ***Exercise 2.36:*** *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be the reciprocal pair of basis from exercise 2.32, and find the components with respect to each of these bases for each of the following vectors:*

$$\mathbf{i} \quad , \quad \mathbf{j} \quad \textit{and} \quad 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} \quad .$$