20

Partial Differential Equations II: Solving (Homogeneous) PDE Problems

20.1 Problems with Two Variables

Putting It All Together

Let us go back to the sort of problem we were considering at the end of chapter 18, that of finding a solution to a (separable) homogeneous partial differential equation involving two variables x and t which also satisfied suitable boundary conditions (at $x = a$ and $x = b$) as well as some sort of initial condition(s). In particular, we were considering the following heat flow problem on a rod of length L:

Find the solution $u = u(x,t)$ to the heat equation

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t \tag{20.1a}$$

that also satisfies the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0 \qquad \text{for} \quad 0 < t \tag{20.1b}$$

and the initial conditions

$$u(x,0) = u_0(x) \qquad \text{for} \quad 0 < x < L \tag{20.1c}$$

where u_0 is some known function.

Recall that, using separation of variables, we found a set of separable partial solutions

$$u_k(x,t) = c_k\phi_k(x)g_k(t) \qquad \text{for} \quad k = \ldots, 3, 4, 5, \ldots$$

where the c_k's are arbitrary constants, the ϕ_k's (along with corresponding λ_k's) are solutions to some eigenvalue problem, and the g_k's satisfy the "other" problem. We call these "partial" solutions because they satisfied the partial differential equation and the boundary conditions, but *not* the initial conditions.

In our example, we found the set of "partial solutions"

$$u_k(x,t) = c_k \sin\left(\frac{k\pi}{L}x\right)e^{-6\lambda_k t}$$

with

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2 \quad \text{and} \quad k = 1, 2, 3, \ldots \quad .$$

Here, each

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2 \quad \text{and} \quad \phi_k(x) = \sin\left(\frac{k\pi}{L}x\right)$$

is a eigen-pair for the eigen-problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \qquad \text{for} \quad 0 < x < L$$
$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0 \quad .$$

Notice that this eigen-problem is a Sturm-Liouville problem with weight function $w(x) = 1$ and corresponding inner product

$$\langle\, f \mid g \,\rangle = \int_0^L f^*(x)g(x) \cdot 1 \cdot dx = \int_0^L f^*(x)g(x)\, dx \quad .$$

By the Sturm-Liouville theory, we know the set of all these ϕ_k's is a complete orthogonal set, and, if f is any "reasonable" function on $(0, L)$, then

$$f(x) = \sum_{k-1}^{\infty} C_k\, \phi_k(x) = \sum_{k-1}^{\infty} C_k \sin\left(\frac{k\pi}{L}x\right) \tag{20.2a}$$

where

$$C_k = \frac{\langle\, \phi_k \mid f \,\rangle}{\|\phi\|^2} = \frac{\displaystyle\int_0^L \sin\left(\frac{k\pi}{L}x\right) f(x)\, dx}{\displaystyle\int_0^L \sin^2\left(\frac{k\pi}{L}x\right) dx} = \frac{2}{L}\int_0^L \sin\left(\frac{k\pi}{L}x\right) f(x)\, dx \quad . \tag{20.2b}$$

In general, we should expect the eigen-problem yielding the ϕ_k's in our list of separable solutions

$$u_k(x,t) = c_k\phi_k(x)g_k(t) \qquad \text{for} \quad k = \cdots$$

to be a Sturm-Liouvlle problem with some corresponding weight function $w(x)$ and corresponding inner product

$$\langle\, f \mid g \,\rangle = \int_a^b f^*(x)g(x)w(x)\, dx \quad .$$

The Sturm-Liouville theory will (usually) assure us that the ϕ_k's form a complete orthogonal set of functions, and that, if f is any "reasonable" function, then

$$f(x) = \sum_k C_k\, \phi_k(x) \qquad \text{with} \quad C_k = \frac{\langle\, \phi_k \mid f \,\rangle}{\|\phi_k\|^2} \quad . \tag{20.3}$$

So what?

Well, recall again that each of the u_k's satisfies both the given homogeneous partial differential equation and the given boundary conditions (but not necessarily the initial conditions). Moreover, because the partial differential equation is homogeneous and the boundary conditions are “suitable”, we have a principle of superposition telling us that any (possibly infinite) linear combination of these partial solutions

$$u(x,t) \;=\; \sum_k u_k(x,t) \;=\; \sum_k c_k \phi_k(x) g_k(t)$$

also will satisfy the partial differential equation and boundary conditions. So all we need to do is to set $u(x,t)$ equal to such a linear combination (as above) and determine the c_k's so that this linear combination, with $t = 0$, satisfies the initial conditions — and we can use equation set (20.3) to do this.

For our heat flow example, applying the principle of superposition yields

$$u(x,t) \;=\; \sum_k u_k(x,t) \;=\; \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{L}\,x\right) e^{-6\lambda_k t} \qquad \text{with} \quad \lambda_k \;=\; \left(\frac{k\pi}{L}\right)^2$$

as a ‘general’ formula for a function satisfying both the partial differential equation and the given boundary conditions.

Plugging $t = 0$ into this series and recalling that $u(x,0) = u_0(x)$ reduces this infinite series for $u(x,t)$ to an infinite series for the initial temperature distribution,

$$u_0(x) \;=\; u(x,0) \;=\; \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{L}\,x\right) e^{-6\lambda_k 0} \;=\; \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{L}\,x\right) \quad .$$

Thus, as noted in equation set (20.2), we must have

$$c_k \;=\; \frac{\langle\, \phi_k \mid u_0 \,\rangle}{\|\phi_k\|^2} \;=\; \frac{2}{L}\int_0^L \sin\left(\frac{k\pi}{L}x\right) u_0(x)\,dx \quad .$$

And thus, the solution to our original heat flow problem is

$$u(x,t) \;=\; \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{L}\,x\right) e^{-6\lambda_k t}$$

where

$$\lambda_k \;=\; \left(\frac{k\pi}{L}\right)^2 \qquad , \qquad c_k \;=\; \frac{2}{L}\int_0^L \sin\left(\frac{k\pi}{L}x\right) u_0(x)\,dx \quad ,$$

and $u_0(x)$ is whatever our initial temperature distribution was.

Let's finish by finally picking an specific initial temperature distribution, and solving for $u(x,t)$.

!►Example 20.1: *Find the solution $u = u(x,t)$ to the heat equation*

$$\frac{\partial u}{\partial t} - 6\,\frac{\partial^2 u}{\partial x^2} \;=\; 0 \qquad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < t$$

that also satisfies the boundary conditions

$$u(0,t) \;=\; 0 \qquad \text{and} \qquad u(1,t) \;=\; 0 \qquad \text{for} \quad 0 < t$$

and the initial conditions

$$u(x,0) = \begin{cases} 1 & \text{if } 0 < x < 1/2 \\ 0 & \text{if } 1/2 < x < 1 \end{cases} \quad .$$

Here, $L = 1$ *, and, applying what we've already derived,*

$$u(x,t) = \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{L}x\right) e^{-6\lambda_k t} = \sum_{k=1}^{\infty} c_k \sin(k\pi x)\, e^{-6\lambda_k t}$$

where

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2 = k^2\pi^2$$

and

$$\begin{aligned} c_k &= \frac{2}{L}\int_0^L \sin\left(\frac{k\pi}{L}x\right) u_0(x)\,dx \\ &= \frac{2}{1}\int_0^1 \sin(k\pi x) \begin{Bmatrix} 1 & \text{if } 0 < x < 1/2 \\ 0 & \text{if } 1/2 < x < 1 \end{Bmatrix} dx \\ &= 2\left\{\int_0^{1/2} \sin(k\pi x)\cdot 1\,dx + \int_{1/2}^1 \sin(k\pi x)\cdot 0\,dx\right\} \\ &= 2\left\{-\frac{1}{k\pi}\cos(k\pi x)\Big|_0^{1/2} + 0\right\} \\ &= \frac{2}{k\pi}\left\{-\cos\left(\frac{k\pi}{2}\right) + \cos(0)\right\} = \frac{2}{k\pi}\left[1 - \cos\left(\frac{k\pi}{2}\right)\right] \quad . \end{aligned}$$

So,

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} c_k \sin(k\pi x)\, e^{-6\lambda_k t} \\ &= \sum_{k=1}^{\infty} \frac{2}{k\pi}\left[1 - \cos\left(\frac{k\pi}{2}\right)\right] \sin(k\pi x)\, e^{-6\pi^2 k^2 t} \\ &= \frac{2}{1\pi}\left[1 - \cos\left(\frac{1\pi}{2}\right)\right] \sin(1\pi x)\, e^{-6\pi^2 1^2 t} \\ &\quad + \frac{2}{2\pi}\left[1 - \cos\left(\frac{2\pi}{2}\right)\right] \sin(2\pi x)\, e^{-6\pi^2 2^2 t} \\ &\qquad + \frac{2}{3\pi}\left[1 - \cos\left(\frac{3\pi}{2}\right)\right] \sin(3\pi x)\, e^{-6\pi^2 3^2 t} \\ &\qquad\quad + \frac{2}{4\pi}\left[1 - \cos\left(\frac{4\pi}{2}\right)\right] \sin(4\pi x)\, e^{-6\pi^2 4^2 t} + \cdots \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{1\pi}\,[1]\sin(1\pi x)\,e^{-6\pi^2 1^2 t} \\
&\qquad + \frac{2}{2\pi}\,[2]\sin(2\pi x)\,e^{-6\pi^2 2^2 t} \\
&\qquad\qquad + \frac{2}{3\pi}\,[1]\sin(3\pi x)\,e^{-6\pi^2 3^2 t} \\
&\qquad\qquad\qquad + \frac{2}{4\pi}\,[0]\sin(4\pi x)\,e^{-6(\pi^2 4^2 t} + \cdots \\
&= \frac{2}{1\pi}\sin(1\pi x)\,e^{-6\pi^2 1^2 t} + \frac{4}{2\pi}\sin(2\pi x)\,e^{-6\pi^2 2^2 t} + \frac{2}{3\pi}\sin(3\pi x)\,e^{-6\pi^2 3^2 t} \\
&\qquad + \frac{2}{5\pi}\sin(5\pi x)\,e^{-6\pi^2 5^2 t} + \frac{4}{6\pi}\sin(6\pi x)\,e^{-6\pi^2 6^2 t} + \frac{2}{7\pi}\sin(7\pi x)\,e^{-6\pi^2 7^2 t} \\
&\qquad + \cdots \quad .
\end{aligned}$$

Summary

To solve a partial differential equation problem consisting of a (separable) homogeneous partial differential equation involving variables x and t, suitable boundary conditions at $x = a$ and $x = b$, and some initial conditions:

1. First use the separation of variables method to obtain a list of separable functions[1]

$$u_k(x,t) = c_k \phi_k(x) g_k(t) \qquad \text{for} \quad k = \cdots$$

satisfying both the partial differential equation and the boundary conditions. Do *not* even think about the initial conditions yet. Initial conditions are dealt with last.

To determine the ϕ_k's, you will solve a Sturm-Liouville problem. Be sure to note the corresponding weight function $w(x)$ and corresponding inner product

$$\langle\, f \mid g \,\rangle = \int_a^b f^*(x) g(x) w(x)\,dx \quad .$$

Keep in mind that the ϕ_k's form a complete orthogonal set of functions.[2] Thus, for any "reasonable function" f,

$$f(x) = \sum_k C_k\,\phi_k(x) \qquad \text{with} \quad C_k = \frac{\langle\, \phi_k \mid f \,\rangle}{\|\phi_k\|^2} \quad . \tag{20.4}$$

2. Then set

$$u(x,t) = \sum_k c_k\,\phi_k(x) g_k(t) \quad .$$

Use this formula with your initial conditions and equation/formula set (20.4) to find the values for the c_k's. This infinite series formula for $u(x,t)$ is your solution to the entire partial differential equation problem.

---

[1] We've made a slight notational change. In chapter 18 we included an arbitrary constant in the formula for ϕ_k. Now we are assuming ϕ_k is a single chosen eigenfunction as discussed in the chapter on Sturm-Liouville problems, and are waiting until the end to stick in the arbitrary constant.

[2] If the eigenvalues from the eigen-problem are not simple, remember to determine an orthogonal pair of corresponding eigenfunctions for each eigenvalue. This is usually easily done, and is only likely to arise with periodic boundary conditions.

The above is the basic standard approach to solving many of the partial differential equation problems normally encountered in physics and engineering. In practice, you may have to make some obvious modifications, such as using different symbols for the variables. Also, as written, the above best describes the situation when the "other problem" is of first order (as in our example). In that case, the solution to the other problem for each eigenvalue λ involves just one arbitrary constant. If, as may well happens, the other problem is of second order, then, for each eigenvalue λ_k, the solution to the other problem will be of the form

$$\alpha_k g_k(t) \;+\; \beta_k h_k(t)$$

where g_k and h_k are an independent pair of solutions to this ordinary differential equation. In this case, the list of separable solutions to the partial differential equation and boundary conditions is best written as

$$a_k\phi_k(x)g_k(t) \quad \text{and} \quad b_k\phi_k(x)g_k(t) \qquad \text{for} \quad k = \cdots \quad ,$$

and for u we take

$$u(x,t) \;=\; \sum_k \left[a_k\phi_k(x)g_k(t) + b_k\phi_k(x)h_k(t)\right]$$

or, equivalently,

$$u(x,t) \;=\; \sum_k a_k\phi_k(x)g_k(t) \;+\; \sum_k b_k\phi_k(x)h_k(t) \quad .$$

You still use the initial conditions (usually two initial conditions, in this case) along with equation/formula (20.4) to find each constant a_k and b_k. It's a little more work, of course, but typically turns out to be relatively simple and straightforward.

What About Convergence?

If the given partial differential equation problem has a solution $u(x,t)$ for $t > 0$, then, for each positive value of t, the solution formula

$$u(x,t) \;=\; \sum_k c_k\,\phi_k(x)g_k(t)$$

found by the above procedure is guaranteed to at least converge in norm. This follow from the completeness of the set of ϕ_k's. What can be said about convergence beyond this depends largely on the g_k's.

For example, in the series solution

$$u(x,t) \;=\; \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{L}\,x\right) e^{-6\lambda_k t} \qquad \text{with} \quad \lambda_k \;=\; \left(\frac{k\pi}{L}\right)^2$$

we derived to our heat flow problem, the g_k's are very rapidly decreasing exponentials that decay even more rapidly as k gets larger. Using this and the fact that

$$|c_k| \;=\; \left|\frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{k\pi}{L}x\right)\,dx\right| \;\leq\; \cdots \;\leq\; \frac{2}{L}\int_0^L |u_0(x)|\,dx$$

you should be able to show that, for each $t > 0$,

$$u(x,t) \;=\; \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi}{L}\,x\right) e^{-6\lambda_k t} \qquad \text{with} \quad \lambda_k \;=\; \left(\frac{k\pi}{L}\right)^2$$

converges absolutely and uniformly on the interval $(0, L)$. You should even be able to get a usable error estimate for using the first few terms instead of all terms. This, in turn, tells us that, yes, indeed, our problem has a well-defined (and very well-behaved) solution for all $t > 0$.

On the other hand, some problems do not lead to such nicely behaved functions of t, and you might not have a series that converges uniformly or absolutely. Often, you will end up with the functions of t being sine and/or cosine functions. Convergence in norm and pointwise convergence at points of continuity can be verified via classical "Fourier analysis" theory, or, by more elementary means if the initial conditions are 'nice enough' to ensure that the coefficients shrink sufficiently rapidly as the index increases to guarantee uniform convergence.

In fact, it is possible to have problems leading to divergent series for $t > 0$, which in turn, means that the problem has no solution. Students are rarely exposed to such problems, but you might want to consider our sample problem with the heat equation replaced by the "backward" heat equation

$$\frac{\partial u}{\partial t} + 6\frac{\partial^2 u}{\partial x^2} = 0 \quad .$$

And that is all I will say about the convergence of these series solution for now. For a better discussion, take a good course in partial differential equations.

20.2 Higher Dimensional Problems

General Comments

Extending our discussions to higher dimensional problems is straightforward, especially when we can carry out all the needed separatations. For example, suppose our solution u is a function of three variables, $u = u(x, y, t)$ on the rectangle

$$\mathcal{R} = \{ (x, y) : a < x < b \text{ and } \alpha < y < \beta \} \quad .$$

For convenience, assume the initial condition is simply

$$u(x, y, 0) = u_0(x, y) \qquad \text{for} \quad (x, y) \text{ in } \mathcal{R}$$

where u_0 is some known function. Doing the first separation may yield partial solutions of the form

$$\Psi_k(x, y) g_k(t) \qquad \text{for} \quad k = \cdots \quad .$$

with the Ψ_k's being eigenfunctions from some two-dimensional eigen-problem. With luck this eigenfunction problem will also be separable, and, for each k, we may get

$$\Psi_k(x, y) = \sum_n c_{k,n} \Phi_{k,n}(x, y) \qquad \text{with} \quad \Phi_{k,n}(x, y) = \phi_{k,n}(x)\psi_{k,n}(y) \qquad \text{for} \quad n = \cdots$$

where the $\phi_{k,n}(x)$'s and $\psi_{k,n}(y)$'s are specific eigenfunctions from some pair of one-dimensional eigen-problems corresponding to weight functions $w(x)$ and $\mu(y)$, respectively. Straightforward extensions of the theory we developed will (often) show that these $\Phi_{k,n}$'s form a complete orthogonal set of functions on $\mathcal{R}$ using the two-dimensional inner product

$$\langle\, u(x, y) \mid v(x, y) \,\rangle = \int_{x=a}^{b} \int_{y=\alpha}^{\beta} u^*(x, y) v(x, y) W(x, y) \, dy \, dx$$

where the two-dimensional weight function W is simply

$$W(x, y) = w(x)\mu(y) \quad .$$

The completeness of this set of functions then means that, for any 'reasonable' function $f(x, y)$ on $\mathcal{R}$,

$$f(x, y) = \sum_k \sum_n C_{k,n}\Phi_{k,n}(x, y) = \sum_k \sum_n C_{k,n}\phi_{k,n}(x)\psi_{k,n}(y) \tag{20.5a}$$

where, using the above inner product,

$$C_{k,n} = \frac{\langle \Phi_{k,n}(x, y) \mid f(x, y) \rangle}{\|\Phi_{k,n}\|^2} = \frac{\langle \phi_{k,n}(x)\psi_{k,n}(y) \mid f(x, y) \rangle}{\|\phi_{k,n}(x)\psi_{k,n}(y)\|^2} \quad . \tag{20.5b}$$

The solution to the full problem is then given by

$$u(x, y) = \sum_k \sum_n c_{k,n}\Phi_{k,n}(x, y)g_k(t) = \sum_k \sum_n c_{k,n}\phi_{k,n}(x)\psi_{k,n}(y)g_k(t) \quad ,$$

and the initial condition with this becomes

$$u_0(x, y) = u(x, y, 0) = \sum_k \sum_n c_{k,n}\phi_{k,n}(x)\psi_{k,n}(y)g_k(0)$$

and the $c_{k,n}$'s can be found using equation set (20.5) (with $C_{k,n} = c_{k,n}g_k(0)$).

For examples to illustrate the above, we'll go to the homework handouts.

?►Exercise 20.1: *Do problem F in Homework Handout IX.*

For a somewhat more challenging example, and one that leads to more interesting Sturm-Liouville problems with weight functions other than $w(x) = 1$, we'll turn to the heat flow on a disk problem problem from the homework.

Heat Flow on a Disk (Part I)
The Complete Problem

Let us consider finding the temperature distribution as a function of time and position, $u = u(\boldsymbol{x}, t)$, on a disk of radius a when the initial temperature distribution is known and the temperature on the boundary of the disk is kept at zero degrees. In terms of polar coordinates, this means finding $u = u(r, \theta, t)$ satisfying the partial differential equation

$$\frac{\partial u}{\partial t} - \kappa\nabla^2 u = 0 \qquad \text{for} \quad 0 \le r < a \quad \text{and} \quad t > 0$$

(where κ is some positive constant depending on the material making up the disk), along with the boundary conditions

$$u(a, \theta, t) = 0 \qquad \text{for} \quad t > 0$$

as well as the initial condition

$$u(r, \theta, 0) = u_0(r, \theta) \qquad \text{for} \quad 0 \le r < a \quad \text{and} \quad t > 0$$

for some known function u_0 .

We might not explicitly mention it, but θ can either be treated as any real number or limited to $0 \leq \theta \leq 2\pi$. Obviously, though, the solution must satisfy

$$u(r, 0, t) = u(r, 2\pi, t) \qquad \text{and} \qquad \left.\frac{\partial u}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial u}{\partial \theta}\right|_{\theta=2\pi} \quad .$$

These will end up being boundary conditions for functions of θ . We will also find that we need a boundary condition at $r = 0$, the center of the disk. There is no physical reason to expect it to be any particular value, but it certainly must be some finite value at each t . So all we will insist is that

$$|u(0, \theta, t)| < \infty \qquad \text{for each} \quad t > 0 \quad .$$

This will turn out to be sufficient.

The last few boundary conditions might be considered implicit or even "hidden" since they probably are not explicitly stated with the original problem. In practice, you might not realize these are boundary conditions worth noting until you try to figure out the eigen-problems that the separation of variables leads to.

In summary, our complete problem is to find $u = u(r, \theta, t)$ satisfying

$$\frac{\partial u}{\partial t} - \kappa \nabla^2 u = 0 \qquad \text{for} \quad 0 \leq r < a \quad \text{and} \quad t > 0 \quad , \tag{20.6a}$$

boundary conditions

$$u(a, \theta, t) = 0 \qquad \text{and} \qquad |u(0, \theta, t)| < \infty \tag{20.6b}$$

and

$$u(r, 0, t) = u(r, 2\pi, t) \qquad \text{and} \qquad \left.\frac{\partial u}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial u}{\partial \theta}\right|_{\theta=2\pi} \quad , \tag{20.6c}$$

and initial condition

$$u(r, \theta, 0) = u_0(r, \theta) \quad . \tag{20.6d}$$

Keep in mind that κ and a are known positive constants, $u_0(r, \theta)$ is a known function, and

$$0 \leq r < a \quad , \quad 0 \leq \theta \leq 2\pi \quad \text{and} \quad t > 0 \quad .$$

You may recognize this problem. It was in the homework (more than once).

Separating the Variables

In the homework, you should have showed that, letting

$$u(r, \theta, t) = \Psi(r, \theta) g(t)$$

leads to the two-dimensional eigen-problem consisting of the partial differential equation

$$\nabla^2 \Psi = -\lambda \Psi$$

with boundary conditions

$$\Psi(a, \theta) = 0 \qquad \text{and} \qquad |\Psi(0, \theta)| < \infty \quad ,$$

and

$$\Psi(r,0) = \Psi(r,2\pi) \qquad \text{and} \qquad \left.\frac{\partial\Psi}{\partial\theta}\right|_{\theta=0} = \left.\frac{\partial\Psi}{\partial\theta}\right|_{\theta=2\pi} \quad .$$

The other equation is

$$\frac{dg(t)}{dt} = -\lambda\kappa g(t) \quad .$$

Thus, whatever the first separation constant λ ends up being, the corresponding g is

$$g(t) = g_\lambda(t) = \text{constant}_\lambda \cdot e^{-\lambda\kappa t} \quad . \tag{20.7}$$

After letting

$$\Psi(r,\theta) = \phi(r)\psi(\theta)$$

and using the polar coordinate formula for the Laplacian, you discovered that the above two-dimensional eigen-problem "separates" into two one-dimensional eigen-problems related by a second separation constant μ. The one involving θ is

$$\frac{d^2\psi(\theta)}{d\theta^2} = -\mu\psi(\theta) \tag{20.8}$$

with

$$\psi(0) = \psi(2\pi) \qquad \text{and} \qquad \left.\frac{d\psi}{d\theta}\right|_{\theta=0} = \left.\frac{d\psi}{d\theta}\right|_{\theta=2\pi} \quad . \tag{20.9}$$

This one is easily solved. You should have gotten

$$\mu = \mu_m = m^2 \qquad \text{for} \quad m = 0,\ 1,\ 2,\ 3,\ \ldots \tag{20.10a}$$

with corresponding eigenfunctions

$$\psi_0(\theta) = \text{constant} = A_0 \tag{20.10b}$$

and

$$\psi_m(\theta) = A_m\cos(m\theta) + B_m\sin(m\theta) \qquad \text{for} \quad m = 1,\ 2,\ 3,\ \ldots \quad . \tag{20.10c}$$

The eigen-problem involving $\phi(r)$ is more challenging. The ordinary differential equation for ϕ arising from this second separation of variables is

$$r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \lambda r^2\phi = \mu\phi \quad .$$

Note that this equation has both of the separation constants: the λ from the first separation, which is yet undetermined, and the μ from the second separation, whose values have been determined and are listed in equation (20.10a), above. Putting this differential equation into self-adjoint form, using the above values for μ, and recalling the appropriate boundary conditions, we end up with the Sturm-Liouville problems

$$\frac{d}{dr}\left[r\frac{d\phi}{dr}\right] - \frac{m^2}{r}\phi = -\lambda r\phi \tag{20.11a}$$

with

$$|\phi(0)| < \infty \qquad \text{and} \qquad \phi(a) = 0 \tag{20.11b}$$

for $m = 0,\ 1,\ 2,\ 3,\ \ldots$. Following standard conventions, we will call this the *radial problem* on the disk. Solving this will lead to a bunch more math.

?►*Exercise 20.2:* *Verify any of the above that you have not already derived in previous homework.*

?►*Exercise 20.3:* *Show that the eigenvalue λ in equation (20.11a) must be positive. (Use the Rayleigh quotient.)*

While the radial problem (for each m) is not quite a regular Sturm-Liouville problem (as officially defined in the previous chapter), it can be shown that the results of the mega-theorem on regular Sturm-Liouville problems, theorem 19.5 on page 19–18 still apply. For each positive integer m there is an increasing sequence of eigenvalues

$$\lambda_{m1} < \lambda_{m2} < \lambda_{m3} < \lambda_{m4} < \cdots$$

and a corresponding complete orthogonal set of eigenfunctions

$$\{ \phi_{m1}, \phi_{m2}, \phi_{m3}, \phi_{m4}, \ldots \} \quad .$$

From an exercise above, we know $\lambda_{m1} > 0$. Later, we will confirm that these eigenvalues are all simple. The orthogonality of the eigenfunctions is with respect to the weight function $w(r) = r$ on the interval $(0, a)$. From the Sturm-Liouville theory we at least know the graphs of these ϕ_{mk}'s "wiggle" above and below the R–axis with the number of wiggles increasing as the index k increases. Part of the work ahead of us is to get a better idea of the graphs of these functions.

Getting back to our original problem: All the above, along with the theory we've developed, tells us that the solution u to our "heat flow on a disk" problem can be written as

$$u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} c_{mk}\, \phi_{mk}(r) \psi_m(\theta) g_{mk}(t) \quad . \tag{20.12}$$

?►*Exercise 20.4* **a:** *Write out the above summation as completely as possible using the above formulas for the $g_{mk}(t)$'s and $\psi_m(\theta)$'s, and keeping in mind that the λ_{mk}'s and $\phi_{mk}(r)$'s have yet to be determined. I would suggest splitting the summation in to convenient pieces.*

b: *What, once we've found the $\phi_{mk}(r)$'s, is/are the formulas for the c_{mk}'s ?*

The Radial Problem, Slightly Generalized and Simplified

Now let's start looking at the Sturm-Liouville problem

$$\frac{d}{dr}\left[r \frac{d\phi}{dr} \right] - \frac{\nu^2}{r}\phi = -\lambda r \phi$$

with

$$|\phi(0)| < \infty \qquad \text{and} \qquad \phi(a) = 0 \quad .$$

where ν is some nonnegative real number. (For the "heat flow on a disk" problem, ν is an integer, but we may want to consider cases where ν might not be an integer.)

Using Rayleigh's quotient, it is easily verified that each eigenvalue λ for this problem is positive. This allows us to hide λ via a cheap trick:

$$\text{Let} \quad z = \sqrt{\lambda}\, r \qquad \left(\text{equivalently,} \quad r = \frac{z}{\sqrt{\lambda}}\right) \quad .$$

And then let

$$y(z) = \phi(r) = \phi\left(\frac{1}{\sqrt{\lambda}} z\right) \qquad \left(\text{equivalently,} \quad \phi(r) = y\left(\sqrt{\lambda}\, r\right)\right)$$

where (λ, ϕ) is any solution to the above Sturm-Liouville problem. *You* can then show that our Sturm-Liouville problem converts to

$$\frac{d}{dz}\left[z\frac{dy}{dz}\right] - \frac{\nu^2}{z} y = -zy \tag{20.13a}$$

with

$$|y(0)| < \infty \qquad \text{and} \qquad y\left(\sqrt{\lambda}\, a\right) = 0 \quad . \tag{20.13b}$$

?►***Exercise 20.5:*** *Assume λ is a positive constant, and let ϕ and y be two functions related by*

$$\phi(r) = y\left(\sqrt{\lambda}\, r\right) \quad .$$

Show that

$$\frac{d}{dr}\left[r\frac{d\phi}{dr}\right] - \frac{\nu^2}{r}\phi = -\lambda r\phi \quad \Longleftrightarrow \quad \frac{d}{dz}\left[z\frac{dy}{dz}\right] - \frac{\nu^2}{z} y = -zy \quad .$$

Equation (20.13a), above, is *Bessel's equation of order* ν. It can also be written as

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - \nu^2\right) y = 0 \quad . \tag{20.14}$$

This is an easier form to work with when using, say, the method of Frobenius to derive a series solution.

Do observe that, to solve our Sturm-Liouville problem for (λ, ϕ), we merely have to find the solution to equation (20.13a) that satisfies $|y(0)| < \infty$, and then find all points

$$z = z_1,\ z_2,\ z_3,\ \ldots$$

for which

$$y(z_k) = 0 \quad .$$

We then set

$$\sqrt{\lambda_k}\, a = z_k \qquad \left(\text{so} \quad \lambda_k = \left(\frac{z_k}{a}\right)^2\right)$$

and

$$\phi_k(r) = y\left(\sqrt{\lambda_k}\, r\right) = y\left(\frac{z_k r}{a}\right) \quad .$$

So now the issue is "What are the solutions to Bessel's equation?"