18

Partial Differential Equations I: Basics and Separable Solutions

We now turn our attention to differential equations in which the "unknown function to be determined" — which we will usually denote by u — depends on two or more variables. Hence the derivatives are partial derivatives with respect to the various variables.

(By the way, it may be a good idea to quickly review the *A Brief Review of Elementary Ordinary Differential Equations*, Appendex A of these notes. We will be using some of the material discussed there.)

18.1 Intro and Examples

Simple Examples

If we have a horizontally stretched string vibrating up and down, let

$$u(x,t) \;=\; \text{the vertical position at time } t \text{ of the bit of string at horizontal position } x \;,$$

and make some almost reasonable assumptions regarding the string, the universe and the laws of physics, then we can show that $u(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

where c is some positive constant dependent on the physical properties of the stretched string. This equation is called the *one-dimensional wave equation* (with no external forces).

If, instead, we have a uniform one-dimensional heat conducting rod along the X–axis and let

$$u(x,t) \;=\; \text{the temperature at time } t \text{ of the bit of rod at horizontal position } x \;,$$

then, after applying suitable assumptions about heat flow, etc., we get the *one-dimensional heat equation*

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f(x,t) \quad .$$

Here, κ is a positive constant dependent on the material properties of the rod, and $f(x,t)$ describes the thermal contributions due to heat sources and/or sinks in the rod.

The physicists in the class, of course, are also well acquainted with *Schrödinger's equation*

$$i\hbar\frac{\partial u}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial x^2} = V(x)u$$

where $\hbar$ and m are positive constants and $V(x)$ is some potential energy function.

Similar problems involving two- and three-dimensional objects lead to similar partial differential equations with ∇^2 replacing $\partial^2 u/\partial x^2$.[1] Thus, we have the *multi-dimensional wave equation*,

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \quad ,$$

and the *multi-dimensional heat equation*,

$$\frac{\partial u}{\partial t} - \kappa\nabla^2 u = f(\boldsymbol{x}, t) \quad .$$

If, in either of these, we reach an "equilibrium" or "steady-state" (i.e., $\partial u/\partial t = 0$), then these equations reduce to either *Laplace's equation*

$$\nabla^2 u = 0$$

or *Poisson's equation*

$$\nabla^2 u = f(\boldsymbol{x}) \quad .$$

Basic Terminology

The *order* of a partial differential equation is the order of the highest derivative explicitly appearing. In practice, most partial differential equations of interest are second order (a few are first order and a very few are fourth order). We will concentrate on second-order "linear" equations.[2]

A second-order partial differential equation (in variables x_1, x_2, …, x_n) is said to be *linear* if it can be written as

$$\sum_{jk} a_{jk}\frac{\partial^2 u}{\partial x_k \partial x_j} + \sum_{l} b_l \frac{\partial u}{\partial x_l} + cu = f \quad . \tag{18.1}$$

where f, c and the a_{jk}'s and b_l's are constants or are functions of the variables (but not u). This equation is said to be *homogeneous* if (and only if) $f \equiv 0$. We will concentrate our initial attention on homogeneous equations because they are a little easier to deal with and because we must solve a homogeneous equation anyway to get the complete solution to any nonhomogeneous linear equation.

For convenience, we'll occasionally let L denote the differential operator defined by the left side of equation (18.1),

$$L = \sum_{jk} a_{jk}\frac{\partial^2}{\partial x_k \partial x_j} + \sum_{l} b_l \frac{\partial}{\partial x_l} + c \quad .$$

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[1] Remember: $\nabla^2 u$ is the divergence of the gradient of u. In Cartesian coordinates,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cdots \quad .$$

[2] Take MA 506 or MA 526 to learn how to reduce first-order, linear partial differential equations to first-order, linear ordinary differential equations.

That is, for any sufficiently differentiable function w,

$$L[w] = \sum_{jk} a_{jk} \frac{\partial^2 w}{\partial x_k \partial x_j} + \sum_l b_l \frac{\partial w}{\partial x_l} + cw \quad .$$

Using this, equation (18.1) can be written more succinctly as

$$L[u] = f \quad ,$$

and, if it is homogeneous, as

$$L[u] = 0 \quad .$$

You should be aware that second-order linear partial differential equations are often classified as being "hyperbolic", "parabolic" or "elliptic". Crudely speaking:

Hyperbolic partial differential equations are partial differential equations like the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad .$$

Parabolic partial differential equations are partial differential equations like the heat equation,

$$\frac{\partial u}{\partial t} - \kappa \nabla^2 u = 0 \quad .$$

Elliptic partial differential equations are partial differential equations like Laplace's equation,

$$\nabla^2 u = 0 \quad .$$

For an intelligent discussion of the "classification of second-order partial differential equations", take a true partial differential equation course (MA 506 or MA 526-626).

Linearity/Principle of Superposition

Letting

$$L = \sum_{jk} a_{jk} \frac{\partial^2}{\partial x_k \partial x_j} + \sum_l b_l \frac{\partial}{\partial x_l} + c \quad ,$$

it is easily verified that, given a bunch of constants — c_1, c_2, c_3, $\ldots$ — and a corresponding bunch of sufficiently differentiable functions — u_1, u_2, u_3, $\ldots$ — then

$$L[c_1u_1 + c_2u_2 + c_3u_3 + \cdots] = c_1L[u_1] + c_2L[u_2] + c_3L[u_3] + \cdots \quad .$$

Thus, L is a *linear* differential operator.

In particular, if u_1, u_2, u_3, $\ldots$ are all solutions to the homogeneous equation

$$L[u] = 0 \quad ,$$

then

$$\begin{aligned} L[c_1u_1 + c_2u_2 + c_3u_3 + \cdots] &= c_1L[u_1] + c_2L[u_2] + c_3L[u_3] + \cdots \\ &= c_1 \cdot 0 + c_2 \cdot 0 + c_3 \cdot 0 + \cdots \\ &= 0 \quad . \end{aligned}$$

This gives us

Theorem 18.1 (principle of superposition)
Any linear combination of solutions to a homogeneous linear partial differential equation is also a solution to that homogeneous partial differential equation.

We will use this often, even with linear combinations involving infinitely many terms (and, at times, slop over issues of the convergence of the resulting infinite series).

At this point we should spend a few seconds to observe that

$$L[0] = \sum_{jk} a_{jk} \frac{\partial^2 0}{\partial x_k \partial x_j} + \sum_l b_l \frac{\partial 0}{\partial x_l} + c \cdot 0 = 0 \quad .$$

So the constant function $u = 0$ is a solution to every homogeneous linear partial differential equation. This not-so-exciting solution is often called the *trivial solution*. Our main interest, of course, will be in the *non*trivial solutions.

General Solutions

In general, we cannot find "general solutions" (i.e., relatively simple formulas describing all possible solutions) to second-order partial differential equations.[3] The one notable exception is with the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad .$$

Using a clever change of variables, it can be shown that this has the general solution

$$u(x,t) = f(x - ct) + g(x + ct) \tag{18.2}$$

where f and g are arbitrary sufficiently differentiable functions of a single variable.[4]

?►*Exercise 18.1:* *Verify that, if $f(s)$ and $g(s)$ are any two twice-differentiable functions of one variable, then*

$$u(x,t) = f(x - ct) + g(x + ct)$$

satisfies

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad .$$

Observe that, at any given time t, the graph of

$$y = f(x - ct)$$

[3] General solutions to first-order linear partial differential equations can often be found.

[4] Letting $\xi = x + ct$ and $\eta = x - ct$ the wave equation simplifies to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \quad .$$

Integrating twice then gives you $u = f(\eta) + g(\xi)$, which is formula (18.2) after the change of variables.

is just the graph of $y = f(x)$ shifted to the right by ct. Thus, the $f(x+ct)$ part of formula (18.2) can be viewed as a "fixed shape" traveling to the right with speed c. Likewise, the $g(x+ct)$ part of formula (18.2) can be viewed as a "fixed shape" traveling to the left with speed c. Unsurprisingly, these are generally known as *traveling waves*.

?►Exercise 18.2: *Illustrate these last few statements by sketching*

$$u(x,t) = f(x-ct)$$

as a function of x *at* $t=0$, $t=1$, $t=2$ *and* $t=3$, *using, say,* $c=2$ *and either* $f(s)=\arctan(s)$ *or*

$$f(s) = \begin{cases} 1 & \text{if} \quad -1 < s < 0 \\ 0 & \text{otherwise} \end{cases} \quad .$$

?►Exercise 18.3: *Let* $f(s)$ *be any twice-differentiable function of one variable and let* $\mathbf{n}$ *be any unit vector. Verify that*

$$u(\mathbf{x},t) = f(\mathbf{n}\cdot\mathbf{x} - ct)$$

is a solution to the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \quad .$$

(This is a plane wave *solution —* $f(\mathbf{n}\cdot\mathbf{x}-ct)$ *remains constant on planes perpendicular to* $\mathbf{n}$ *and traveling with speed* c *in the direction of* $\mathbf{n}$*.)*

18.2 Separation of Variables for Partial Differential Equations (Part I)

Separable Functions

A function of N variables

$$u(x_1, x_2, \ldots, x_N)$$

is *separable* if and only if it can be written as a product of two functions of different variables,

$$u(x_1, x_2, \ldots, x_N) = g(x_1, \ldots, x_k)\, h(x_{k+1}, \ldots, x_N) \quad .$$

It is *completely separable* if and only if it can be written as a product of N functions, each of which is a function of just one variable,

$$u(x_1, x_2, \ldots, x_N) = g_1(x_1)\, g_2(x_2)\, g(x_3) \cdots g_N(x_N) \quad .$$

!►Example 18.1: *The following functions are all separable:*

$$u(x,t) = e^{-6t}\sin(x)$$

$$u(x, y, t) = \sqrt{x^2 + y^2}\left(\frac{y}{x}\right)\sin(3t)$$

$$u(r, \theta, t) = r\tan(\theta)\sin(3t)$$

with the first and the last being completely separable.

In applications, our "functions" are often really scalar fields, and, to simplify our work, we often try to find coordinate systems under which these fields are given by separable functions.

Finding Separable Solutions

A first step towards solving many partial differential equation problems is to find all possible separable solutions to a given homogeneous linear partial differential equation (i.e., all solutions to the partial differential equation given by separable functions). Here is a procedure for finding all such solutions. For simplicity, we will explicitly discuss equations involving just two variables x and t, though as will be noted, the method is easily extended to equations involving more variables. To illustrate its use, we'll go ahead and find all separable solutions to the simple one-dimensional heat equation

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 0 \quad . \tag{18.3}$$

1. Assume the solution $u(x, t)$ can be written as

$$u(x, t) = g(x)h(t) \quad ,$$

plug this into the partial differential equation, and 'compute the derivatives'.

Letting $u(x, t) = g(x)h(t)$ *in*

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 0 \quad . \tag{18.4}$$

yields

$$\frac{\partial}{\partial t}[g(x)h(t)] - 6\frac{\partial^2}{\partial x^2}[g(x)h(t)] = 0 \quad .$$

which, after differentiating, is

$$g(x)h'(t) - 6g''(x)h(t) = 0 \quad .$$

2. Using good algebra, rearrange the last equation obtained to get

$$\text{formula of } t \text{ only} = \text{formula of } x \text{ only} \tag{18.5}$$

Some side notes:

(a) Dividing through by $g(x)h(t)$ is usually a good idea.

(b) For reasons that won't be clear for a while, it is usually a good idea to move any 'floating' constants to the side with the t variable.

(c) If you can get the equation into form (18.5), then your partial differential equation is said to be *separable*. Otherwise, the partial differential equation is "not separable" and this approach leads to a disappointing dead end. For the rest of this discussion, we will assume the equation *is* separable.

In our example:

$$g(x)h'(t) - 6g''(x)h(t) = 0$$

$$\Longrightarrow \qquad \frac{g(x)h'(t) - 6g''(x)h(t)}{g(x)h(t)} = \frac{0}{g(x)h(t)}$$

$$\Longrightarrow \qquad \frac{h'(t)}{h(t)} - 6\frac{g''(x)}{g(x)} = 0$$

$$\Longrightarrow \qquad \frac{h'(t)}{h(t)} = 6\frac{g''(x)}{g(x)}$$

$$\Longrightarrow \qquad \frac{h'(t)}{6h(t)} = \frac{g''(x)}{g(x)} \quad .$$

3. "Observe" that the only way we can have

$$\text{formula of } t \text{ only} = \text{formula of } x \text{ only}$$

is for both sides to be equal to a single constant (see exercise 18.4 on page 18–10). Because it will later simplify our work slightly, we will denote this constant by "$-\lambda$". This λ is the *separation constant* (and is totally unknown at this point).

After making this observation:

(a) Write this fact down using the separation constant.

(b) Observe further that this gives us a pair of *ordinary* differential equations, each involving λ.

(c) Write out that pair of ordinary differential equations in as simplified a form as seems reasonable.

Continuing our example, we have

$$\frac{h'(t)}{6h(t)} = \frac{g''(x)}{g(x)} = -\lambda \quad .$$

That is,

$$\frac{h'(t)}{6h(t)} = -\lambda \qquad \text{and} \qquad \frac{g''(x)}{g(x)} = -\lambda \quad .$$

After a little algebra, these become

$$h'(t) = -6\lambda h(t) \qquad \text{and} \qquad g''(x) + \lambda g(x) = 0 \quad .$$

4. The pair of ordinary differential equations just found form a 'system' to be solved for all possible values of λ. If you have no other conditions to satisfy (e.g., "boundary conditions" — to be discussed later), go ahead and solve this system for all possible λ's, keeping in mind the following:

(a) For bookkeeping, subscript the solutions found for each λ with that λ.

(b) You will probably have to consider different ranges of λ corresponding to the different 'types' of solutions to one or both of the equations.

(c) We will later discover that, for all cases of interest, λ is real. So we will limit the possible values of λ to the real numbers.

In our example, we have

$$h_\lambda{}'(t) = -6\lambda h_\lambda(t) \qquad \text{and} \qquad g_\lambda{}''(x) + \lambda g_\lambda(x) = 0$$

for each λ *in* $\mathbb{R}$.

The first equation,

$$h_\lambda{}'(t) = -6\lambda h_\lambda(t) \quad ,$$

is a simple separable and linear first-order ordinary differential equation whose solution is readily found to be

$$h_\lambda(t) = C_\lambda e^{-6\lambda t} \qquad \text{for each } \lambda \text{ in } \mathbb{R}$$

where C_λ *is an arbitrary constant.*

The other equation,

$$g_\lambda{}''(x) + \lambda g_\lambda(x) = 0 \quad ,$$

is a second-order, homogeneous linear differential equation with constant coefficients. Solving it starts by assuming $g_\lambda(x) = e^{rx}$. *Plugging this into the differential equation yields the characteristic equation*

$$r^2 + \lambda = 0 \quad ,$$

which, in turn, requires that

$$r = \pm\sqrt{-\lambda} \quad . \tag{18.6}$$

The basic form for the corresponding $g_\lambda(x)$ *depends on whether* λ *is negative, zero or positive. Each case must be considered:*

i. *For* $\lambda < 0$, *we have that* $-\lambda > 0$. *For convenience, let* $\nu = \sqrt{-\lambda}$. *Then*

$$r = \pm\nu \quad ,$$

and

$$g_\lambda(x) = \alpha_\lambda e^{\nu x} + \beta_\lambda e^{-\nu x}$$

where α_λ *and* β_λ *are arbitrary constants, and, as already stated,* $\nu = \sqrt{-\lambda}$.

ii. *If* $\lambda = 0$, *the differential equation for* g_λ *reduces to*

$$g_0{}''(x) = 0 \quad .$$

Integrating this twice yields

$$g_0(x) = \alpha_0 x + \beta_0 \quad .$$

iii. *If* $\lambda > 0$, *then it is convenient to let* $\nu = \sqrt{\lambda}$, *so that formula (18.6) becomes*

$$r = \pm\sqrt{-\lambda} = \pm i\nu \quad .$$

While the general formula for g_λ can be written in terms of complex exponentials, it is better to recall that these complex exponentials can be written in terms of sines and cosines, and that the general formula for g_λ can then be given as

$$g_\lambda(x) = \alpha_\lambda \cos(\nu x) + \beta_\lambda \sin(\nu x)$$

where, again, α_λ and β_λ are arbitrary constants, and, as already stated, $\nu = \sqrt{\lambda}$.

5. For each value of λ, write out the corresponding separable solution to the partial differential equation

$$u_\lambda(x,t) = g_\lambda(x)h_\lambda(t) \quad .$$

using the functions g_λ and h_λ just found. (Remember that the product of arbitrary constants is another arbitrary constant. Use this to simplify your formulas.)

For our example, we have three cases:

i. For $\lambda < 0$,

$$\begin{aligned} u_\lambda(x,t) &= g_\lambda(x)h_\lambda(t) \\ &= \left[\alpha_\lambda e^{\nu x} + \beta_\lambda e^{-\nu x}\right] C_\lambda e^{-6\lambda t} \\ &= \left[A_\lambda e^{\nu x} + B_\lambda e^{-\nu x}\right] e^{-6\lambda t} \end{aligned}$$

where A_λ and B_λ are arbitrary constants, and $\nu = \sqrt{-\lambda}$.

ii. For $\lambda = 0$,

$$\begin{aligned} u_\lambda(x,t) &= u_0(x,t) \\ &= g_0(x)h_0(t) \\ &= [\alpha_0 x + \beta_0]\, C_0 e^{-6\cdot 0\cdot t} \\ &= A_0 x + B_0 \end{aligned}$$

where A_0 and B_0 are arbitrary constants.

iii. For $\lambda > 0$,

$$\begin{aligned} u_\lambda(x,t) &= g_\lambda(x)h_\lambda(t) \\ &= [\alpha_\lambda \cos(\nu x) + \beta_\lambda \sin(\nu x)]\, C_\lambda e^{-6\lambda t} \\ &= [A_\lambda \cos(\nu x) + A_\lambda \sin(\nu x)]\, e^{-6\lambda t} \end{aligned}$$

where A_λ and B_λ are arbitrary constants, and $\nu = \sqrt{\lambda}$.

Thus, the full set of separable solutions to

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 0$$

is given by the set of all $u_\lambda(x,t)$'s where, for each $\lambda \in \mathbb{R}$,

$$u_\lambda(x,t) = \begin{cases} \left[A_\lambda e^{\sqrt{-\lambda}x} + B_\lambda e^{-\sqrt{-\lambda}x}\right] e^{-6\lambda t} & \text{if } \lambda < 0 \\ A_0 x + B_0 & \text{if } \lambda = 0 \\ \left[A_\lambda \cos(\sqrt{\lambda}x) + B_\lambda \sin(\sqrt{\lambda}x)\right] e^{-6\lambda t} & \text{if } \lambda > 0 \end{cases}$$

and the A_λ*'s and* B_λ*'s are all arbitrary constants.*

?►Exercise 18.4: *In step 3 on page 18–7 it was "observed" that the only way we can have*

$$\text{formula of } t \text{ only } = \text{ formula of } x \text{ only}$$

is for each side to be equal to a constant. Verify this. More precisely, assume H *and* G *are each functions of a single variable which satisfy*

$$H(t) = G(x)$$

where t *and* x *are two different variables. Show that there must be a constant* c *such that*

$$H(t) = c = G(x) \qquad \text{for all } t \text{ and } x \quad .$$

Hint: Start by taking the partial derivative of $H(t) = G(x)$ *with respect to either* t *or* x *.*

Further Notes on this Procedure

1. Most problems involving partial differential equations include additional conditions that will allow us to reject many of the separable solutions. We will soon modify the above procedure to avoid computing separable solutions that will later be rejected.

(By the way, one way to reject solutions is to claim that some are not "physically reasonable". For example, since it may be argued that the temperature in the rod cannot increase with time, the solutions of the form

$$\left[A_\lambda e^{\sqrt{-\lambda}x} + B_\lambda e^{-\sqrt{-\lambda}x}\right]e^{-6\lambda t} \qquad \text{with} \quad \lambda < 0$$

may be rejected as "physically unreasonable". This is a poor way to reject solutions because (1) it assumes your mathematical model is a good model and (2) maybe the rod is getting hotter.)

2. With the obvious modifications, the above procedure works with partial differential equations involving variables other than x and t, and with partial differential equations involving more than two variables. When dealing with the later, you should try to separate out one variable at a time. That is, first assume

$$u(x_1, x_2, \ldots, x_N) = g_1(x_1, x_2, \ldots, x_{N-1})\, h_N(x_N) \quad .$$

"Separating" this yields an ordinary differential equation for h_N and a partial differential equation for g_1 — both involving the separation constant λ_N. Then separate the partial differential equation for g_1, assuming

$$g_1(x_1, x_2, \ldots, x_{N-1}) = g_2(x_1, x_2, \ldots, x_{N-2})\, h_{N-1}(x_{N-1}) \quad ,$$

and obtaining an ordinary differential equation for h_{N-1} and a partial differential equation for g_2 — both involving another separation constant λ_{N-1}. Then "separate out" the partial differential equation for g_2, … .

18.3 Boundary and Initial Conditions

In practice, a problem involving a partial differential equation usually consists of that differential equation along with additional "boundary" and/or "initial" conditions that must also be satisfied.

!►Example 18.2: *If we are modeling the heat flow in a rod of length L, visualized as laying on the X–axis from $x = 0$ to $x = L$, and let*

$$u(x, t) = \text{the temperature at time } t \text{ of the bit of rod at horizontal position } x ,$$

Then we may require u to satisfy all of the following

1. *The heat equation*

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t$$

where κ is a positive constant dependent on the material properties of the rod. (This assumes we have no heat sources or sinks in the rod.)

2. *The boundary conditions*

$$u(0, t) = T_0 \qquad \text{and} \qquad u(L, t) = T_L \qquad \text{for all} \quad t > 0$$

where T_0 and T_L are the known temperatures at the ends of the rod. This assumes that we control the temperatures at these ends. If, instead, we were to insulate the ends so that no heat can flow in or out, then 'it can be shown' that the appropriate boundary conditions are

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad \text{and} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \qquad \text{for all} \quad t > 0 \quad .$$

Other boundary conditions are also possible.

3. *An initial condition of the form*

$$u(x, 0) = u_0(x) \qquad \text{for} \quad 0 < x < L$$

where $u_0(x)$ is a known function giving the initial temperature distribution throughout the length of the rod.

In practice:

Initial conditions (ICs) are the things known about the desired solution to the partial differential equation at a particular time.

Boundary conditions (BCs) are the things known about the desired solution to the partial differential equation at the edges (boundaries) of the physical object of interest.

(When time is not involved, as in Laplace's equation, these concepts may have to be slightly modified.)

Be aware that, sometimes, determining just what are the appropriate boundary conditions and initial conditions can be a very nontrivial problem.

18.4 Separation of Variables for Partial Differential Equations (Part II)

Superposition and the Role of Boundary and Initial Conditions

Recently, we saw how to find all the separable solutions to a given homogeneous partial differential equation. As we will soon see, it is the *boundary conditions* that determine which of these separable solutions are relevant and which are not. Ultimately, the idea will be to find a bunch of separable functions $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$, $\ldots$, that satisfy *both* the partial differential equation and the boundary conditions, and then express the solution of real interest as

$$u(x,t) = \sum_k c_k \, u_k(x,t) \tag{18.7}$$

where the c_k's are constants chosen so that the summation, $u(x,t)$, satisfies whatever initial conditions we may have (just how the c_k's can be so chosen will be discussed later). Remember, by the principle of superposition, the (possibly infinite) linear combination of solutions to the differential equation is also a solution to the differential equation. If it also satisfies the boundary conditions, then this summation is the desired solution.

This raises an issue: Can we be sure that the summation in formula (18.7) satisfies the boundary conditions whenever each $u_k(x,t)$ satisfies the given boundary conditions? In general, the answer is "obviously not".

!►*Example 18.3:* *Suppose one boundary condition is*

$$u(0,t) = 50 \qquad \text{for} \quad t > 0 \quad .$$

If each $u_k(x,t)$ satisfies this, then

$$u(0,t) = \sum_k c_k \, u_k(0,t) = \sum_k c_k 50 = 50 \sum_k c_k \quad .$$

So we will have $u(x,t)$ satisfying the above boundary condition if and only if

$$\sum_k c_k = 1 \quad ,$$

which is highly unlikely since, as stated above, the c_k's are being chosen to satisfy a completely different set of conditions, namely, the initial conditions.

To ensure that

$$u(x,t) = \sum_k c_k u_k(x,t)$$

satisfies the given boundary conditions whenever each $u_k(x,t)$ does, we will restrict ourselves to boundary conditions for which a 'principle of superposition' also holds. That is, we will only allow boundary conditions such that, if $u_1(x,t)$ and $u_2(x,t)$ satisfy the given boundary conditions, then so does every linear combination of them,

$$c_1 u_1(x,t) + c_2 u_2(x,t) \quad .$$

For convenience, and because no better term comes to mind, we will call these types of boundary conditions *suitable*.

!►Example 18.4: *Clearly, the boundary condition given in the previous example is not 'suitable'. Suppose, however, a boundary condition is*

$$u(0,t) = 0 \qquad \text{for} \quad t > 0 \quad .$$

If $u_1(x,t)$ *and* $u_2(x,t)$ *are any two functions satisfying this boundary condition, and* c_1 *and* c_2 *are any two constants, then*

$$c_1 u_1(0,t) + c_2 u_2(0,t) = c_1 \cdot 0 + c_2 \cdot 0 = 0 \quad .$$

So

$$u(0,t) = 0 \qquad \text{for} \quad t > 0$$

is a 'suitable' boundary condition.

Observe that, if $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$, ... , *are a bunch of functions that satisfy the above boundary condition, and*

$$u(x,t) = \sum_k c_k\, u_k(x,t)$$

where the c_k*'s are conveniently chosen constants, then*

$$u(0,t) = \sum_k c_k\, u_k(0,t) = \sum_k c_k \cdot 0 = 0 \quad ,$$

as desired.

It is important to note that, if $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$, ... , satisfy any single 'suitable' boundary condition (as defined above), then so will any linear combination of these functions,

$$u(x,t) = \sum_k c_k\, u_k(x,t) \quad ,$$

even if there are infinitely many terms in the series (assuming the series converges, of course).

Here are some 'suitable' boundary conditions (for a function with the variable x) that we will employ:

homogeneous/regular boundary conditions: A boundary condition at $x = a$ is said to be *homogeneous* or *regular* if and only if it can be described by

$$\alpha u\Big|_{x=a} + \beta \frac{\partial u}{\partial x}\Big|_{x=a} = 0$$

where α and β are constants with at least one being nonzero.

Special (and rather common) cases are

$$u(a,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}\Big|_{x=a} = 0 \quad .$$

boundedness boundary conditions: This is where we simply say that a solution does not "blow up" at a point $x = a$. That is,

$$\left| u|_{x=a} \right| < \infty \quad .$$

Such conditions at "$r = 0$" are particularly important in problems involving polar or spherical coordinates.

periodic boundary conditions: A *periodic* boundary condition matches the behavior of the solution at two distinct points $x = a$ and $x = b$. In particular,

$$u|_{x=a} = u|_{x=b} \qquad \text{and} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=a} = \left.\frac{\partial u}{\partial x}\right|_{x=b}$$

are two periodic boundary conditions. In practice, these arise naturally when using polar or spherical coordinates.

(More general periodic boundary conditions can be described, but the two above are the only ones of significant interest to us.)

?► ***Exercise 18.5:*** *Verify that each of the boundary conditions just described is 'suitable'. That is, for each of the boundary conditions just described, show that any linear combination*

$$u = c_1 u_1 + c_2 u_2$$

satisfies the given boundary condition whenever u_1 and u_2 satisfies that boundary condition.

The Full Separation of Variables Procedure

Here we describe in detail the separation of variables procedure when the problem involves a function u of two variables x and t, and suitable boundary conditions are given for $u(x,t)$ at two distinct values of x.[5] This procedure, which is just an extension of that described in section 18.2, yields all the nontrivial separable solutions to a given partial differential equation which also satisfy the given boundary conditions. This does not actually solve the problem you are likely to have — it is just one big step towards solving that problem. We will also introduce a few new concepts that will require further discussion later. These concepts will be central to constructing the final solution using the separable solutions found here.

As suggested a few pages ago, extending these ideas to deal with other partial differential equation problems is straightforward, and will be largely left for you to work out as the need arises.

To illustrate the procedure, we will consider the following heat flow problem on a rod of length L:

Find the solution $u = u(x,t)$ to the heat equation

$$\frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{for} \quad 0 < x < L \quad \text{and} \quad 0 < t \tag{18.8a}$$

that also satisfies the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0 \qquad \text{for} \quad 0 < t \tag{18.8b}$$

and the initial conditions

$$u(x,0) = u_0(x) \qquad \text{for} \quad 0 < x < L \tag{18.8c}$$

where u_0 is some known function (precisely what u_0 is will not be important in these discussions).

---

[5] Mathematically, it's the conditions at 'two distinct values of x' that mean we are talking about 'boundary conditions' and not 'initial' conditions. You'll see why later.

Here, now, is the complete set of steps in doing “separation of variables”:

1. Assume the solution $u(x,t)$ can be written as

$$u(x,t) = \phi(x)h(t) \quad .$$

(Yes, I’ve changed the name of the function of x — no good reason except that it meshes better with notation we’ll be using later.)

2. Identify the boundary conditions, plug $u(x,t) = \phi(x)h(t)$ into those conditions, and determine the corresponding boundary conditions for ϕ.

In our example, the boundary conditions are

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0 \qquad \text{for} \quad 0 < t \quad .$$

Replacing $u(x,t)$ *with* $\phi(x)h(t)$ *in the first gives*

$$\phi(0)h(t) = 0 \qquad \text{for} \quad 0 < t \quad .$$

This means that either

$$\phi(0) = 0 \qquad \text{or} \qquad h(t) = 0 \quad .$$

If $h(t) = 0$ *then we get*

$$u(x,t) = \phi(x)h(t) = \phi(x) \cdot 0 = 0 \quad ,$$

which not very interesting. So, to avoid just getting the trivial solution to our partial differential equation, we will instead require that

$$\phi(0) = 0 \quad .$$

Plugging $u(x,t) = \phi(x)h(t)$ *in the second boundary condition gives*

$$\phi(L)h(t) = 0 \qquad \text{for} \quad 0 < t \quad .$$

Again, this means that either

$$\phi(L) = 0 \qquad \text{or} \qquad h(t) = 0 \quad .$$

And, again, to avoid triviality, we will require

$$\phi(L) = 0 \quad .$$

Thus, our $\phi(x)$ *must, itself, satisfy the two boundary conditions*

$$\phi(0) = 0 \qquad \text{and} \qquad \phi(L) = 0 \quad . \tag{18.9}$$

3. Plug $u(x,t) = \phi(x)h(t)$ into the partial differential equation, simplify and rearrange (still using good algebra) to get that

$$\text{formula of } t \text{ only} = \text{formula of } x \text{ only} \quad ,$$

with all the “floating constants” on the side with the t.

In our example, with $u(x,t) = \phi(x)h(t)$,

$$
\begin{aligned}
&& \frac{\partial u}{\partial t} - 6\frac{\partial^2 u}{\partial x^2} &= 0 \\
\Longrightarrow && \frac{\partial}{\partial t}[\phi(x)h(t)] - 6\frac{\partial^2}{\partial x^2}[\phi(x)h(t)] &= 0 \\
\Longrightarrow && \phi(x)h'(t) - 6\phi''(x)h(t) &= 0 \\
\Longrightarrow && \frac{\phi(x)h'(t) - 6\phi''(x)h(t)}{\phi(x)h(t)} &= \frac{0}{\phi(x)h(t)} \\
\Longrightarrow && \frac{h'(t)}{h(t)} - 6\frac{\phi''(x)}{\phi(x)} &= 0 \\
\Longrightarrow && \frac{h'(t)}{h(t)} &= 6\frac{\phi''(x)}{\phi(x)} \\
\Longrightarrow && \frac{h'(t)}{6\,h(t)} &= \frac{\phi''(x)}{\phi(x)} \quad .
\end{aligned}
$$

(This step requires, of course, that the partial differential equation be separable. If it isn't, stop — separation of variables isn't applicable.)

4. Set each side of the last equation equal to $-\lambda$ (remember, λ is the 'separation constant'), and then write out the two resulting ordinary differential equations (simplifying and rearranging as seems appropriate).

In our example,

$$\frac{h'(t)}{6h(t)} = -\lambda = \frac{\phi''(x)}{\phi(x)} \quad .$$

So we have the ordinary differential equations

$$\frac{h'(t)}{6h(t)} = -\lambda \qquad \text{and} \qquad \frac{\phi''(x)}{\phi(x)} = -\lambda \quad ,$$

which we rewrite as

$$h'(t) = -6\lambda h(t) \qquad \text{and} \qquad \phi''(x) + \lambda\phi(x) = 0 \quad .$$

(Keep in mind that the two ordinary differential equations are not independent of each other — they are linked by the common value λ .)

5. Under the ordinary differential equation for ϕ , write out the boundary conditions obtained for ϕ in step 2.

For our example, the boundary conditions for ϕ *are given by equation set (18.9). Placing them with the ordinary differential equation for* ϕ *gives*

$$
h'(t) = -6\lambda h(t) \qquad \text{and} \qquad
\begin{aligned}
\phi''(x) + \lambda\phi(x) &= 0 \\
\phi(0) &= 0 \\
\phi(L) &= 0
\end{aligned}
$$

Commentary: We have now decomposed our "partial differential equation with boundary conditions" into two ordinary differential equation problems. These two problems are related to each other only through the common constant λ. The ordinary differential equation problem with the boundary conditions is called either the *boundary-value problem* or the *eigenvalue/eigenfunction problem* (*eigenproblem* for short) or the *Sturm-Liouville problem*, depending on the text/instructor/time of day. The other ordinary differential equation will be officially called *the other problem*.

In many ways, finding and solving the eigenproblem is the most important element of this procedure. A solution to the eigenproblem consists of *both* a value for λ and a corresponding function $\phi = \phi_\lambda(x)$ that, together, satisfy the differential equation and the boundary conditions. The value λ is called an *eigenvalue* for the problem, and the function ϕ is called an *eigenfunction* for the problem.[6,7]

Back to the separation of variables procedure:

6. Find all solutions (λ, ϕ_λ) to the eigenproblem. Typically, you find the general solution ϕ_λ to the ordinary differential equation for arbitrary real values of λ, and then use the boundary conditions to determine the possible values for λ.

In our example, the eigenproblem is

$$\phi''(x) + \lambda\phi(x) = 0 \qquad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0 \quad .$$

From the last time we dealt with this differential equation, we know there are three cases to consider: $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. We now need to reconsider these cases, taking into account the required boundary conditions:

$\lambda < 0$: *The general solution to the differential equation when $\lambda < 0$ was found to be*

$$\phi_\lambda(x) = \alpha_\lambda e^{\nu x} + \beta_\lambda e^{-\nu x}$$

where α_λ and β_λ are arbitrary constants and $\nu = \sqrt{-\lambda}$. Applying the first boundary condition:

$$0 = \phi_\lambda(0) = \alpha_\lambda e^{\nu \cdot 0} + \beta_\lambda e^{-\nu \cdot 0} = \alpha_\lambda + \beta_\lambda \quad ,$$

which tells us that

$$\beta_\lambda = -\alpha_\lambda \quad ,$$

and thus,

$$\phi_\lambda(x) = \alpha_\lambda e^{\nu x} - \alpha_\lambda e^{-\nu x} = \alpha_\lambda \big[e^{\nu x} - e^{-\nu x} \big] \quad .$$

Combining this with the boundary condition at $x = L$ yields

$$0 = \phi_\lambda(L) = \alpha_\lambda \big[e^{\nu L} - e^{-\nu L} \big] \quad .$$

[6] As we will see, the boundary-value problem really is an eigenvalue problem with a linear differential operator. In this sense, though, it will be $-\lambda$, not λ which is the eigenvalue — a rather unfortunate perversion of well-known terminology.

[7] We will examine the eigenproblem in great detail in the next chapter.

Now $e^{\nu L} > e^0 = 1$ *and* $e^{-\nu L} < e^0 = 1$ *. So*

$$e^{\nu L} - e^{-\nu L} > 0 \quad ,$$

and the only way we can have

$$0 = \alpha_\lambda \left[e^{\nu L} - e^{-\nu L} \right]$$

is for $\alpha_\lambda = 0$ *. Thus, the only solution to the eigenproblem with* $\lambda < 0$ *is*

$$\phi_\lambda(x) = 0\left[e^{\nu L} - e^{-\nu L} \right] = 0 \quad ,$$

the trivial solution (which we don't really care about).

$\boldsymbol{\lambda = 0}$ *The general solution to the differential equation when* $\lambda = 0$ *was found to be*

$$\phi_0(x) = \alpha_0 x + \beta_0$$

where α_0 *and* β_0 *are arbitrary constants. Applying the first boundary condition gives us*

$$0 = \phi_0(0) = \alpha_0 \cdot 0 + \beta_0 = \beta_0 \quad .$$

Combined with the boundary condition at $x = L$ *, we then have*

$$0 = \phi_0(L) = \alpha_0 L + 0 = \alpha_0 L \quad ,$$

which says that $\alpha_0 = 0$ *(since* $L > 0$ *). Thus, the only solution to the differential equation that satisfies the boundary conditions when* $\lambda = 0$ *is*

$$\phi_0(x) = 0 \cdot x + 0 = 0 \quad ,$$

again, just the trivial solution.

$\boldsymbol{\lambda > 0}$**:** *The general solution to the differential equation when* $\lambda > 0$ *was found to be*

$$\phi_\lambda(x) = \alpha_\lambda \cos(\nu x) + \beta_\lambda \sin(\nu x)$$

where α_λ *and* β_λ *are arbitrary constants and* $\nu = \sqrt{\lambda}$ *. Applying the first boundary condition:*

$$\begin{aligned} 0 &= \phi_\lambda(0) \\ &= \alpha_\lambda \cos(\nu \cdot 0) + \beta_\lambda \sin(\nu \cdot 0) \\ &= \alpha_\lambda \cdot 1 + \beta_\lambda \cdot 0 = \alpha_\lambda \quad , \end{aligned}$$

which tells us that

$$\phi_\lambda(x) = \beta_\lambda \sin(\nu x) \quad .$$

With the boundary condition at $x = L$ *, this gives*

$$0 = \phi(L) = \beta_\lambda \sin(\nu L) \quad .$$

To avoid triviality, we want β_λ *to be nonzero. So, for the boundary condition at* $x = L$ *to hold, we must have*

$$\sin(\nu L) = 0 \quad ,$$

which means that

$$\nu L \;=\; \text{an integral multiple of } \pi \quad ,$$

Moreover, since $\nu = \sqrt{\lambda} > 0$, νL *must be a positive integral multiple of* π *. Thus, we have a list of allowed values of* ν *,*

$$\nu_k \;=\; \frac{k\pi}{L} \qquad \text{with} \quad k \;=\; 1,\, 2,\, 3,\, \ldots \quad ,$$

a corresponding list of allowed values for λ *,*

$$\lambda_k \;=\; (\nu_k)^2 \;=\; \left(\frac{k\pi}{L}\right)^2 \qquad \text{with} \quad k \;=\; 1,\, 2,\, 3,\, \ldots$$

(these are the eigenvalues), and a corresponding list of $\phi(x)$*'s (the corresponding eigenfunctions),*

$$\phi_k(x) \;=\; \beta_k \sin(\nu_k x) \;=\; \beta_k \sin\left(\frac{k\pi}{L}x\right) \qquad \text{with} \quad k \;=\; 1,\, 2,\, 3,\, \ldots$$

where the β_k*'s are arbitrary constants. For our example, these are the only nontrivial eigenfunctions.*

Commentary: As we will see when we discuss the eigenproblem in more detail, you should always end up with an indexed list of eigenvalues — λ_1 , λ_2 , λ_3 , ... — and an indexed list of corresponding eigenfunctions — ϕ_1 , ϕ_2 , ϕ_3 , ... The eigenfunctions will involve arbitrary constants, and the indexing will usually start at 0 or 1 .

7. Now solve the other problem for each λ_k , obtaining the corresponding $h_k(t)$'s .

   In our example, the other problem is

$$h'(t) \;=\; -6\lambda h(t) \quad .$$

   No matter what real value λ *is, the general solution to this differential equation is*

$$h(t) \;=\; Ce^{-6\lambda t}$$

   where C *is an arbitrary constant. Using the* λ_k*'s found above, we have, for each positive integer* k *,*

$$h_k(t) \;=\; C_k e^{-6\lambda_k t} \;=\; C_k e^{-6k^2\pi^2 t/L^2}$$

   where C_k *is an arbitrary constant.*

8. For each eigenvalue λ_k , write out the corresponding separable solution to the partial differential equation,

$$u_k(x,t) \;=\; \phi_k(x)h_k(t) \quad ,$$

   combining arbitrary constants as appropriate. (It's also a good idea to state the range for the indexing.)

For our example,

$$u_k(x,t) = \phi_k(x)h_k(t) = \beta_k \sin\left(\frac{k\pi}{L}x\right) \cdot C_k e^{-6k^2\pi^2 t/L^2} \quad .$$

Letting $c_k = \beta_k \cdot C_k$ *and recalling that the* k*'s are all the positive integers, we see that*

$$u_k(x,t) = c_k \sin\left(\frac{k\pi}{L}x\right) e^{-6k^2\pi^2 t/L^2} \qquad \text{with} \quad k = 1, 2, 3, \ldots$$

describes all the separable solutions to our partial differential equation that also satisfy the given boundary conditions.

This ends the separation of variables part of solving a partial differential equation problem. What we do with these separable solutions will be developed in the next few chapters.

A Few Notes

1. The separation of variables procedure is the first mega-step in solving many partial differential equation problems arising in applications — it leads to a list of suitable 'partial solutions' to our problem. The second mega-step — constructing *the* solution from the partial solutions — requires understanding some of the great properties of the set of solutions to the eigenproblem. Our next big goal is to develop and describe these properties.

2. When you have more than two variables, do the separation 'one variable at a time', as discussed in the notes on page 18–10. The first time through, the eigenproblem will involve another partial differential equation with boundary conditions. Solving that problem will then involve further separation of variables.