A

A Brief Review of Elementary Ordinary Differential Equations

At various points in the material we will be covering, we will need to recall and use material normally covered in an elementary course on ordinary differential equations. In these notes, we will *very briefly* review the main topics that will be needed later. For more complete discussions of these topics, see your old introductory text on ordinary differential equations, or see your instructor's treatment of these topics.[1]

A.1 Basic Terminology

Recall: A *differential equation* (often called a "de") is an equation involving derivatives of an unknown function. If the unknown can be assumed to be a function of only one variable (so the derivatives are the "ordinary" derivatives from Calc. I), then we say the differential equation is an ordinary differential equation (ode). Otherwise, the equation is a partial differential equation (pde). Our interest will just be in odes. In these notes, the variable will usually be denoted by x and the unknown function by y or $y(x)$.

Recall, also, for any given ordinary differential equation:

1. The *order* is the order of the highest order derivative of the unknown function explicitly appearing in the equation.
2. A *solution* is any function (or formula for a function) that satisfies the equation.
3. A *general solution* is a formula that describes all solutions to the equation. Typically, the general solution to a k^{th} order ode contains k arbitrary/undetermined constants.
4. Typically, a "differential equation problem" consists of a differential equation along with some auxiliary conditions the solution must also satisfy (e.g., "initial values" for the solution). In practice you usually find the general solution first, and then choose values for the "undetermined constants" so that the auxiliary conditions are satisfied.

[1] available online at `www.uah.edu/math/howell/DEtext`.

A.2 Some "Analytic" Methods for Solving First-Order ODEs

(Warning: Here, the word "analytic" just means that the method leads to exact formulas for solutions, as opposed to, say, a numerical algorithm that gives good approximations to particular solutions at fixed points. Later in this course, the word "analytic" will mean something else.)

Separable Equations*

A first-order ode is *separable* if it can be written as

$$\frac{dy}{dx} = g(x)h(y) \quad .$$

Such a de can be solved by the following procedure:

1. Get it into the above form (i.e., the derivative equaling the product of a function of x (the $g(x)$ above), with a function of y (the above $h(y)$).
2. Divide through by $h(y)$ (but also consider the possibility that $h(y) = 0$).
3. Integrate both sides with respect to x (don't forget an arbitrary constant).
4. Solve the last equation for $y(x)$.

!►*Example A.1:* *Consider finding the general solution to*

$$\frac{dy}{dx} = 2x\left(y^2 + 1\right) \quad .$$

Going through the above steps:

$$\frac{1}{y^2+1}\frac{dy}{dx} = 2x$$

$$\Longrightarrow \qquad \int \frac{1}{y^2+1}\frac{dy}{dx}\,dx = \int 2x\,dx$$

$$\Longrightarrow \qquad \arctan(y) = x^2 + c$$

$$\Longrightarrow \qquad y = \tan\left(x^2 + c\right) \quad .$$

Linear Equations†

A first-order ode is said to be *linear* if it can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

where $p(x)$ and $q(x)$ are known functions of x . Such a differential equation can be solved by the following procedure:

* see, also, chapter 4 of `www.uah.edu/math/howell/DEtext`

† see, also, chapter 5 of `www.uah.edu/math/howell/DEtext`

1. Get it into the above form.
2. Compute the *integrating factor*
$$\mu(x) = e^{\int p(x)\,dx}$$
(don't worry about arbitrary constants here).
3. *(a)* Multiply the equation from the first step by the integrating factor.
   (b) Observe that, by the product rule, the left side of the resulting equation can be rewritten as $\frac{d}{dx}[\mu y]$, thus giving you the equation
$$\frac{d}{dx}[\mu(x)\,y(x)] = \mu(x)\,q(x) \quad .$$
4. Integrate both sides of your last equation with respect to x, and solve for $y(x)$. Don't forget the arbitrary constant.

!►*Example A.2:* *Consider finding the general solution to*
$$x\frac{dy}{dx} + 4y = 21x^3 \quad .$$
Dividing through by x gives
$$\frac{dy}{dx} + \frac{4}{x}y = 21x^2 \quad .$$
So the integrating factor is
$$\mu(x) = e^{\int p(x)\,dx} = e^{\int 4/x\,dx} = e^{4\ln x} = x^4 \quad .$$
Multiplying the last differential equation above by this integrating factor and then continuing as described in the procedure:
$$x^4\left[\frac{dy}{dx} + \frac{4}{x}y\right] = x^4\left[21x^2\right]$$
$$\Longrightarrow \qquad x^4\frac{dy}{dx} + 4x^3y = 21x^6 \quad .$$
But, by the product rule,
$$\frac{d}{dx}\left[x^4y(x)\right] = x^4\frac{dy}{dx} + 4x^3y \quad ,$$
and so we can rewrite our last differential equation as
$$\frac{d}{dx}\left[x^4y(x)\right] = 21x^6 \quad .$$
This can be easily integrated and solved:
$$\int \frac{d}{dx}\left[x^4y(x)\right]dx = \int 21x^6\,dx$$
$$\Longrightarrow \qquad x^4y(x) = 3x^7 + c$$
$$\Longrightarrow \qquad y(x) = \frac{3x^7 + c}{x^4}$$
$$\Longrightarrow \qquad y(x) = 3x^3 + cx^{-4} \quad .$$

Two notes on this method:

1. The formula for the integrating factor $\mu(x)$ is actually derived from the requirement that

$$\frac{d}{dx}[\mu(x)\,y(x)] = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu\frac{dy}{dx} + \mu p y \quad ,$$

which is the "observation" made in step 3b of the procedure. This means that μ must satisfy the simple differential equation

$$\frac{d\mu}{dx} = \mu p \quad .$$

2. Many texts state a formula for $y(x)$ in terms of $p(x)$ and $q(x)$. The better texts also state that memorizing and using this formula is stupid.

Other Methods

Other methods for solving first-order ordinary differential equations include the integration of exact equations, and the use of either clever substitutions or more general integrating factors to reduce "difficult" equations to either separable, linear or exact equations. See a good de text if you are interested.

A.3 Higher-Order Linear Differential Equations

Basics[‡]

An N^{th} order differential equation is said to be *linear* if it can be written in the form

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2}y'' + a_{N-1}y' + a_N y = f$$

where f and the a_k's are known functions of x (with $a_0(x)$ not being the zero function). The equation is said to be *homogeneous* if and only if f is the zero function (i.e., is always 0).

Recall that, if the equation is homogeneous, then we have "linearity", that is, whenever y_1 and y_2 are two solutions to a homogeneous linear differential equation, and a and b are any two constants, then $y = ay_1 + by_2$ is another solution to the differential equation. In other words, the set of solutions to a homogeneous linear differential equations is a *vector space* of functions. (Isn't it nice to see vector spaces again?)

Recall further, that

1. The general solution to an N^{th} order linear homogeneous ordinary differential equation is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x)$$

where the c_k's are arbitrary constants and

$$\{y_1,\ y_2,\ \ldots,\ y_N\}$$

[‡] see, also, chapter 12, sections 1 — 3, and chapter 14 of www.uah.edu/math/howell/DEtext

is a linearly independent set of solutions to the homogeneous de. (i.e., $\{y_1, y_2, \ldots, y_N\}$ is a basis for the N-dimensional space of solutions to the homogeneous differential equation.)

2. A general solution to an N^{th} order linear *non*homogeneous ordinary differential equation is given by

$$y(x) = y_p(x) + y_h(x)$$

where y_p is any particular solution to the nonhomogeneous ordinary differential equation and y_h is a general solution to the corresponding homogeneous ode.

In "real" applications, N is usually 1 or 2. On rare occasions, it may be 4, and, even more rarely, it is 3. Higher order differential equations can arise, but usually only in courses on differential equations. Do note that if $N = 1$, then the differential equation can be solved using the method describe for first order linear equations (see page A–2).

Notes About Linear Independence

Recall that a set of functions

$$\{y_1(x),\ y_2(x),\ \ldots,\ y_N(x)\}$$

is linearly independent if and only if none of the y_k's can be written as a linear combination of the other y_k's. There are several ways to test for linear independence. The one usually discussed in de texts involves the corresponding *Wronskian* $W(x)$, given by

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ {y_1}' & {y_2}' & {y_3}' & \cdots & {y_n}' \\ {y_1}'' & {y_2}'' & {y_3}'' & \cdots & {y_n}'' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ {y_1}^{(n-1)} & {y_2}^{(n-1)} & {y_3}^{(n-1)} & \cdots & {y_n}^{(n-1)} \end{vmatrix} .$$

The test is that the set of N solutions

$$\{y_1(x),\ y_2(x),\ \ldots,\ y_N(x)\}$$

to some given N^{th}-order homogeneous linear differential equation is linearly independent if and only if

$$W(x_0) \neq 0$$

for any point in the interval over which these y_k's are solutions.

This is a highly recommended test when $N > 2$, but, frankly, it is silly to use it when $N = 2$. Then, we just have a pair of solutions

$$\{y_1(x),\ y_2(x)\}$$

and any such pair is linearly independent if and only if neither function is a constant multiple of each other, and THAT is usually obvious upon inspection of the two functions.

Second-Order Linear Homogeneous Equations with Constant Coefficients[§]

Consider a differential equation of the form

$$ay'' + by' + cy = 0$$

where a, b, and c are (real) constants. To solve such an equation, assume a solution of the form

$$y(x) = e^{rx}$$

(where r is a constant to be determined), and then plug this formula for y into the differential equation. You will then get the corresponding characteristic equation for the de,

$$ar^2 + br + c = 0 \quad .$$

Solve the characteristic equation. You'll get two values for r,

$$r = r_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(with the possibility that $r_+ = r_-$). Then:

1. If r_+ and r_- are two distinct real values, then the general solution to the differential equation is
$$y(x) = c_1 e^{r_+ x} + c_2 e^{r_- x}$$
where c_1 and c_2 are arbitrary constants.

2. If $r_+ = r_-$, then r_+ is real and the general solution to the differential equation is
$$y(x) = c_1 e^{r_+ x} + c_2 x e^{r_+ x}$$
where c_1 and c_2 are arbitrary constants. (Note: The $c_2 x e^{r_+ x}$ part of the solution can be derived via the method of "reduction of order".)

3. If r_+ or r_- is complex valued, then they are complex conjugates of each other,
$$r_+ = \alpha + i\beta \qquad \text{and} \qquad r_- = \alpha - i\beta$$
for some real constants α and β. The general solution to the differential equation can then be written as
$$y(x) = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$
where c_1 and c_2 are arbitrary constants. However, because
$$e^{(\alpha \pm i\beta)x} = e^{\alpha x}[\cos(\beta x) \pm i \sin(\beta x)] \quad ,$$
the general solution to the differential equation can also be written as
$$y(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x)$$
where C_1 and C_2 are arbitrary constants. In practice, the later formula for y is usually preferred because it involves just real-valued functions.

[§] see, also, chapter 16 of `www.uah.edu/math/howell/DEtext`

!►Example A.3: *Consider*

$$y'' - 4y' + 13y = 0 \quad .$$

Plugging in $y = e^{rx}$ *, we get*

$$\frac{d^2}{dx^2}\left[e^{rx}\right] - 4\frac{d}{dx}\left[e^{rx}\right] + 13\left[e^{rx}\right] = 0$$

$$\Longrightarrow \qquad r^2 e^{rx} - 4re^{rx} + 13e^{rx} = 0$$

$$\Longrightarrow \qquad r^2 - 4r + 13 = 0 \quad .$$

Thus,

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i \quad .$$

So the general solution to the differential equation can be written as

$$y(x) = c_1 e^{(2+3i)x} + c_2 e^{(2-3i)x}$$

or as

$$y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x) \quad ,$$

with the later formula usually being preferred.

Second-Order Euler Equations[¶]

A second-order *Euler equation*[2] is a differential equation that can be written as

$$ax^2y'' + bxy' + cy = 0$$

where a , b , and c are (real) constants. To solve such an equation, assume a solution of the form

$$y(x) = x^r$$

(where r is a constant to be determined), and then plug this formula for y into the differential equation, and solve for r .

With luck, you will get two distinct real values for r , r_1 and r_2 , in which case, the general solution to the differential equation is

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

where c_1 and c_2 are arbitrary constants.

With less luck, you only complex values for r , or only one value for r . See see your old de text or chapter 19 of `www.uah.edu/math/howell/DEtext` to see what to do in these cases.

[¶] see, also, chapter 19 of `www.uah.edu/math/howell/DEtext`

[2] also called a *Cauchy-Euler equation*

!►Example A.4: *Consider*

$$x^2 y'' + xy' - 9y = 0 \quad .$$

Plugging in $y = x^r$ *, we get*

$$x^2 \frac{d^2}{dx^2}\left[x^r\right] + x\frac{d}{dx}\left[x^r\right] - 9\left[x^r\right] = 0$$

$$\Longrightarrow \qquad x^2\left[r(r-1)x^{r-2}\right] + x\left[rx^{r-1}\right] - 9\left[x^r\right] = 0$$

$$\Longrightarrow \qquad r^2x^r - rx^r + rx^r - 9x^r = 0$$

$$\Longrightarrow \qquad r^2 - 9 = 0$$

$$\Longrightarrow \qquad r = \pm 3 \quad .$$

So the general solution to the differential equation is

$$y(x) = c_1 x^3 + c_2 x^{-3} \quad .$$

Other Methods

For solving more involved homogeneous second-order odes, there is still the method of Frobenius (which we will later discuss in some detail). You may also want to look up the method of reduction of order in your old differential equation text or chapter 13 of `www.uah.edu/math/howell/DEtext` (or see the last problem in the next homework list).

For solving nonhomogeneous second-order odes, you may want to recall the methods of "undetermined coefficients" (aka the "method of guess") and "variation of parameters".

Additional Exercises

A.1. *In this set, all the differential equations are first-order and separable.*

a. *Find the general solution for each of the following:*

i. $\frac{dy}{dx} = xy - 4x$ ***ii.*** $\frac{dy}{dx} = 3y^2 - y^2 \sin(x)$

iii. $\frac{dy}{dx} = xy - 3x - 2y + 6$ ***iv.*** $\frac{dy}{dx} = \frac{y}{x}$

b. *Solve each of the following initial-value problems.*

i. $\frac{dy}{dx} - 2y = -10$ *with* $y(0) = 8$

ii. $y\frac{dy}{dx} = \sin(x)$ *with* $y(0) = -4$

iii. $x\frac{dy}{dx} = y^2 - y$ *with* $y(1) = 2$

A.2. *In this set, all the differential equations are linear first-order equations.*

a. *Find the general solution for each of the following:*

i. $\frac{dy}{dx} + 2y = 6$ ***ii.*** $\frac{dy}{dx} + 2y = 20e^{3x}$

iii. $\frac{dy}{dx} = 4y + 16x$ ***iv.*** $\frac{dy}{dx} - 2xy = x$

b. *Solve each of the following initial-value problems:*

i. $\frac{dy}{dx} + 5y = e^{-3x}$ *with* $y(0) = 0$

ii. $x\frac{dy}{dx} + 3y = 20x^2$ *with* $y(1) = 10$

iii. $x\frac{dy}{dx} = y + x^2 \cos(x)$ *with* $y\left(\frac{\pi}{2}\right) = 0$

A.3. *Find the general solution to each of the following second-order linear equations with constant coefficients. Express your solution in terms of real-valued functions only.*

a. $y'' - 9y = 0$ **b.** $y'' + 9y = 0$

c. $y'' + 6y' + 9y = 0$ **d.** $y'' + 6y' - 9y = 0$

e. $y'' - 6y' + 9y = 0$ **f.** $y'' + 6y' + 10y = 0$

g. $y'' - 4y' + 40y = 0$ **h.** $2y'' - 5y' + 2y = 0$

A.4. *Solve the following initial-value problems:*

a. $y'' - 7y' + 10y = 0$ *with* $y(0) = 5$ *and* $y'(0) = 16$

b. $y'' - 10y' + 25y = 0$ *with* $y(0) = 1$ *and* $y'(0) = 0$

c. $y'' + 25y = 0$ *with* $y(0) = 4$ *and* $y'(0) = -15$

A.5. *Find the general solution to each of the following Euler equations on* $(0, \infty)$ *:*

a. $x^2y'' - 5xy' + 8y = 0$

b. $x^2y'' - 2y = 0$

c. $x^2y'' - 2xy' = 0$

d. $2x^2y'' - xy' + y = 0$

A.6. *Solve the following initial-value problems involving Euler equations:*

a. $x^2y'' - 6xy' + 10y = 0$ *with* $y(1) = -1$ *and* $y'(1) = 7$

b. $4x^2y'' + 4xy' - y = 0$ *with* $y(4) = 0$ *and* $y'(4) = 2$

A.7. *Find the general solution to each of the following:*

a. $y' + 2y = 3$

b. $xy' + 2y = 8$

c. $y' + \frac{1}{x}y = 2e^{x^2}$

d. $y'' + a^2y = 0$ *where* a *is a positive constant*

e. $y'' - a^2y = 0$ *where* a *is a positive constant*

f. $y'' + 4y' - 5y = 0$

g. $y'' - 6y' + 9y = 0$

h. $x^2y'' - 6xy' + 10y = 0$

i. $x^2y'' - 9xy' + 25y = 0$ *(See following note)*

Note: For the last one, start by assuming $y = x^r$ *as described for Cauchy-Euler equations. This will lead to one solution. To find the full solution, assume*

$$y(x) = x^r v(x)$$

where r *is the exponent just found and* $v(x)$ *is a function to be determined. Plug this into the differential equation, simplify, and you should get a relatively easy differential equation to solve for* $v(x)$ *(it may help to let* $u(x) = v'(x)$ *at one point). Solve for* v *(don't forget any arbitrary constants) and plug the resulting formula into the above formula for* y *. There, you've just done "reduction of order".*

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

1a i. $y = 4 + A\exp\left(\frac{1}{2}x^2\right)$

1a ii. $y = \big(c - 3x - \cos(x)\big)^{-1}$ and $y = 0$

1a iii. $y = 3 + A\exp\left(\frac{1}{2}x^2 - 2x\right)$

1a iv. $y = Ax$

1b i. $y = 5 + 3e^{2x}$

1b ii. $y = -\sqrt{18 - 2\cos(x)}$

1b iii. $y = 2(2 - x)^{-1}$

2a i. $y = 3 + ce^{-2x}$

2a ii. $y = 4e^{3x} + ce^{-2x}$

2a iii. $y = ce^{4x} - 4x - 1$

2a iv. $y = ce^{x^2} - \frac{1}{2}$

2b i. $\frac{1}{2}\left[e^{-3x} - e^{-5x}\right]$

2b ii. $4x^2 + 6x^{-3}$

2b iii. $x[\sin(x) - 1]$

3a. $y(x) = c_1e^{3x} + c_2e^{-3x}$

3b. $y(x) = c_1\cos(3x) + c_2\sin(3x)$

3c. $y(x) = c_1e^{-3x} + c_2xe^{-3x}$

3d. $y(x) = c_1e^{(-3+3\sqrt{2})x} + c_2e^{(-3-3\sqrt{2})x}$

3e. $y(x) = c_1e^{3x} + c_2xe^{3x}$

3f. $y(x) = c_1e^{-3x}\cos(x) + c_2e^{-3x}\sin(x)$

3g. $y(x) = c_1e^{2x}\cos(6x) + c_2e^{2x}\sin(6x)$

3h. $y(x) = c_1e^{2x} + c_2e^{x/2}$

4a. $3e^{2x} + 2e^{5x}$

4b. $e^{5x} - 5xe^{5x}$

4c. $4\cos(5x) - 3\sin(5x)$

5a. $y = c_1x^2 + c_2x^4$

5b. $y = c_1x^2 + c_2x^{-1}$

5c. $y = c_1 + c_2x^3$

5d. $y = c_1x + c_2\sqrt{x}$

6a. $y = 3x^5 - 4x^2$

6b. $y = 4x^{1/2} - 16x^{-1/2}$

7a. $3 + ce^{-2x}$

7b. $4 + cx^{-2}$

7c. $\left[e^{x^2} + c\right]/x$

7d. $c_1\cos(ax) + c_2\sin(ax)$

7e. $c_1e^{ax} + c_2e^{-ax}$

7f. $c_1e^{x} + c_2e^{-5x}$

7g. $c_1e^{3x} + c_2xe^{3x}$

7h. $c_1x^2 + c_2x^5$

7i. $c_1x^5 + c_2x^5\ln|x|$