# 14

# Complex Analysis I: The Basic Calculus

We now begin an extended study of "analysis" using complex variables in complex functions. We start with the standard basic review and will proceed as quickly as possible to "reconstructing" basic calculus assuming our variable are complex instead of real.

After that, things will start getting interesting.

## 14.1 Complex Numbers*

Recall that $z$ is a *complex number* if and only if it can be written as

$$z = x + iy$$

where $x$ and $y$ are real numbers and $i$ is a "complex constant" satisfying $i^2 = -1$. The *real part* of $z$, denoted by $\text{Re}[z]$, is the real number $x$, while the *imaginary part* of $z$, denoted by $\text{Im}[z]$, is the real number $y$. If $\text{Im}[z] = 0$ (equivalently, $z = \text{Re}[z]$), then $z$ is said to be real. Conversely, if $\text{Re}[z] = 0$ (equivalently, $z = i\,\text{Im}[z]$), then $z$ is said to be imaginary.

The *complex conjugate* of $z = x + iy$, which we will denote by $z^*$, is the complex number $z^* = x - iy$.

In the future, given any statement like "the complex number $z = x + iy$", it should automatically be assumed (unless otherwise indicated) that $x$ and $y$ are real numbers.

The algebra of complex numbers can be viewed as simply being the algebra of real numbers with the addition of a number $i$ whose square is negative one. Thus, choosing some computations that will be of particular interest,

$$zz^* = z^*z = (x - iy)(x + iy) = x^2 - (iy)^2 = x^2 + y^2$$

and

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2} = \frac{z^*}{zz^*} \quad .$$

We will often use the easily verified facts that, for any pair of complex numbers $z$ and $w$,

$$(z + w)^* = z^* + w^* \qquad \text{and} \qquad (zw)^* = (z^*)(w^*) \quad .$$

* This section is stolen from *Fundamentals of Fourier Analysis* by Howell, with the author's permission. It's just a slightly expanded version of the initial review of complex variables in chapter 3.

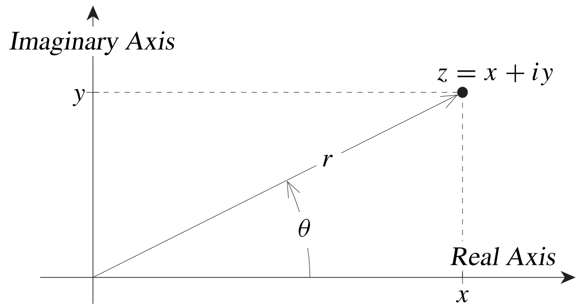


**Figure 14.1:** Coordinates in the complex plane for $z = x + iy$, where $x > 0$ and $y > 0$.

The set of all complex numbers is denoted by $\mathbb{C}$. By associating the real and imaginary parts of the complex numbers with the coordinates of a two-dimensional Cartesian system, we can identify $\mathbb{C}$ with a plane (called, unsurprisingly, the complex plane). This is illustrated in figure 14.1. Also indicated in this figure are the corresponding polar coordinates $r$ and $\theta$ for $z = x + iy$. The value $r$, which we will also denote by $|z|$, is commonly referred to as either the *magnitude*, the *absolute value*, or the *modulus* of $z$, while $\theta$, sometimes denoted by $\arg(z)$, is commonly called either the *argument*, the *polar angle*, or the *phase* of $z$. It is easily verified that

$$r = |z| = \sqrt{x^2 + y^2} = \sqrt{z^* z} \quad ,$$

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta) \quad .$$

From this it follows that the complex number $z = x + iy$ can be written in *polar form*,

$$z = x + iy = r[\cos(\theta) + i\sin(\theta)] \quad ,$$

and that

$$|x| \leq |z| \qquad \text{and} \qquad |y| \leq |z| \quad .$$

Observe that the polar angle for a nonzero complex number $z$ (i.e., $\theta = \arg(z)$) is not unique. If

$$z = |z|[\cos(\theta_0) + i\sin(\theta_0)] \quad ,$$

then any $\theta$ differing from $\theta_0$ by an integral multiple of $2\pi$ is another polar angle for $z$. This is readily seen by considering how little figure 14.1 changes if the $\theta$ there is increased by an integral multiple of $2\pi$. It is also clear that these are the only polar angles for $z$. We will refer to the polar angle $\theta$ with $0 \leq \theta < 2\pi$ as the *principal argument* (or *principal polar angle*) and denote it by $\operatorname{Arg}(z)$.[1] That is,

$$\operatorname{Arg}(z) = \theta \quad \iff \quad z = |z|[\cos(\theta) + i\sin(\theta)] \quad \text{and} \quad 0 \leq \theta < 2\pi \quad .$$

Note that

$$\arg(z) = \operatorname{Arg}(z) + 2n\pi \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \quad ,$$

[1] Some authors prefer using $(-\pi, \pi]$ as the range for the principal argument. We will actually use whichever of these two ranges is most convenient at the time.

It is instructive to look at the polar form of the product of two complex numbers. So let $z$ and $w$ be two complex numbers with polar forms

$$z = r[\cos(\theta) + i\sin(\theta)] \quad \text{and} \quad w = \rho[\cos(\phi) + i\sin(\phi)] \quad .$$

Multiplying $z$ and $w$ together gives

$$\begin{aligned} zw &= (r[\cos(\theta) + i\sin(\theta)])(\rho[\cos(\phi) + i\sin(\phi)]) \\ &= r\rho\,([\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)] + i\,[\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)]) \quad , \end{aligned}$$

which, using well-known trigonometric identities, simplifies to

$$zw = r\rho[\cos(\theta+\phi) + i\sin(\theta+\phi)] \quad . \tag{14.1}$$

From this it immediately follows that $|zw| = |z|\,|w|$ and that a polar angle of $zw$ can be found by adding the polar angles of $z$ and $w$.

?► ***Exercise 14.1:*** *Let $N$ be any positive integer and let $z$ be any complex number with polar angle $\theta$. Show that $\left|z^N\right| = |z|^N$ and that $N\theta$ is a polar angle for $z^N$.*

"Finally", recall how trivial it is to verify that

$$|z+w| \leq |z| + |w| \tag{14.2}$$

whenever $z$ and $w$ are two real numbers. This inequality also holds if $z$ and $w$ are any two complex numbers. Basically, it is an observation about the triangle in the complex plane whose vertices are the points $0$, $z$, and $z+w$. Sketch this triangle and you will see that the sides have lengths $|z|$, $|w|$, and $|z+w|$. The observation expressed by inequality (14.2) is that no side of the triangle can be any longer than the sum of the lengths of the other two sides. Because of this, inequality (14.2) is usually referred to as the (basic) *triangle inequality* (for complex numbers).

## 14.2 Sets of Complex Numbers

A number of specific types of subsets of the complex numbers are important in complex analysis. For convenience, here is a list of the ones we'll be dealing with most, along with a brief discussion of each:

*Sets of isolated points in* $\mathbb{C}$: Often, these will be points at which given functions are ill defined, such as the set $\{i, -i\}$, which is the set of points where $(z^2+1)^{-1}$ "blows up".

*Curves in* $\mathbb{C}$: Unless otherwise stated, it should always be assumed that any given curve is reasonably smooth (i.e., has well-defined tangents at every point other than, possibly, a finite number of corners), and that we don't have silly things happening such as a curve having infinite length inside some finite rectangle. These curves may be oriented, especially those associated with integrals (more on this later).

*Loops or Closed Curves in* $\mathbb{C}$: If we refer to a curve as being either a *closed curve* or a *loop*, then we mean that it can be viewed as starting and ending at the same finite point.

*Simple Loops or Closed Curves in* $\mathbb{C}$: A closed curve (or loop) is *simple* if it doesn't cross itself; that is, has no points at which one piece intersects another. A circle is a simple loop; a figure-eight is not.

*Regions in* $\mathbb{C}$: Unless otherwise stated, whenever we refer to $\mathcal{R}$ (or any other symbol) as being a "region" in $\mathbb{C}$, assume that region is either all of $\mathbb{C}$, or is a subset of $\mathbb{C}$ with a well-defined boundary consisting of curves and/or a few isolated points. In addition, we will automatically assume our regions are 'open' sets, which means that the set does not include any points in its boundary.[2,3]

By the way, in complex analysis it is standard to also use the word *domain* as a synonym for "region", as well as to indicate "the set of all values of $z$ that can be plugged into a given function". Thus, the statement that "the domain of $f$ is a domain" really is saying something about the set of all $z$ for which $f(z)$ is defined.

*Simply-Connected Regions in* $\mathbb{C}$: A region is said to be *simply connected* if and only if it "is in one piece" and "has no holes". The region enclosed by any simple loop is simply connected; the region outside any simple loop is not.

*Multiply-Connected Regions in* $\mathbb{C}$: A region that is not simply connected is said to be *multiply connected.*

## 14.3 Functions

### Basics

We will be interested in complex-valued functions of complex variables. The value of any such function $f$ at any point $z$ of its domain, therefore, has real and imaginary parts. Traditionally, we let

$$u(x,y) = \mathrm{Re}[f(x+iy)] \qquad \text{and} \qquad v(x,y) = \mathrm{Im}[f(x+iy)] \quad .$$

Unsurprisingly, $u$ and $v$ are known, respectively, as the real and the imaginary parts of $f$. Note that $u$ and $v$ are the real-valued functions of two real variables such that

$$f(x+iy) = u(x,y) + iv(x,y) \quad .$$

This allows us to import anything we know about "real-valued functions of two real variables" into the theory of complex-valued functions of complex variables.

A few classes of complex functions are worth a brief mention. These include:

*Constant functions*, such as

$$f(z) = 3 \quad , \quad g(z) = 3+2i \quad \text{and, our favorite,} \quad h(z) = 0 \quad .$$

[2] A closed set contains all of its boundary.
[3] The entire complex plane is both open and closed, or, as some like to say, 'clopen'.

*Polynomials*, such as

$$f(z) = 3z^2 + 8z - 2 \quad \text{and} \quad g(z) = 827z^{3,425} - 2 \quad .$$

(Recall that, using complex numbers, any polynomial $P$ can, in theory, be written in 'completely factored form'; that is,

$$P(z) = A(z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_k)^{n_k} \quad ,$$

where the $z_j$'s are the distinct roots of $P$ (i.e., the different solutions of $P(z) = 0$), each $n_j$ is the 'multiplicity' of $z_j$, and the sum of the $n_j$'s equals the degree of $P(z)$.)

*Rational functions*, such as

$$f(z) = \frac{1}{z^2 + 1} \quad \text{and} \quad g(z) = \frac{3z^2 + 8z - 2}{827z^{3,425} - 2} \quad .$$

(Points in $\mathbb{C}$ where these functions "blow up" will be of particular interest to us.)

*Convergent power series*, such as

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^k \quad \text{for all} \quad z \in \mathbb{C} \qquad \text{and} \qquad \sum_{k=0}^{\infty} z^k \quad \text{for all} \quad |z| < 1 \quad .$$

A few other functions are worth a bit more discussion. Since some of these are "multi-valued", let us pause briefly to clarify this concept and to discuss some general issues regarding these functions.

## Single-Valued Versus Multi-Valued Functions

A function $f$ is *single valued* if, for each $z$ in its domain, there is a single, unambiguous value for $f(z)$. Strictly speaking, most standard definitions of "function" require a function to be single valued. However, there are function-like things which are not single valued. Things into which you plug in a particular $z$ and get a set of numbers as the output. The basic example of this is the argument "function", arg. Remember, $\arg(z)$ is any polar angle for $z$; that is,

$$\theta = \arg(z) \quad \Longleftrightarrow \quad z = |z| [\cos(\theta) + i \sin(\theta)] \quad .$$

But if $\theta_0$ is one polar angle for $z$, so is $\theta_0 + n2\pi$ for any integer $n$. So, to be complete, we may write (for $z \neq 0$)

$$\arg z = \theta_0 + n2\pi$$

where

$$\theta_0 \text{ is any single value satisfying } z = |z| [\cos(\theta_0) + i \sin(\theta_0)]$$

and

$$n = 0, \pm 1, \pm 2, \pm 3, \ldots \quad .$$

Such things are commonly referred to as *multi-valued functions*. It is a slight abuse of terminology, but no one seems to mind.

We will see that multi-valued functions play a significant role in complex analysis. This is largely because the argument function arises, explicitly or implicitly, in many basic computations.

Any multi-valued function $f$ can be converted to a single-valued function $f_0$ by restricting its output appropriately. For example, by restricting the output of $\arg(z)$ to values in the interval $[0, 2\pi)$, we get the *principal argument function* of $z$, $\text{Arg}(z)$.[4] This is a single-valued function because, for any given $z \neq 0$, there can only be one value for $\arg(z)$ in $[0, 2\pi)$. For any $z \neq 0$, the relation between $\arg(z)$ and $\text{Arg}(z)$ is

$$\arg z = \text{Arg}(z) + 2n\pi \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \quad .$$

When we so restrict a multi-valued function $f$ to obtain a single-valued function $f_0$, we are said to be *choosing a branch* of $f$, with $f_0$ being *that branch*. For example, the principal argument function, Arg, is a "branch" of arg.[5]

Unfortunately, when we "choose a branch", we are typically trading an ambiguous function for a function that is definitely *dis*continuous on certain curves. Consider the principal argument function at points near the positive real axis. The principal arguments of the points just above this (half) line are close to $0$ and approach $0$ as these points approach the positive real axis. However, the principal arguments of the points just below the the positive real axis are close to $2\pi$ and approach $2\pi$ as these points approach the positive real axis. Consequently, if we move a point $z$ across the positive real axis, $\text{Arg}(z)$ suddenly jumps by $2\pi$, instead of smoothly changing value. So $\text{Arg}(z)$, as we have defined it, is discontinuous across the positive real axis.

Curves on which a particular "branch" $f_0(z)$ is discontinuous are called *branch cuts* or *cut lines for the branch*, and the endpoints of these curves are called *branch points*. Branch cuts and branch points typically are used for the boundaries of the regions on which our chosen branch $f_0$ is well defined. For the principal argument function (as defined above), the positive real axis is the (only) branch cut, and the point $z = 0$ is the only branch point (unless you count $z = \infty$, which is sometimes done). Typically, different "branches" for a particular multi-valued function will have different branch cuts. Indeed, deciding which curves you want as branch cuts may help determine the specific branch you wish to choose. As we will later see, however, the branch points are invariably points at which the basic multi-valued function somehow behaves very badly (consider, what is $\text{Arg}(0)$ ?). As a result, all the branches of any multi-valued function will have the same branch points.

The whole concept of "branches" is sometimes confusing when first encountered. Don't fret it. "Choosing a branch" for multi-valued function is just like deciding $\sqrt{9}$ refers to the positive square root of 9, instead of both square roots. Okay, not "just like", but similar. And the concept will become clearer as we see more examples and need to deal with multi-valued functions. Then you can start worrying about the 'deep questions' such as "should we consider the cut lines as part of the domain of a branch?"

## Exponential and Trigonometric Functions

Recall that, for any $z = x + iy$, the complex exponential of $z$ is given by

$$\exp(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos(y) + i \sin(y)] \quad .$$

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[4] Alternatively, many define $\text{Arg}(z)$ by restricting the output of $\arg(z)$ to values in the interval $(-\pi, \pi]$. On occasion, we'll find this definition preferable.

[5] If you must attempt to make sense of this terminology, think of choosing a "branch in a road" rather than a "branch in a tree".

It's "easily verified" that the algebraic and differential properties of $e^z$ are the same as for the real exponential. In particular, if $A$ and $B$ are any complex constants,

$$e^{A+B} = e^A e^B \qquad \text{and} \qquad \frac{d}{dt} e^{At} = A e^{At} \quad .$$

More precisely, the above formula for $e^z$ was derived to ensure these properties held.[6]

It is worth noting that, if $\theta$ is a real value, then we have

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \qquad \text{and} \qquad e^{-i\theta} = \cos(\theta) - i\sin(\theta) \quad .$$

Solving this pair of equations for the sines and cosines, we get the well-beloved formulas

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad .$$

This relates the trigonometric functions to the complex exponentials. Indeed, many computations involving trigonometric functions, including the derivation of many trigonometric identities are much more easily done via the complex exponentials

**!►Example 14.1:** *Let $n$ be any integer. Then*

$$[\cos(\theta) + i\sin(\theta)]^n = \left[e^{i\theta}\right]^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta) \quad .$$

*Cutting out the middle leaves us with De Moivre's formula,*

$$[\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta) \quad .$$

The relations between the complex exponentials and the sines and cosines also give us a natural way to extend the definition of the sine and cosine functions to being functions on the entire complex plane. For each $z \in \mathbb{C}$ we simply define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad .$$

The other other trigonometric functions can then be defined as functions of $z$ through their relations with $\sin(z)$ and $\cos(z)$. For example,

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{e^{iz} - e^{-iz}}{i\left[e^{iz} + e^{-iz}\right]} \quad .$$

**?►Exercise 14.2:** *Let $\cosh(z)$ and $\sinh(z)$ be defined using the regular exponential formulas for the hyperbolic sines and cosines, and find the relation between these functions and the functions*

$$\cos(iz) \qquad \text{and} \qquad \sin(iz) \quad .$$

**?►Exercise 14.3:** *Recall the 'polar form' for complex numbers, and convince yourself that it is the same as*

$$z = |z|\, e^{i\theta} \qquad \text{where} \qquad \theta = \arg z \quad .$$

[6] See the subsections on the complex exponentials in section 3.1 of the notes.

**?►Exercise 14.4:** *Using the observation from the last exercise, find all three solutions to*

$$z^3 = 1 \quad .$$

*Also find the three solutions to*

$$z^3 = -1 \quad .$$

## The Natural Logarithm

The natural logarithm function can be extended from being a function on $(0, \infty)$ to a function on the entire nonzero complex plane (i.e., all nonzero complex numbers) by using the polar form of the complex numbers along with the assumption that the standard algebraic rules for logs hold:

$$\ln(z) = \ln(|z|\, e^{i\theta}) = \ln|z| + \ln e^{i\theta} = \ln|z| + i\theta \qquad \text{where} \quad \theta = \arg(z) \quad .$$

So, for any nonzero complex value $z$, we define

$$\ln(z) = \ln|z| + i\arg(z) \quad . \tag{14.3}$$

Note that this is equivalent to

$$\ln(z) = \ln|z| + i[\text{Arg}(z) + 2\pi n] \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad . \tag{14.3$'$}$$

Because $\arg(z)$ is multi-valued, so is $\ln(z)$, with the imaginary part of the natural log only defined up to an arbitrary integral multiple of $2\pi$. In practice, this difficulty is often eliminated by restricting the allowed values of the argument to being in either $[0, 2\pi)$ or $(-\pi, \pi)$, depending on the problem. That is, we choose the "branch"

$$\ln(z) = \ln|z| + i\,\text{Arg}(z) \quad .$$

Unsurprisingly, this function is sometimes called the principal branch of the natural logarithm.

It should also not be very surprising to discover that a favorite relation between the exponential function and the natural log remains valid even when using complex numbers.

**?►Exercise 14.5:** *Let $z$ be any nonzero complex number. Show that, even with the natural logarithm being multi-valued,*

$$e^{\ln(z)} = z \quad .$$

*Why, however, can we not say that $\ln(e^z) = z$?*

## Square Roots and Such

Recall that, for any positive real number $x$ and any real value $\gamma$,

$$x^\gamma = e^{\ln(x^\gamma)} = e^{\gamma \ln(x)} \quad .$$

Now that we have the complex exponential and natural logarithm defined on $\mathbb{C}$ (excluding $0$), we can extend the above to being a definition for the function $z^\gamma$:

$$z^\gamma = e^{\gamma \ln(z)} \qquad \text{for} \quad z \neq 0 \quad . \tag{14.4}$$

In this definition, $\gamma$ need not be real, it can be any complex number.

For the moment, assume $\gamma$ is real. Observe that

$$\begin{aligned} z^\gamma = e^{\gamma \ln(z)} &= e^{\gamma[\ln|z| + i(\operatorname{Arg}(z) + n2\pi)]} \\ &= e^{\gamma \ln|z|} e^{\gamma i \operatorname{Arg}(z)} e^{\gamma i n 2\pi} \\ &= |z|^\gamma e^{\gamma i \operatorname{Arg}(z)} e^{i\gamma 2n\pi} \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \quad . \end{aligned}$$

If $\gamma$ is an integer, the $e^{i\gamma 2n\pi}$ factor reduces to $1$. Otherwise, the value of the $e^{i\gamma 2n\pi}$ factor depends on the choice of $n$. Consequently, we see that $z^\gamma$ is single-valued if and only if $\gamma$ is an integer. Otherwise, $z^\gamma$ is multi-valued.

**?►Exercise 14.6:** *Verify that, if $\gamma = 1/k$ for some positive integer $k$ and $z \neq 0$, then*

**a:** *$z^\gamma$ has $k$ distinct values, and*

**b:** $(z^\gamma)^k = z$ .

**?►Exercise 14.7:** *Compute the following. Remember, these are multi-valued.*

$$1^{1/2} \quad , \quad 1^{1/3} \quad , \quad 1^{1/4} \quad , \quad i^{1/2} \quad , \quad (-1)^{1/2} \quad , \quad 8^{1/3} \quad \textit{and} \quad i^\pi \quad .$$

We probably will not need to deal with $z^\gamma$ when $\gamma$ is not real, but you should play with it yourself, for fun and enlightenment.

**?►Exercise 14.8:** *Compute the following:*

$$1^i \quad , \quad 2^i \quad \textit{and} \quad i^i \quad .$$

*(Note: $1^i \neq 1$.)*

## 14.4 Derivatives

For all the following, $z = x + iy$ and

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) \quad .$$

In all of the following computations, unless otherwise specified, we will restrict ourselves to some region in the complex plane on which $u$ and $v$ are continuous and have partial derivatives.

### Partial Derivatives

The partial derivatives of $f$ are inherited directly from the corresponding partials for $u$ and $v$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[u(x, y) + iv(x, y)] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[u(x,y) + iv(x,y)] = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \quad .$$

We should recall the basic definition of partial derivatives, and note that the above is equivalent to

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x + iy) - f(x + iy)}{\Delta x}$$

and

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x + i[y + \Delta y]) - f(x + iy)}{\Delta y} \quad .$$

## The Complex Derivative
### Definition and Basic Properties

The complex derivative of $f$ is defined by

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad ,$$

provided this limit exists. If the limit does exist for a given choice of $z$, then we say $f$ is differentiable at that $z$ (but see the comments on terminology a few paragraphs from here).

Since this limit is formally exactly the same as the limit in the elementary calculus definition of the derivative (but with $z$ now assumed complex), you can be sure that all the basic rules derived in the elementary calculus course using that limit still apply. In particular, the chain, product, and quotient rules all remain valid, and

$$\frac{d}{dz} z^n = nz^{n-1}$$

for any integer $n$. Moreover, you can go ahead and trust me that

$$\frac{d}{dz}e^z = e^z \quad , \quad \frac{d}{dz}\sin(z) = \cos(z) \quad , \quad \frac{d}{dz}\cos(z) = -\sin(z) \quad , \quad \frac{d}{dz}\ln(z) = \frac{1}{z} \quad , \quad \text{etc.}$$

(Actually, verifying some of these may be homework.)

In "complex analysis", we rarely say that a given function $f$ is "differentiable". This is possibly to avoid confusion over whether we are referring to the existence of the complex derivative, $f'$, or the partials. Or it may be because we actually need something a little stronger. Anyway, we say

$f$ is *analytic* on a set of points $\mathcal{S}$

if and only if

$f$ is differentiable at every point in some open set containing $\mathcal{S}$ .

In particular:

*1.* $f$ is analytic on an open region $\mathcal{R}$ if and only if $f$ is differentiable at every point on $\mathcal{R}$. (So, on open regions, "analyticity" is synonymous with "differentiability".)

2. $f$ is analytic at a point $z_0$ if and only if $f$ is analytic on some open region containing $z_0$. (So, at a point, "analyticity" is a stronger requirement than "differentiability".)

The words "holomorphic" and "regular" are used by some instead of "analytic". Do note that this definition of "analytic" is different from the definition used when we were dealing with functions represented by power series. Later, we will find a link between the two definitions.

Finally, you may hear of a function being *entire*. That is simply the adjective for a function that is analytic on all of $\mathbb{C}$.

As we will see, a major part of the study of "complex variables" ends up being the study of functions that are either entire, or else are analytic at all but a few isolated points on the complex plane.

## Derivatives, Partial Derivatives, and the Cauchy-Riemann Equations

Rewriting the above limit definition for $f'(z)$ with $z = x + iy$ and $\Delta z = \Delta x + i\Delta y$, we get

$$f'(z) = \lim_{\Delta x + i\Delta y \to 0} \frac{f([x+iy] + [\Delta x + i\Delta y]) - f(x+iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x + i\Delta y \to 0} \frac{f(x + \Delta x + i[y + \Delta y]) - f(x+iy)}{\Delta x + i\Delta y} \quad , \tag{14.5}$$

Keep in mind that $\Delta z = \Delta x + i\Delta y$ can approach $0$ from any direction in the complex plane, and, if the above limit exists, then the limit's value cannot depend on the direction along which $\Delta z$ approaches $0$. In particular, if we compute the limit by letting $\Delta y \to 0$ and then letting $\Delta x \to 0$, we should get the same value as we had first let $\Delta x \to 0$ and then had $\Delta y \to 0$. Using this fact, we can easily get the relation between the derivative $df/dz$ and the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$.

Letting $\Delta y \to 0$ first in formula (14.5) and then letting $\Delta x \to 0$ gives

$$f'(z) = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{f(x + \Delta x + i[y + \Delta y]) - f(x+iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x + iy) - f(x+iy)}{\Delta x} = \frac{\partial f}{\partial x} \quad .$$

On the other hand, computing the limit in formula (14.5) by first letting $\Delta x \to 0$ yields

$$f'(z) = \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{f(x + \Delta x + i[y + \Delta y]) - f(x+iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{f(x + i[y + \Delta y]) - f(x+iy)}{i\Delta y} = \frac{1}{i}\frac{\partial f}{\partial y} = -i\frac{\partial f}{\partial y} \quad .$$

After cutting out the middle of the above two computations for $f'(z)$ and glancing back at the section on partial derivatives, we find that

$$f'(z) = \frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[u(x,y) + iv(x,y)] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{14.6}$$

and

$$f'(z) = \frac{df}{dz} = -i\frac{\partial f}{\partial y} = -i\frac{\partial}{\partial y}[u(x,y) + iv(x,y)] = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad . \tag{14.7}$$

A number of observations can be made based on equations (14.6) and (14.7). First of all, observe what we get when we solve these equations for the partial derivatives of $f$, namely,

$$\frac{\partial f}{\partial x} = f'(z) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = if'(z) \quad .$$

That is (to be a little more explicit),

$$\frac{\partial}{\partial x} f(x+iy) = f'(x+iy) \qquad \text{and} \qquad \frac{\partial}{\partial y} f(x+iy) = if'(x+iy) \quad ,$$

which is exactly what you should expect to get using the chain rule for derivatives. Indeed, we could have defined $f'$ to be the function such that the equations immediately above hold, instead of using the limit definition. Either way, it's probably easier to remember the relations between $f'$ and its partial derivatives as examples of the chain rule than to attempt memorizing them or recreating the derivation above.

From the relations between $f$ and its partial derivatives described above, we also get relations between the various partial derivatives. In particular, we have

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \quad . \tag{14.8}$$

Written in terms of the partials of $u$ and $v$, this is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad .$$

By isolating the real and imaginary parts, we see that this last equation is completely equivalent to the system

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad . \tag{14.9}$$

This pair of equations is the (in)famous *Cauchy-Riemann equations*. We've just shown that they must hold whenever $f'(z)$ exists. Thus, it immediately follows that, if the Cauchy-Riemann equations do not hold at a point, then $f$ is not differentiable at that point. Moreover, using a standard result from (advanced) calculus regarding directional derivatives of real functions of two real variables, you can show that the limit defining $f'(z)$ exists at each $z$ in a region if, in that region, the Cauchy-Riemann equations hold and the partial derivatives are continuous.[7] This gives the following theorem and test for analyticity:

***Theorem 14.1 (test for analyticity)***

*Let*

$$f(x+iy) = u(x,y) + iv(x,y)$$

*be a complex function on an open region $\mathcal{R}$ (and assume the partial derivatives of $u$ and $v$ are continuous on $\mathcal{R}$). Assume further that at least one of the following statements is true:*

1. *Throughout $\mathcal{R}$,*

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \quad .$$

[7] See page 472 of Arfken, Weber and Harris for an attempt at a proof. Unfortunately, the meaning of the notation must be guessed at, and the proof is based on an equation that, itself, is not trivial to verify.

2. *The Cauchy-Riemann equations,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad ,$$

*hold throughout* $\mathcal{R}$ *.*

3. $f$ *is analytic on* $\mathcal{R}$ *.*

*Then all of the above statements are true and, at every* $z$ *in* $\mathcal{R}$ *,*

$$f'(z) = \frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \quad .$$

Some notes about this theorem:

*1.* Traditionally, this theorem is stated just using the Cauchy-Riemann equations, not the equivalent expression

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \quad ,$$

which, in turn, comes from the "chain rule requirement" that

$$\frac{\partial}{\partial x} f(x+iy) = f'(x+iy) \qquad \text{and} \qquad \frac{\partial}{\partial y} f(x+iy) = if'(x+iy) \quad .$$

Frankly, remembering this "chain rule requirement" and, if necessary, deriving the Cauchy-Riemann equations (or their equivalents) may be preferable to memorizing the Cauchy-Riemann equations.

*2.* The continuity assumption on the partial derivatives of $u$ and $v$ allows us to use 'traditional' methods from the theory of real analysis to show the analyticity of $f$ . They also explicitly exclude such choices for $u$ and $v$ as

$$u(x, y) = \sqrt[3]{x+y} \quad .$$

Later, however, we will develop tools that will allow us to completely prove this theorem without requiring these partial derivatives be continuous. That is the reason this assumption was placed in parenthesis — in the end, it's not really needed. That is also the reason we won't really worry about a more rigorous proof of this theorem at this time.

*3.* We now have two ways to verify the analyticity of a function over a region. One is to directly confirm the existence and "well-definiteness" of the two-dimensional limit defining the derivative at each point of the region. The other is to compute the partial derivatives and apply the test in theorem 14.1. Which method you use probably depends on the function at hand and whether or not you've been ordered to use the Cauchy-Riemann equations. (And see the next discussion about identifying analytic functions that follows.)

*4.* The Cauchy-Riemann equations are not just used in verifying that a given formula defines an analytic function. They combine with other formulas to yield surprisingly useful general results. We will see at least one example in the next section.

*5.* Finally, it should be noted that by telling you that a given complex-valued formula involving $x$ and $y$ describes an analytic function, the above theorem is also telling you that this formula can be written more simply in terms of just $z$ with $z = x + iy$ .

**?►Exercise 14.9:** *Consider the following functions of* $z = x + iy$ *:*

$$f(z) = z^* \quad , \quad \mathrm{Re}(z) \quad , \quad \mathrm{Im}(z) \quad , \quad |z| \quad \text{and} \quad \arg(z) \quad .$$

**a:** *Find the real and imaginary parts of each, and compute their partial derivatives.*[8]

**b:** *Verify that these functions are not analytic anywhere on the complex plane.*

**?►Exercise 14.10:** *Using the definition for the natural logarithm (formula (14.3) on page 14–8), along with results from the last exercise, compute the partial derivatives of* $\ln(z)$ *and then verify that*

**a:** $\ln(z)$ *is analytic at every nonzero point of the complex plane.*

**b:** $\frac{d}{dz}\ln(z) = \frac{1}{z}$ .

*What about the issue of single-/multi-valuedness?*

**?►Exercise 14.11:** *Let* $\gamma$ *be any complex constant. Using the definition for* $z^\gamma$ *(formula (14.4) on page 14–8), along with results from the last exercise, compute the partial derivatives of* $z^\gamma$ *and then verify that*

**a:** $z^\gamma$ *is analytic at every nonzero point of the complex plane.*

**b:** $\frac{d}{dz}z^\gamma = \gamma z^{\gamma-1}$ .

*What about the issue of single-/multi-valuedness?*

**?►Exercise 14.12:** *Let*

$$f(x + iy) = u(x, y) + iv(x, y)$$

*be an analytic function on an open region* $\mathcal{R}$ *. Use the Cauchy-Riemann equations to show that the real and imaginary parts,* $u$ *and* $v$ *, satisfy*

$$\nabla^2 u = 0 \quad \text{and} \quad \nabla^2 v = 0 \quad \text{on} \quad \mathcal{R} \quad .$$

**?►Exercise 14.13:** *Let*

$$f(x + iy) = u(x, y) + iv(x, y)$$

*be an analytic function on an open region* $\mathcal{R}$ *.*

**a:** *Use the Cauchy-Riemann equations to show that the real and imaginary parts,* $u$ *and* $v$ *, satisfy*

$$(\nabla u) \cdot (\nabla v) = 0 \quad \text{on} \quad \mathcal{R} \quad .$$

**b:** *Suppose that a level curve for* $u$ *and a level curve for* $v$ *intersect at some point* $z_0$ *in* $\mathcal{R}$ *. What does the fact that* $(\nabla u) \cdot (\nabla v) = 0$ *tell you about how these two curves intersect at* $z_0$ *?*

---

[8] Even though $\arg(z)$ is multi-valued, you should find that its partial derivatives are single-valued. For computing these partial derivatives, you might make use of the observation that

$$\arg(z) = \begin{cases} \arctan(y/x) & \text{if} \quad x \neq 0 \\ \mathrm{arccot}(x/y) & \text{if} \quad y \neq 0 \end{cases} \quad .$$

## Recognizing Analytic Functions

It is not necessary to appeal to the Cauchy-Riemann equations every time you need to decide if a given function is or is not analytic. Keep in mind that those equations are to determine if a given complex formula of $x$ and $y$ determines a differentiable function of $z = x + iy$, treating $z$ as a single variable. And keep in mind that the "differentiability" results usually taught in calculus using the derivative

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

all have direct analogs using the complex derivative

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad .$$

Using these analogs (or the Cauchy-Riemann equations, if you must), it is trivial to verify that any function given by

$$f(z) = \text{constant} \times z$$

is analytic everywhere. It is also easy to verify that, if $f$ and $g$ are analytic on some region $\mathcal{R}$, then so are the sum and the product, $f + g$ and $fg$, while the quotient $^f/_g$ is analytic everywhere in $\mathcal{R}$ that $g$ is nonzero. From this, it easily follows that every polynomial of $z$ is analytic everywhere in $\mathbb{C}$, and every function of the form

$$\frac{\text{one polynomial of } z}{\text{another polynomial of } z}$$

is analytic everywhere the denominator is not zero.

It is also true that any function defined by a power series of $z$ is analytic on the "disk of convergence". This is a straightforward extension of the above observations about polynomials and the theorem on the calculus of power series (theorem 12.17 on page 12–32). Thus, some of our favorite functions, including

$$e^z \quad , \quad \sin(z) \quad \text{and} \quad \cos(z)$$

are analytic everywhere on $\mathbb{C}$.

And don't forget that you just showed $z^\gamma$ and $\ln(z)$ are analytic at nonzero points of $\mathbb{C}$ (though you may want to restrict yourself to branches of these functions).

Almost finally, it's not hard to show that the composition of two analytic functions $h(z) = g(f(z))$ is analytic at every $z_0$ where $f$ is analytic, provided $g$ is also analytic at $f(z_0)$. For example, using

$$g(z) = \frac{1}{z} \quad \text{and} \quad f(z) = \sin(z) \quad ,$$

we get the function

$$h(z) = g(f(z)) = \frac{1}{\sin(z)} \quad .$$

This function is analytic everywhere $\sin(z) \neq 0$.

Finally, of course, all reasonable combinations of the above (e.g., a linear combination of functions defined by power series composed with other analytic functions) are analytic on "suitable regions".

But don't forget that there are functions that are not analytic anywhere. As you recently verified in exercises,

$$z^* \quad , \quad \mathrm{Re}[z] \quad , \quad \mathrm{Im}[z] \quad , \quad |z| \qquad \text{and} \qquad \mathrm{Arg}(z)$$

are not analytic.

## 14.5 Complex Integration Over Curves

### Basics

In complex analysis, "integration" is usually 'integration of functions over curves".

Let $C$ be an oriented curve in the complex plane from point $z_S$ to point $z_E$ (as in figure 14.2a), and let

$$z(t) = x(t) + iy(t) \qquad \text{for} \quad t_S \le t \le t_E \quad .$$

be any parametrization of this curve. Then the line integral of a function $f$ over $C$, denoted by

$$\int_C f(z)\,dz \quad ,$$

can be defined as either the limit of the "obvious" Riemann sums, or by the equation

$$\int_C f(z)\,dz = \int_{t_S}^{t_E} f(z(t))\frac{dz}{dt}\,dt \quad .$$

As with the line integrals discussed last term (see section 10.1), the two definitions are equivalent, and the value of the integral does not depend on which of the many possible parameterizations is used. Also, as before, we'll let $-C$ denote the same curve as $C$ but with the opposite orientation, and note that

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz \quad .$$

Also, if we put a little circle on the integral symbol, as in

$$\oint_C f(z)\,dz \quad .$$

then it should be understood that $C$ is a closed curve.[9]

**!►*Example 14.2:*** *Let $C$ be the unit circle oriented counterclockwise (see figure 14.2b), and let $f(z) = 1/z$. Here, the curve is parametrized by*

$$z(\theta) = x(\theta) + iy(\theta) = \cos(\theta) + i\sin(\theta) = e^{i\theta} \qquad \text{for} \quad 0 \le \theta \le 2\pi \quad .$$

*So,*

$$dz = \frac{dz}{d\theta}\,d\theta = \frac{de^{i\theta}}{d\theta}\,d\theta = ie^{i\theta}\,d\theta \quad ,$$

*and*

$$\oint_C f(z)\,dz = \oint_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\,d\theta = \int_0^{2\pi} i\,d\theta = i2\pi \quad .$$

[9] The use of "$\oint$" instead of "$\int$" when integrating around a closed curve is traditional, but but not required.

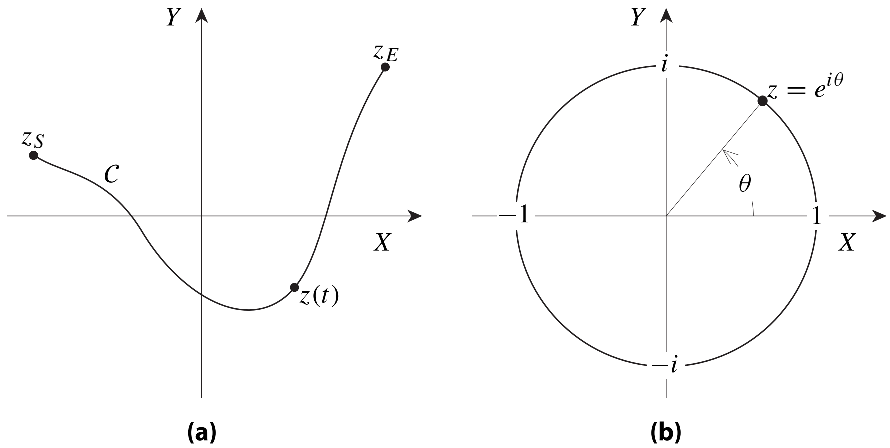


**Figure 14.2:** **(a)** An oriented curve $C$ in the complex plane starting at $z_S$ and $z_E$, and **(b)** the unit circle in the complex plane.

It turns out that the computation and result from the last exercise will be of considerable importance later. So will the following extensions of that exercise:

**?►*Exercise 14.14:*** *For the following, assume $z_0$ is any point in the complex plane and $R$ is any positive value, and let $C$ be the circle of radius $R$ centered at $z_0$ and oriented counterclockwise.*

**a:** *Sketch the figure, analogous to figure 14.2b, describing $C$, and convince yourself that this circle is parametrized by*

$$z(\theta) = z_0 + Re^{i\theta} \qquad \text{for} \quad 0 \le \theta \le 2\pi \quad .$$

**b:** *Let $m$ be integer (positive, negative, or zero), show that*

$$\oint_C (z - z_0)^m \, dz = \begin{cases} i2\pi & \text{if} \quad m = -1 \\ 0 & \text{if} \quad m \neq -1 \end{cases} \quad .$$

*What if $m$ is not an integer, say $m = 1/2$ ?*

Since we are already experts at integrating real-valued functions of two variables over curves in the plane (from last term), let us explicitly develop the connection between the integrals being discussed here and the line integrals discussed last term. So let $C$ be an oriented curve in the complex plane with parametrization

$$z(t) = x(t) + iy(t) \qquad \text{for} \quad t_S \le t \le t_E \quad ,$$

and let $f(z) = u(x, y) + iv(x, y)$. Note that

$$\begin{aligned} f\frac{dz}{dt} &= [u + iv]\left[\frac{dx}{dt} + i\frac{dy}{dt}\right] \\ &= \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right) + i\left(v\frac{dx}{dt} + u\frac{dy}{dt}\right) \\ &= [u\mathbf{i} - v\mathbf{j}] \cdot \underbrace{\left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right]}_{dr/dt} + i[v\mathbf{i} + u\mathbf{j}] \cdot \underbrace{\left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right]}_{dr/dt} \quad . \end{aligned}$$

From this, we almost immediately get the next lemma:

***Lemma 14.2***
*Let $f(z) = u(x, y) + iv(x, y)$ be defined on some oriented curve $C$ in the complex plane. Then*

$$\int_C f(z)\,dz = \int_C \mathbf{G}\cdot d\boldsymbol{r} + i\int_C \mathbf{H}\cdot d\boldsymbol{r}$$

*where $\mathbf{G}$ and $\mathbf{H}$ are the two-dimensional (real) vector fields given by*

$$\mathbf{G}(x, y) = u(x, y)\mathbf{i} - v(x, y)\mathbf{j} \qquad \text{and} \qquad \mathbf{H}(x, y) = v(x, y)\mathbf{i} + u(x, y)\mathbf{j} \quad .$$

The above lemma gives us a way to quickly derive and verify results about complex line integrals using what we already learned last term about line integrals. One of those results is the following, often overlooked, theorem:

***Theorem 14.3 (Complex Fundamental Theorem of Integration)***
*Assume $F(z)$ is a single-valued, analytic function in a region containing an oriented curve $C$. Then*

$$\int_C F'(z)\,dz = F(z_E) - F(z_S) \tag{14.10}$$

*where $z_S$ is the point at which $C$ starts, and $z_E$ is the point at which $C$ ends.*

*PROOF (partial):* Let $U$ and $V$ be the real and imaginary parts of $F$,

$$F(x + iy) = U(x, y) + iV(x, y) \quad .$$

Then observe (after using lemma 14.2, the Cauchy-Riemann equations, and what we learned about integrals of conservative vector fields[10]) that

$$\int_C F'(z)\,dz = \cdots = \int_C \nabla U\cdot d\mathbf{r} + i\int_C \nabla V\cdot d\mathbf{r} = \cdots = F(z_E) - F(z_S) \quad .$$ ■

**?►*Exercise 14.15:*** *Fill in the details of the above proof.*

The above theorem can be applied when $F$ is multi-valued (e.g., $\ln(z)$ ) provided you use appropriate branches of $F$. If you can arrange it so that the curve $C$ does not touch any branch cut lines, then you can just use that one branch. Otherwise, you may have to employ limits or even split the integral up and use different branches. Basically, just remember to "follow $C$" to get the correct relation between $F(z_E)$ and $F(z_S)$ in equation (14.10).

**!►*Example 14.3:*** *Consider*

$$\int_C \frac{1}{z}\,dz$$

*where $C = C_{0,\theta_E}$ is the part of the unit circle oriented counterclockwise starting at $z_S = 1 = e^{i\cdot 0}$ and going to some point $z_E = e^{i\theta_E}$ for some $\theta_E > 0$.*

[10] see section 10.2 of last term's notes.

*If* $0 < \theta_E < 2\pi$ *, we can use any branch of the natural logarithm with formula*

$$\ln(z) = \ln|z| + i\,\mathrm{Arg}(z) + i2n\pi$$

*where* $n$ *is any single integer and* $0 \leq \mathrm{Arg}(z) < 2\pi$ *, along with the fact that*

$$\frac{1}{z} = \frac{d}{dz}\ln(z) \quad .$$

*Then*

$$\begin{aligned}\int_C \frac{1}{z}\,dz &= \int_{C_{\theta_E}} \frac{d}{dz}\ln(z)\,dz \\ &= \ln(z_E) - \ln(z_S) \\ &= \ln\left(e^{i\theta_E}\right) - \ln\left(e^{i\cdot 0}\right) \\ &= \left[\ln 1 + i\,\mathrm{Arg}\left(e^{i\theta_E}\right) + i2n\pi\right] - \left[\ln 1 + i\,\mathrm{Arg}\left(e^{i\cdot 0}\right) + i2n\pi\right] \\ &= [i\theta_E + i2n\pi] - [i\cdot 0 + i2n\pi] = i\theta_E \quad .\end{aligned}$$

*Note that the choice of* $n$ *is totally irrelevant here; that constant term cancels itself out.*

*If the curve* $C$ *is the complete unit circle,* $C_{0,2\pi}$ *, then we must take into account the fact that* $C$ *is given by* $z = e^{i\theta}$ *with* $\theta$ *going from* $0$ *to* $2\pi$ *by letting* $\theta_E \to 2\pi$ *in the last computations,*

$$\oint_C \frac{1}{z}\,dz = \lim_{\theta_E \to 2\pi} \int_{C_{0,\theta_E}} \frac{1}{z}\,dz = \lim_{\theta_E \to 2\pi} i\theta_E = i2\pi \quad .$$

*Finally, if* $2\pi < \theta_E < 4\pi$ *, then* $C = C_1 + C_2$ *where*

$$C_1 = C_{0,2\pi} \qquad \text{and} \qquad C_2 = C_{2\pi,\theta_E} \quad ,$$

*and*

$$\int_C \frac{1}{z}\,dz = \int_{C_1} \frac{1}{z}\,dz + \int_{C_2} \frac{1}{z}\,dz = i2\pi + [i\theta_E - i2\pi] = i\theta_E \quad .$$

*(For our work, however, there will rarely be any need to have a curve wrap around onto itself.)*

As illustrated by the last example, if $F$ is multi-valued, then we can get

$$F(z_E) - F(z_S) \neq 0 \qquad \text{even though} \qquad z_E = z_S \quad .$$

Of course, if $F$ is single valued, then

$$z_E = z_S \quad \Longrightarrow \quad F(z_E) = F(z_S)$$

and, thus, we must then have

$$\oint_C F'(z)\,dz = 0$$

whenever $C$ is a closed curve.