Homework Handout IX

Note: You've already done much of the work for the problems in this set in earlier homework sets. Feel free to make use of what you've already done!

A 1. Find the solution to the following "heat flow" problem

$$u_t - 3u_{xx} = 0 \quad , \quad 0 < x < 2$$

$$u(0,t) = 0 \quad \text{and} \quad u(2,t) = 0 \quad , \quad t \geq 0$$

$$u(x,0) = u_o(x) \quad , \quad 0 < x < 2$$

for each of the following choices of $u_o(x)$.

a. $u_o(x) = 5\sin(\pi x)$

b. $u_o(x) = 50$

c. $u_o(x) = 2x$

d. $u_o(x) = \begin{cases} 50 & , \quad 0 < x < 1 \\ 0 & , \quad 1 < x < 2 \end{cases}$

2. (optional, but interesting) In the previous part you found the solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\lambda_n t} \quad .$$

In practice, it is very difficult to completely compute an infinite sum. So, for each positive integer N , let S_N denote the " N^{th} partial sum approximation to u ",

$$S_N(x,t) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\lambda_n t} \quad .$$

For each of the $u_0(x)$'s given above, graph the 10^{th} partial sum approximation, $S_N(x,t)$, to the corresponding solution, $u(x,t)$, for each of the following values of t :

$$t = 0 \ , \quad t = 0.1 \ , \quad t = 0.2 \ , \quad t = 0.3 \ , \quad t = 0.4 \ , \quad t = 0.5 \quad .$$

Use the graphs drawn to visualize how the temperature distribution throughout the rod is changing with time. Do NOT attempt to draw these graphs by hand! Do it on a computer using a "math package" such as MathCad, Maple, or Mathematica.

B 1. Find the infinite series formula for the solution to the following heat flow problem:

$$u_t = \kappa u_{xx} \quad , \quad 0 < x < L$$
$$u_x(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0 \quad , \quad t \geq 0$$
$$u(x,0) = u_o(x) \quad , \quad 0 < x < L$$

(In deriving this formula, be sure to

i. identify the eigenvalue / eigenfunction problem, and

ii. state the appropriate orthogonality relation.)

2. Using your results from part ***1***, find the solution to the heat flow problem in part ***1*** (assuming $\kappa = 2$ and $L = 3$) corresponding to each of the following choices for the initial temperature distribution, u_o :

a. $u_o(x) = 5\sin(\pi x)$

b. $u_o(x) = 50$

c. $u_o(x) = 2x$

d. $u_o(x) = \begin{cases} 50 & , \quad 0 < x < \frac{3}{2} \\ 0 & , \quad \frac{3}{2} < x < 3 \end{cases}$

3. (optional) Graph (as in problem ***A 1***) the 10^{th} partial sum approximation to each of the above solutions at times $t = 0,\ 0.1,\ 0.2,\ 0.3,\ 0.4$ and 0.5 .

C 1, 2 & 3. Repeat the previous problem using the following heat flow problem:

$$u_t = \kappa u_{xx} \quad , \quad 0 < x < L$$
$$u(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0 \quad , \quad t \geq 0$$
$$u(x,0) = u_o(x) \quad , \quad 0 < x < L$$

D. Consider the basic "vibrating string between two fixed endpoints" problem:

$$u_{tt} - c^2 u_{xx} = 0 \quad , \quad 0 < x < L$$

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \quad , \quad t \geq 0$$

$$u(x,0) = u_o(x) \quad \text{and} \quad u_t(x,0) = v_o(x) \quad , \quad 0 < x < L$$

where u_o and v_o are known functions, and c is a positive constant.

1. Find the infinite series formula for the solution to this problem.

2. Let $c = 8$ and $L = 8$ and solve this vibrating string problem for each of the following choices of u_o and v_o (try to visualize the initial conditions and the resulting behavior of the string):

a. $u_o(x) = 5\sin\left(\frac{\pi}{4}x\right) \quad , \quad v_o(x) = 0$

b. $u_o(x) = 0 \quad , \quad v_o(x) = 5\sin\left(\frac{\pi}{4}x\right)$

c. $u_o(x) = 5\sin\left(\frac{\pi}{4}x\right) \quad , \quad v_o(x) = 5\sin\left(\frac{\pi}{4}x\right)$

d. $u_o(x) = \begin{cases} 0 & , \quad 0 < x < 2 \\ 1 & , \quad 2 < x < 6 \\ 0 & , \quad 6 < x < 8 \end{cases} \quad , \quad v_o(x) = 2$

e. $u_o(x) = \begin{cases} x & , \quad 0 < x < 4 \\ 8 - x & , \quad 4 < x < 8 \end{cases} \quad , \quad v_o(x) = 0$

E. Now consider the "vibrating string between two fixed endpoints with air resistence" problem:

$$u_{tt} - c^2 u_{xx} = -\kappa u_t \quad , \quad 0 < x < L$$

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \quad , \quad t \geq 0$$

$$u(x,0) = u_o(x) \quad \text{and} \quad u_t(x,0) = v_o(x) \quad , \quad 0 < x < L$$

where u_o and v_o are known functions, and c and κ are positive constants.

Find the infinite series formula for the solution to this problem. You may assume that the medium is "thin" (see/recall problem ***B*** of *Homework Handout VII*).

F. Find the infinite series formula for the solution $u = u(x, y, t)$ to the following "heat flow on a rectangle" problem:

$$u_t - 9\nabla^2 u = 0 \quad \text{for} \quad 0 < x < 2\ ,\ 0 < y < 5\ ,\ t \geq 0$$

$$u(0, y, t) = 0 \quad \text{and} \quad u(2, y, t) = 0 \quad \text{for} \quad t \geq 0\ ,\ 0 < y < 5$$

$$u(x, 0, t) = 0 \quad \text{and} \quad u(x, 5, t) = 0 \quad \text{for} \quad t \geq 0\ ,\ 0 < x < 2$$

$$u(x, y, 0) = 25 \quad \text{for} \quad 0 < x < 2\ ,\ 0 < y < 5$$

G. Let's go back to the "heat flow problem on a two-dimensional disk of radius a":

$$u_t - \kappa\nabla^2 u = 0 \qquad \text{for} \quad x^2 + y^2 < a^2 \ \text{and} \ t > 0$$

$$u(x, y, t) = 0 \qquad \text{if} \quad x^2 + y^2 = a^2 \ \text{and} \ t > 0$$

$$u(x, y, 0) = u_0(x, y) \qquad \text{for} \quad x^2 + y^2 < a^2$$

You originally considered this way back in problem ***D*** of *Homework Handout VII*. There you separated the problem into three "smaller" ode problems. We now have all the pieces we need to finish solving this problem. So

1. Write out (again) those three ode problems along with any appropriate boundary conditions, and describe the solutions to each.

2. Write out, as completely as possible, the series solution to this problem. Your series will involve exponentials, sines and cosines, and Bessel functions. You will also need to refer to the zeroes for the Bessel functions (which you may simply denote by z_{m1}, z_{m2}, z_{m3}, $\ldots$). Be sure to give the complete integral formulas for computing the constants in the series, *but do **not** actually try to compute them!*

3. To what does the series formula reduce when you can assume the solution u is radially symmetric; that is, when u depends only on time t and the radial distance r, and not on the polar angle θ? Note that, in this case, the initial condition can be written as

$$u(r, t) = f(r) \qquad \text{for} \quad r < a$$

where $f(r)$ is some known function (related to u_0).

Some of the Answers

A 1a. $u(x,t) \;=\; 5\, e^{-3\pi^2 t} \sin(\pi x)$

b. $u(x,t) \;=\; \sum_{n=1}^{\infty} \frac{100}{n\pi} \left[1 - (-1)^n\right] e^{-3\lambda_n t} \sin\left(\frac{n\pi}{2}x\right)$ where $\lambda_n = \left(\frac{n\pi}{2}\right)^2$

c. $u(x,t) \;=\; \sum_{n=1}^{\infty} (-1)^{n+1} \frac{8}{n\pi} e^{-3\lambda_n t} \sin\left(\frac{n\pi}{2}x\right)$ where $\lambda_n = \left(\frac{n\pi}{2}\right)^2$

d. $u(x,t) \;=\; \sum_{n=1}^{\infty} \frac{100}{n\pi} \left[1 - \cos\left(\frac{n\pi}{2}\right)\right] e^{-3\lambda_n t} \sin\left(\frac{n\pi}{2}x\right)$ where $\lambda_n = \left(\frac{n\pi}{2}\right)^2$

B 1. Here $\lambda = 0$ does lead to a nontrivial eigenfunction (a constant function).
The orthogonality relation(s) is (are):

If n and m are positive integers, then

$$\int_{x=0}^{L} \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx \;=\; 0 \qquad \text{if } n \neq m$$

and

$$\int_{x=0}^{L} 1 \cdot \cos\left(\frac{n\pi}{L}x\right) dx \;=\; 0$$

The formula for the solution is $u(x,t) \;=\; c_o + \sum_{n=1}^{\infty} c_n\, e^{-\kappa \lambda_n t} \cos\left(\frac{n\pi}{L}x\right)$

where $c_o = \frac{1}{L} \int_{x=0}^{L} u_o(x)\, dx$

and, for $n = 1, 2, 3, \ldots$,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad , \quad c_n = \frac{2}{L} \int_{x=0}^{L} u_o(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

B 2. In the following $\lambda_n = \left(\frac{n\pi}{L}\right)^2$.

a. $u(x,t) = \frac{10}{3\pi} + \sum_{n=1}^{\infty} c_n \, e^{-2\lambda_n t} \cos\left(\frac{n\pi}{3}x\right)$

where $c_n = \frac{30}{\pi}\left[\frac{(-1)^n + 1}{9 - n^2}\right]$ if $n \neq 3$, and $c_3 = 0$.

b. $u(x,t) = 50$

c. $u(x,t) = 3 + \sum_{n=1}^{\infty} 3\left(\frac{2}{n\pi}\right)^2 [(-1)^n - 1] \, e^{-2\lambda_n t} \cos\left(\frac{n\pi}{3}x\right)$

d. $u(x,t) = 25 + \sum_{n=1}^{\infty} \frac{100}{n\pi} \sin\left(\frac{n\pi}{2}\right) e^{-2\lambda_n t} \cos\left(\frac{n\pi}{3}x\right)$

C 1. $u(x,t) = \sum_{n=1}^{\infty} c_n \, e^{-\kappa\lambda_n t} \sin\left(\frac{\mu_n \pi}{2L}x\right)$

where $\mu_n = n^{th}$ positive odd integer $= 2n - 1 = 1, 3, 5, \ldots ,$

$$\lambda_n = \left(\frac{2n - 1}{2L}\pi\right)^2 \quad \text{and} \quad c_n = \frac{2}{L}\int_{x=0}^{L} u_o(x) \sin\left(\frac{\mu_n \pi}{2L}x\right) dx$$

D 1. $u(x,t) = \sum_{n=1}^{\infty} \left[a_n \sin\left(\frac{nc\pi}{L}t\right) + b_n \cos\left(\frac{nc\pi}{L}t\right) \right] \sin\left(\frac{n\pi}{L}x\right)$

where

$$b_n = \frac{2}{L}\int_{x=0}^{L} u_o(x) \sin\left(\frac{n\pi}{L}x\right) dx \quad \text{and} \quad a_n = \frac{2}{nc\pi}\int_{x=0}^{L} v_o(x) \sin\left(\frac{n\pi}{L}x\right) dx$$