Homework Handout VIII

Note: L always denotes some finite distance in the following problems.

A. Compute the following, assuming the interval is $(0, 3)$ and the weight function is $w(x) = 1$.

1. $\langle\, x \,|\, \sin(2\pi x) \,\rangle$ ***2.*** $\langle\, x^2 \,|\, 9 + i8x \,\rangle$

3. $\langle\, 9 + i8x \,|\, x^2 \,\rangle$ ***4.*** $\langle\, e^{i2\pi x} \,|\, x \,\rangle$

5. $\|x\|$ ***6.*** $\|9 + i8x\|$

7. $\|\sin(2\pi x)\|$ ***8.*** $\|e^{i2\pi x}\|$

B. Repeat the above problem, but using the interval $(0, 1)$ and the weight function $w(x) = x$.

C. Verify that each of the following sets of functions is orthogonal on $(0, L)$ and with respect to the weight function $w(x) = 1$.

1. $\left\{ \sin\left(\frac{k\pi}{L}\,x\right) \;:\; k = 1, 2, 3, \dots \right\}$ ***2.*** $\left\{ \cos\left(\frac{k\pi}{L}\,x\right) \;:\; k = 1, 2, 3, \dots \right\}$

3. $\left\{ e^{i2\pi \frac{k}{L} x} \;:\; k = 0, \pm 1, \pm 2, \pm 3, \dots \right\}$

D. Determine a value for β so that $\left\{ e^{i2\pi x^2} , e^{i2\pi\beta x^2} \right\}$ is an orthogonal set on $(0, 1)$ with weight function $w(x) = x$.

E. In each of the following, we have a function on $(0, L)$ along with a set of functions $\{\phi_1(x), \phi_2(x), \phi_3(x), \dots\}$ known to be orthogonal on $(0, L)$ with respect to the weight function $w(x) = 1$. Consider, now, the corresponding generalized Fourier series for f,

$$\text{G.F.S.}[f]\big|_x = \sum_{k=1}^{\infty} c_k\, \phi_k(x) \qquad \text{with} \quad c_k = \frac{\langle\, \phi_k \,|\, f \,\rangle}{\|\phi_k\|^2} \quad .$$

1. Assuming each ϕ_k is given by $\phi_k(x) = \sin\left(\frac{k\pi}{L}\,x\right)$, verify that the $\text{G.F.S.}[f]\big|_x$ reduces to the classical Fourier sine series for f , i.e.,

$$\sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}\,x\right) \qquad \text{with} \quad b_k = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{k\pi}{L}\,x\right) dx \quad .$$

2. To what does the generalized Fourier series for f reduce when the ϕ_k's are given by

$$\phi_k(x) = \cos\left(\frac{k\pi}{L}x\right) \quad ?$$

3. To what does the generalized Fourier series for f reduce when the ϕ_k's are given by

$$\phi_k(x) = e^{i2\pi\frac{k}{L}x} \quad ?$$

(WARNING: Your answers to ***b*** and ***c*** will not quite be the classical Fourier cosine or exponential series — some terms will be missing!)

F. Assume $\{\phi_1(x), \phi_2(x), \phi_3(x), \dots\}$ is an orthogonal set of functions and $f(x)$ is any suitably integrable function. Let

$$\text{G.F.S.}[f]|_x = \sum_{k=1}^{\infty} c_k\,\phi_k(x) \qquad \text{with} \quad c_k = \frac{\langle\,\phi_k\,|\,f\,\rangle}{\|\phi_k\|^2} \quad ,$$

1. Show that, for each positive integer N,

$$\left\|f - \sum_{k=1}^{N} c_k\,\phi_k\right\|^2 = \|f\|^2 - \sum_{k=1}^{N} |c_k|^2\,\|\phi_k\|^2 \quad .$$

2. Using the above, verify that, in general,

$$\|f\|^2 \geq \sum_{k=1}^{\infty} |c_k|^2\,\|\phi_k\|^2 \qquad \text{(Bessel's inequality)},$$

and that, if $\{\phi_1(x), \phi_2(x), \phi_3(x), \dots\}$ is complete, then

$$\|f\|^2 = \sum_{k=1}^{\infty} |c_k|^2\,\|\phi_k\|^2 \qquad \text{(Bessel's equality)}.$$

3. Assume $\{\phi_1(x), \phi_2(x), \phi_3(x), \dots\}$ is complete,

$$\text{G.F.S.}[g]|_x = \sum_{k=1}^{\infty} a_k\,\phi_k(x) \qquad \text{and} \qquad \text{G.F.S.}[h]|_x = \sum_{k=1}^{\infty} b_k\,\phi_k(x) \quad .$$

Show that

$$\langle\,g\,|\,h\,\rangle = \sum_{k=1}^{\infty} a_k^*\,b_k\,\|\phi_k\|^2 \quad .$$

G. Using the interval $(0, L)$ and weight function $w(x) = 1$, show that the set

$$\{\phi_1(x),\ \phi_2(x),\ \phi_3(x),\ \dots\} = \left\{ \cos\left(\frac{k\pi}{L}x\right) : k = 1, 2, 3, \dots \right\}$$

is not complete by showing that

$$f(x) \neq \sum_{k=1}^{\infty} c_k \cos\left(\frac{n\pi}{L}x\right) \qquad \text{with} \qquad c_k = \frac{\langle \phi_k \,|\, f \rangle}{\|\phi_k\|^2}$$

when $f(x) = 1$ everywhere on $(0, L)$.

H. It can be shown that the set

$$\{\phi_1(x),\ \phi_2(x),\ \phi_3(x),\ \dots\} = \left\{ \sin\left(\frac{k\pi}{L}x\right) : k = 1, 2, 3, \dots \right\}$$

is complete (on the interval $(0, L)$ with weight function $w(x) = 1$). The "Fourier series" for a function f using this set is the classic sine series, $\sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right)$ (as described in problem ***E1***). Find the sine series for each of the following:

1. $f(x) = 1$

2. $f(x) = x$

3. $f(x) = \begin{cases} x & , \quad 0 \leq x \leq \frac{L}{2} \\ L - x & , \quad \frac{L}{2} \leq x \leq L \end{cases}$

Also, for each, sketch and compare the graphs of

$$f(x) \quad , \quad \sum_{k=1}^{5} b_k \sin\left(\frac{k\pi}{L}x\right) \quad , \quad \sum_{k=1}^{10} b_k \sin\left(\frac{k\pi}{L}x\right) \quad \text{and} \quad \sum_{k=1}^{25} b_k \sin\left(\frac{k\pi}{L}x\right)$$

(use a computer and appropriate software such as Mathcad, Maple, Mathematical, etc.)

I. Convert each of the following to self-adjoint form (treat λ and μ as constants with λ being the "eigenvalue"). Also, for the last three, be sure to clearly identify the formulas for $p(x)$, $q(x)$, and, especially, $w(x)$.

1. $\phi'' + 8\phi' + 3x\phi = 0$

2. $\phi'' + 4x\phi' + 3\lambda x\phi = 0$

3. Legendre's equation: $(1 - x^2)\phi'' - 2x\phi' + \lambda\phi = 0$

4. Bessel's equation: $x^2\phi'' + x\phi' + [\mu x^2 - \lambda]\phi = 0$

J. Five sets of boundary conditions are given below for an eigen-problem on a finite interval (a,b) whose ode is in the form

$$\frac{d}{dx}\left[p(x)\,\frac{d\phi}{dx}\right] + q(x)\,\phi = -\lambda w(x)\,\phi \quad .$$

Verify that each set is "Sturm-Liouville appropriate" by verifying that the right-hand side of the appropriate Green's formula vanishes when $u(x)$ and $v(x)$ satisfy the given conditions. Unless otherwise indicated, assume all functions and their derivatives exist and are finite at $x = a$ and $x = b$.

1. $\phi'(a) = 0$ and $\phi'(b) = 0$

2. $\phi(a) = 0$ and $2\phi(b) + 3\phi'(b) = 0$

3. Any pair of "regular" boundary conditions at a and b; i.e.,

$$A\phi(a) + B\phi'(a) = 0 \quad \text{and} \quad C\phi(b) + D\phi'(b) = 0$$

where A, B, C, and D are constants with A or B (or both) being nonzero, and C or D (or both) being nonzero.

4. $\phi(a) = \phi(b)$ and $\phi'(a) = \phi'(b)$ assuming also that $p(a) = p(b)$

5. $\phi(a)$ is bounded and $\phi'(b) = 0$ assuming also that $p(x) = x - a$

K. (classical Fourier series) Let $f(x)$ be a piecewise smooth function on $(0,L)$. For each of the following three Sturm-Liouville problems,

i. Find all eigenvalues and corresponding eigenfunctions, along with the formula for computing the eigenfunction expansion (general Fourier series) of f using these eigenfunctions. In each case, you should get one of the classical Fourier series (sine, cosine, trigonometric) for $f(x)$ on $(0,L)$.

ii. Then compute the eigenfunction expansion (Fourier series) for each of the following choices of $f(x)$ on $(0,L)$:

$$f(x) = 1 \quad , \quad f(x) = x \quad , \quad f(x) = \sin^2\left(\frac{2\pi}{L}\,x\right)$$

$$\text{and} \qquad f(x) = \begin{cases} x & , \ 0 \le x \le \frac{L}{2} \\ L - x & , \ \frac{L}{2} \le x \le L \end{cases}$$

(Note: You've already done some of this in a previous problem in Handout VI.)

1. $\phi'' = -\lambda\phi$ with bc's $\phi(0) = 0$ and $\phi(L) = 0$ (sine series).

2. $\phi'' = -\lambda\phi$ with bc's $\phi'(0) = 0$ and $\phi'(L) = 0$ (cosine series).

3. $\phi'' = -\lambda\phi$ with bc's $\phi(0) = \phi(L)$ and $\phi'(0) = \phi'(L)$ (full trig. series).