Homework Handout VII

A. For each of the following homogeneous boundary-value problems:

i. "Separate" the problem into the corresponding eigenvalue/eigenfunction problem and the "other" problem.

ii. Solve the eigenvalue/eigenfunction problem (i.e., find all the possible eigenvalues and corresponding eigenfunctions).

iii. Solve the "other" problem for each possible eigenvalue.

iv. Write out all possible separable solutions.

Notes:

1. In each case c, L, and κ denote positive constants and the pde is to be valid for $0 < t$ and $0 < x < L$.

2. Problems a, b, and c correspond to "heat flow" problems for a uniform rod between $x = 0$ and $x = L$ with diffusivity κ. Problem d is a heat flow problem with periodic boundary conditions (pretend the rod is actually a ring of circumference L). The rest are finite vibrating string problems with the string stretched between $x = 0$ and $x = L$.

3. I was lazy and used the "subscript" notation for partial derivatives (i.e., $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc.).

4. No initial values are given because they are not relevant to the computations being done here.

a. $u_t - \kappa u_{xx} = 0$ with $u(0,t) = 0$ and $u(L,t) = 0$

b. $u_t - \kappa u_{xx} = 0$ with $u_x(0,t) = 0$ and $u_x(L,t) = 0$

c. $u_t - \kappa u_{xx} = 0$ with $u(0,t) = 0$ and $u_x(L,t) = 0$

d. $u_t - \kappa u_{xx} = 0$ with $u(0,t) = u(L,t)$ and $u_x(0,t) = u_x(L,t)$

e. $u_{tt} - c^2 u_{xx} = 0$ with $u(0,t) = 0$ and $u(L,t) = 0$

f. $u_{tt} - c^2 u_{xx} = 0$ with $u(0,t) = 0$ and $u_x(L,t) = 0$

B. Let us consider a string stretched between $x = 0$ and $x = L$ (and fixed at the endpoints). If we include the dampening effect on the vertical vibrations due to the friction between the string and the surrounding medium, then the corresponding homogeneous boundary-value problem is

$$u_{tt} - c^2 u_{xx} = -\kappa u_t \quad \text{for} \quad 0 < x < L$$

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

where κ is a nonnegative constant (the "drag coefficient"). In a vacuum $\kappa = 0$. In a fairly thin medium (like air) κ is fairly small. In a thick viscous medium (like molasses) the value of κ is fairly large.

Find the separable solutions to this problem and show that in a thin medium these solutions are exponentially decaying "standing waves" of the form

$$u_n(x,t) = e^{-\kappa t/2} \left[A_n \cos(\gamma_n t) + B_n \sin(\gamma_n t) \right] \sin\left(\frac{n\pi}{L} x\right)$$

where the γ_n's are constants (which you must determine). Also show that in a thick viscous medium some of the separable solutions contain *no* oscillating factors (i.e., no factor of the form $A_n \cos(\gamma_n t) + B_n \sin(\gamma_n t)$). What does this mean physically?

Suggestion: Illustrate the above by specific examples. Assume, for example that $L = \pi$ and $c = 2$ and first find the u_n's corresponding to $\kappa = 1$ (a thin medium) and then find the u_n's corresponding to $\kappa = 12$ (a thick medium). Note: For these choices of L, c, and κ, the "first 4 or so" u_n's should be found separately from the rest of the u_n's (why?).

C. For each of the following homogeneous boundary-value problems, "separate" the given problem into the corresponding eigenvalue/eigenfunction problem and the "other" problem. Just "separate". Do not attempt to actually solve these problems.

a. $u_{tt} - c^2\left(u_{xx} + \frac{1}{x} u_x\right) = 0 \quad \text{for} \quad 0 < x < L$

with $u(0,t) = 0$ and $u(L,t) = 0$

b. $u_{tt} - c^2 u_{xx} + u = 0 \quad \text{for} \quad 0 < x < L$

with $u(0,t) = 0$ and $u_x(L,t) = 0$

c. $u_t - \kappa\left(u_{xx} + \frac{1}{x} u_x\right) = 0 \quad \text{for} \quad 0 < x < L$

with $u(0,t) = 0$ and $u(L,t) = 0$

d. $u_{rr} - \frac{1}{r} u_r - \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{for} \quad 0 < \theta < 2\pi$

with $u(r,0) = u(r,2\pi)$ and $u_\theta(r,0) = u_\theta(r,2\pi)$.

D. Consider the following "heat flow problem on a two-dimensional disk of radius a":

$$u_t - \kappa \nabla^2 u = 0 \qquad \text{for} \quad x^2 + y^2 < a^2 \text{ and } t > 0$$

$$u(x,y,t) = 0 \qquad \text{if} \quad x^2 + y^2 = a^2 \text{ and } t > 0$$

$$u(x,y,0) = u_0(x,y) \qquad \text{for} \quad x^2 + y^2 < a^2$$

a. Do one iteration of the separation of variables procedure, starting with

$$u(x,y,t) = \Psi(x,y)\, g(t) \ ,$$

and derive the eigenvalue problem involving the Helmholtz equation.

b. Express this eigenvalue problem in terms of polar coordinates, (r,θ) .[1]

c. Give arguments that, implicit in our original problem, we have periodic boundary conditions involving θ and a "boundedness" boundary condition at $r = 0$.

d. Continue separating the eigenvalue problem for Ψ and find the functions of θ and the eigenvalue problem for the functions of r .

E. Skim through § 9.4 of AW&H, pages 414–534, which is mainly concerned with separating the Helmholtz equation in various coordinate systems. WARNING: In this section they do not adequately point out the role of the boundary conditions and make totally unjustified assumptions regarding the separation constants.

F. AW&H § 9.4, page 432: (3), 5(just do this in 2 dimensions, not 3)

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[1]Remember, in polar coordinates $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.