Homework Handout V

Note: In the following, assume x and y are real variables while z is a complex varible with $z = x + iy$

A. For each of the following functions,

1. Identify all the singularities and zeroes of each function in the (finite) complex plane, and determine what type of singularity/zero each is (e.g., "pole of order 2", "simple pole", "essential singularity", "zero of order 827").

2. Determine if the function is analytic or has a pole at ∞. If it has a pole, determine its order. It is analytic, find its value at ∞, and if that value is zero, find its order.

3. Compute the residue at each singularity in the (finite) complex plane.

a. $\dfrac{4z-1}{z^2+5z+4}$ ***b***. $\dfrac{1-e^z}{z}$

c. $\dfrac{e^z}{z^2+4}$ ***d***. $\dfrac{e^{3z}}{1-e^z}$

e. $\dfrac{8z^3 + 4}{(z-1)^2}$ ***f***. $\dfrac{8z^3 + 4}{(z-1)^3}$

g. $\dfrac{8z^3 + 4}{(z-1)^4}$ ***h***. $\dfrac{8z^3 + 4}{(z^2+1)^2}$

i. $\dfrac{\sin(iz)}{z^2+1}$ ***j***. $\dfrac{z^2+1}{\sin(iz)}$

B. For each of the following functions,

1. What sort of singularity (if any) does it have at 0?

2. Find its residue at 0.

a. $\exp\left(\dfrac{1}{z}\right)$ ***b***. $\sin\left(\dfrac{1}{z}\right)$

c. $\cos\left(\dfrac{1}{z}\right)$ ***d***. $\dfrac{1}{\sin(z)}$

e. $\dfrac{1}{\sin^2(z)}$ ***f***. $\dfrac{1}{\cos(z)}$

g. $\dfrac{1}{\sin(z^2)}$ $\left(\text{***h***. } \dfrac{1}{\sin(z^3)}\right)$

C. Consider the function $\dfrac{1}{\cosh(\pi z)}$:

1. Where are the singularities of this function and what sort of singularities are they?

2. Compute the residue of this function at one of its singularities. What are the residues at the other singularities?

D. Using residues, evaluate each of the following integrals assuming $\mathcal{C}$ is the circle about 0 of radius 2 (oriented counter-clockwise)

1. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}}{z+i}\,dz$

2. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}}{3z+i}\,dz$

3. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}}{z+3i}\,dz$

4. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}}{z^2+1}\,dz$

5. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}(4z-1)}{z^2+5z+4}\,dz$

6. $\displaystyle\int_{\mathcal{C}} \frac{8z^3+4}{(z-1)^2}\,dz$

7. $\displaystyle\int_{\mathcal{C}} \frac{8z^3+4}{(z-1)^3}\,dz$

8. $\displaystyle\int_{\mathcal{C}} \sin\left(\frac{1}{z}\right)\,dz$

9. $\displaystyle\int_{\mathcal{C}} \cos\left(\frac{1}{z}\right)\,dz$

10. $\displaystyle\int_{\mathcal{C}} \frac{1}{\sin(z)}\,dz$

E. Using residues, evaluate $\displaystyle\int_{\mathcal{C}} \frac{e^{i2\pi z}}{z^2+1}\,dz$ where $\mathcal{C}$ is a simple loop enclosing $-i$ but not $+i$.

F. AW&H § 11.8, page 538: 1, 2, 3 (For #2 you should first convince yourself that, for this integral, "$\int_0^{\pi} = \frac{1}{2}\int_0^{2\pi}$ ").

G. AW&H § 11.8, page 541: 15, 16 (For 16, first convince yourself that, for this integral, "$\int_0^{\infty} = \frac{1}{2}\int_{-\infty}^{\infty}$ "). Also evaluate

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+a^2}\,dx$$

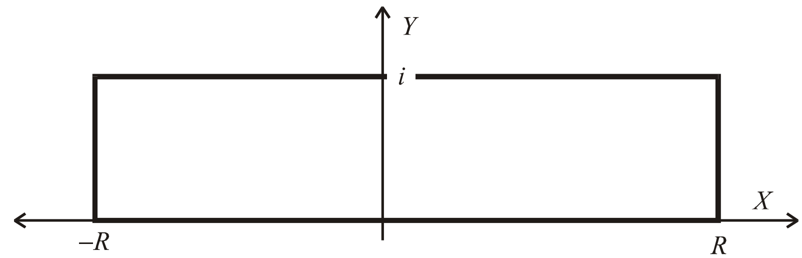
(assuming $a > 0$) and then do 12a & b on page 540 of AW&H.

H. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2-4x+13}$.

I. For the following, let $f(z) = \dfrac{1}{\cosh(\pi z)}$. Recall that, in a problem above, you discovered that this function has a simple pole at $z_0 = \dfrac{1}{2}i$. Now,

1. verify that, for each real x , $f(x+i) = -f(x)$, and then

2. evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\cosh(\pi x)}$ by using the closed curved sketched below along with the facts noted above.



3. Show that, for any real value ω , $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-i2\pi\omega x}}{\cosh(\pi x)}\,dx = \frac{A}{\cosh(\pi\omega)}$. Be sure to determine the value of A .

J. Compute the following:

1. CPV$\displaystyle\int_{-\infty}^{\infty} \frac{e^{i2x}}{x^2-1}\,dx$ *2*. CPV$\displaystyle\int_{-\infty}^{\infty} \frac{e^{-i2x}}{x^2-1}\,dx$

3. CPV$\displaystyle\int_{-\infty}^{\infty} \frac{e^{i2x}}{x^3-1}\,dx$ *4*. CPV$\displaystyle\int_{-\infty}^{\infty} \frac{e^{-i2x}}{x^3-1}\,dx$

K. AW&H § 11.7, page 522: 12

L. AW&H § 11.8, page 539: 8, 9, 11, 12 (To do these correctly, you should either convert the trig functions to complex exponentials or express the integrals as real or imaginary parts of integrals involving complex exponentials, and use Cauchy principal parts. Also, it isn't vital that you use the contour they suggest for problem 15. [Actually, we may have done #12 in class.])

M. AW&H § 11.8, page 543: 24 (Use the contour from the class example on multiply-valued functions and assume

$$x^{-\alpha} = \frac{1}{\sqrt[\alpha]{x}} = \frac{1}{\text{positive } \alpha\text{th root of } x} \quad .)$$

N. Evaluate $\displaystyle\int_0^{\infty} \frac{x^{\alpha}}{(1+x)^2}\,dx$ assuming $-1 < \alpha < 1$. Reduce your answer to a relatively simple, explicitly real-valued formula involving sines and/or cosines.