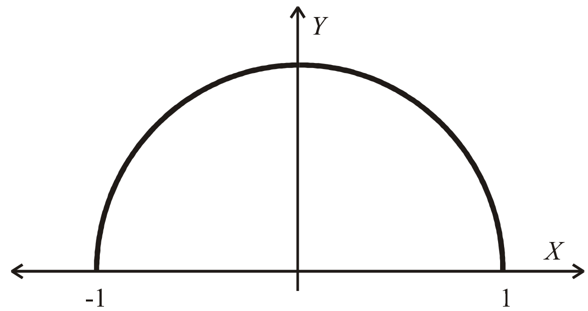
Homework Handout IV

Note: In the following, assume x and y are real variables while z is a complex varible with $z = x + iy$

A. Let $\mathcal{C}$ be the semicircle from $z = 1$ to $z = -1$ sketched to the right.



1. Using the Cauchy integral theorem, convert

$$\int_{\mathcal{C}} \frac{1}{4 + z^2}\, dz$$

to an integral over the interval $(-1, 1)$, and evaluate that integral.

2. Why can't the Cauchy integral theorem be used in a similar fashion to evaluate

$$\int_{\mathcal{C}} \frac{1}{1 + 4\, z^2}\, dz \qquad ?$$

B. Exercise 11.3.6 on page 485 of AW&H.

C. Using the Cauchy integral formulas (and other derived identities as needed), evaluate each of the following integrals assuming $\mathcal{C}$ is the circle about 0 of radius 2 (oriented counter-clockwise)

1. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}}{z - 1}\, dz$

2. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}}{z + i}\, dz$

3. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}}{3z + i}\, dz$

4. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}}{z + 3i}\, dz$

5. $\displaystyle\int_{\mathcal{C}} \frac{e^{4z}}{z^2 + 1}\, dz$ $\left(\text{Hint: Partial fractions, find } A \text{ and } B \text{ w/ } \dfrac{1}{z^2+1} = \dfrac{A}{z+i} + \dfrac{B}{z-i}\right)$

6. $\displaystyle\int_{\mathcal{C}} \frac{1}{z^2 + z}\, dz$ (Partial factions again.)

7. $\displaystyle\int_{\mathcal{C}} \frac{4z - 1}{z^2 + 5z + 4}\, dz$

8. $\displaystyle\int_{\mathcal{C}} \frac{8z^3 + 4}{(z - 1)^2}\, dz$

9. $\displaystyle\int_{\mathcal{C}} \frac{8z^3 + 4}{(z - 1)^3}\, dz$

10. $\displaystyle\int_{\mathcal{C}} \frac{8z^3 + 4}{(z - 1)^4}\, dz$

D. Find all the Laurent series for each of the following functions about the point indicated. Be sure to describe the region over which each series is valid.

1. $\dfrac{1}{z-3}$ $\quad z_0 = 0$

2. $\dfrac{1}{z-3}$ $\quad z_0 = 3$

3. $\dfrac{1}{z^2-3z}$ $\quad z_0 = 0$

4. $\dfrac{1}{z}$ $\quad z_0 = i$

5. $\dfrac{1}{z^2-z}$ $\quad z_0 = 1$

6. $\dfrac{1}{(z-1)^2}$ $\quad z_0 = 0$ $\quad \left(\text{Hint: You can use } \dfrac{d}{dz}\left[\dfrac{1}{z-1}\right] = \dfrac{-1}{(z-1)^2}\right)$

7. $\dfrac{1}{z^2-4z+3}$ $\quad z_0 = 0$ $\quad$ (Hint: Partial fractions!)

8. $\dfrac{1}{z^2-4z+3}$ $\quad z_0 = 1$

9. $\dfrac{e^z}{z-i}$ $\quad z_0 = i$ $\quad$ (Hint: What is the Taylor series for e^z about $z_0 = i$?)