Homework Handout II

A. Consider the differential equation

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + x\,y = 0 \quad .$$

Set $x_0 = 0$ and, using the basic Frobenius method described in the notes, do the following:

1. Find and solve the indicial equation (using $x_0 = 0$).

2. Find the recursion formula corresponding to r_1.

3. Find the series solution corresponding to r_1.

4. Find the recursion formula corresponding to r_2.

5. Find the series solution corresponding to r_2.

6. Write out the general series solution to the above differential equation, and rewrite this in terms of well-known functions (such as sines and cosines).

B. Consider the differential equation

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} - x\,y = 0 \quad .$$

Set $x_0 = 0$ and, using the basic Frobenius method described in the notes, do the following:

1. Find and solve the indicial equation (using $x_0 = 0$).

2. Find the recursion formula corresponding to r_1.

3. Find the series solution corresponding to r_1.

4. Why can't you find a second solution by the basic Frobenius method?

5. What can you say about a second solution to this differential equation when $x \approx 0$?

C. Consider the differential equation

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + \left[1 + x^2\right]y = 0 \quad .$$

Set $x_0 = 0$ and, using the basic Frobenius method described in the notes, do the following:

1. Find and solve the indicial equation (using $x_0 = 0$).

2. Find the recursion formula corresponding to r_1.

3. Find the series solution corresponding to r_1.

4. Why can't you find a second solution by the basic Frobenius method?

5. What can you say about a second solution to this differential equation when $x \approx 0$?

D. In the exercise set at the end of chapter 13 of the lecture notes, do:

1. 3a ***2.*** 3b ***3.*** 3c ***4.*** 3f ***5.*** And others in set 3

E. In the exercise set at the end of chapter 13 of the lecture notes, do all of 4 (on the Hermite equation).

F. In the exercise set at the end of chapter 13 of the lecture notes, do 5a,b,c&d (on the Legendre equation).

G. In the exercise set at the end of chapter 13 of the lecture notes, do 5a,b,c&d (on the Chebyshev equation).

H. Let $\nu \geq 0$. Bessel's equation of order ν is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)y = 0 \qquad .$$

1. Find and solve the indicial equation (using $x_0 = 0$).

2. Find a series solution for each $\nu \geq 0$ about $x_0 = 0$. Also find a second solution when ν is not an integer. What goes "bad" when ν is an integer (try solving with $\nu = 1$, as a start)?

3. What can you say about the second solution when $x \approx 0$ and

a. $\nu = 0$ ***b.*** $\nu = N$ where N is any positive integer.

I. Solve each of the following two ways: once using the methods discussed in the review of differential equations (Appendix A)[1], and once using the basic method of Frobenius:

$$\frac{d^2y}{dx^2} - 4y = 0 \qquad \text{and} \qquad x^2\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 10y = 0$$

There is a moral to this exercise. What's the moral?

J 1. Using the method of Frobenius, find a series solution (about $x_0 = 0$) to

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \left(x^2 - 8\right)y = 0 \qquad .$$

2. Try to find a second solution "via Frobenius". Why does the basic method of Frobenius fail to yield a second solution?

3. What can you say about the second solution when $x \approx 0$?

K. OPTIONAL: § 7.5 of AW&H, page 356 and following: 1, 7, 9

[1]Actually, these are from the exercises of that appendix. So you should have already solved them once.

Some Answers for Homework Handout II

Warning: Theses were done quickly and at odd times. Errors are likely!

Answers to problem ***A*** :

1. $r^2 + r = 0$ with solutions $r_1 = 0$ and $r_2 = -1$.

2. $a_k = -\dfrac{a_{k-2}}{(k+1)k}$.

3. $y_1(x) = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{5!}x^6 + \cdots$.

4. $a_k = -\dfrac{a_{k-2}}{k(k-1)}$.

5. $y_2(x) = x^{-1}\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right]$.

6. $y(x) = c_1 \dfrac{\sin(x)}{x} + c_2 \dfrac{\cos(x)}{x}$.

Answers to problem ***B*** :

1. $r^2 = 0$ with solutions $r_1 = r_2 = 0$

2. $a_k = \dfrac{a_{k-2}}{k^2}$.

3. $y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{2^{2n}(n!)^2} x^{2n}$.

4. $r_1 = r_2$.

5. $y_2(x) \approx a_0 \ln|x|$ and $y_2(x) \to \infty$ as $x \to 0$.

Answers to problem ***C*** :

1. $(r-1)^2 = 0$ with solutions $r_1 = r_2 = 1$

2. $a_k = -\dfrac{a_{k-2}}{k^2}$.

3. $y_1(x) = a_0 x^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$.

4. $r_1 = r_2$.

5. $y_2(x) \approx a_0 x \ln|x|$ and $y_2(x) \to 0$ as $x \to 0$.

Answers to problems ***D*** through ***G*** are given after the end of the referenced exercise set.