Homework Handout I

A. Verify the following:

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

Hint: Rewrite each term using partial fractions, then write out the partial sums, see what cancels, and then take the appropriate limits.

B. Which of the following geometric series converge, and what are their sums?

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \quad , \quad \sum_{k=0}^{\infty} \left(\frac{i}{2}\right)^k \quad , \quad \sum_{k=2}^{\infty} 5\left(\frac{4}{3}\right)^k \quad , \quad \sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^k \quad ,$$

$$\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k \quad , \quad \sum_{k=1}^{\infty} \left(\frac{1+2i}{3}\right)^k \quad , \quad \sum_{k=1}^{\infty} \left(\frac{1-4i}{3}\right)^k$$

C. The geometric series can be defined using terms other than simply numbers, including square matrices. That's what we'll consider here:

Let $\mathbf{R}$ be any square matrix, say, a $M \times M$ matrix. Since it is $M \times M$, so are its positive integral powers,

$$\mathbf{R}^1 = \mathbf{R} \quad , \quad \mathbf{R}^2 = \mathbf{R}\,\mathbf{R} \quad , \quad \mathbf{R}^3 = \mathbf{R}\,\mathbf{R}\,\mathbf{R} \quad , \quad \ldots \quad .$$

For convenience, we'll define $\mathbf{R}^0 = \mathbf{I}$ where $\mathbf{I}$ is the $M \times M$ identity matrix.

The corresponding basic (matrix) geometric series is $\sum_{k=0}^{\infty} \mathbf{R}^k$, and its N^{th} partial sum is $\mathbf{S}_N = \sum_{k=0}^{N} \mathbf{R}^k$. Keeping in mind that these are *matrices*, not numbers:

1. Show that $(\mathbf{I} - \mathbf{R})\mathbf{S}_N = \mathbf{I} - \mathbf{R}^{N+1}$ and that $\mathbf{S}_N(\mathbf{I} - \mathbf{R}) = \mathbf{I} - \mathbf{R}^{N+1}$.

2a. Why can we not then write $\mathbf{S}_N = \dfrac{\mathbf{I} - \mathbf{R}^{N+1}}{\mathbf{I} - \mathbf{R}}$?

b. What formula can we write for $\mathbf{S}_N$ (using the results from part ***a***) assuming $\mathbf{I} - \mathbf{R}$ is invertible?

3. Assume $\mathbf{R}^{N+1} \to \mathbf{0}$ as $N \to \infty$ (where $\mathbf{0}$ is the $M \times M$ zero matrix). Using the results from part ***1,*** show that $\mathbf{I} - \mathbf{R}$ is invertible and that

$$(\mathbf{I} - \mathbf{R})^{-1} = \sum_{k=0}^{\infty} \mathbf{R}^k \quad .$$

$\Bigg($Recall that the norm of $\mathbf{R}$ is $\|\mathbf{R}\| = \sqrt{\langle \mathbf{R} \,|\, \mathbf{R} \rangle} = \sqrt{\mathbf{R} \cdot \mathbf{R}} = \sqrt{\sum_{j=1}^{M} \sum_{k=1}^{M} |R_{jk}|^2}$ where the R_{jk}s are the entries in $\mathbf{R}$. It's not hard to show that

$$\mathbf{R}^{N+1} \to \mathbf{0} \quad \text{as} \quad N \to \infty \qquad \Longleftrightarrow \qquad \|\mathbf{R}\| < 1$$

— provided you know the Schwarz inequality, $\sqrt{\langle \mathbf{R} \,|\, \mathbf{R} \rangle} = |\mathbf{A} \cdot \mathbf{B}| \leq \|\mathbf{A}\| \, \|\mathbf{B}\|$.$\Bigg)$

D. Read/skim pages 1 through 7 in AW&H (or the equivalent sections of your old calculus text — the main thing is to refresh your memory of the elementary tests for convergence, including the integral test) and skim pages 8 – 10. Then read/skim the rest of section 1.1 of AW&H.

E. Using the integral test:

1. Show that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

2. Determine the values of p such that $\sum_{k=1}^{\infty} \frac{1}{k^p}$

i. converges ***ii.*** diverges

F. AW&H, pages 10 &11: 5, 6, (7)

G. Does $\sum_{k=0}^{\infty} x^k$ converge uniformly on $\left(0, \frac{1}{2}\right)$? on $\left(-\frac{3}{4}, \frac{3}{4}\right)$? on $(-1, 0)$?

H. Which of the following converges uniformly on $(-\infty, \infty)$?

$$\sum_{k=1}^{\infty} 2^{-k} \cos(kx) \quad , \quad \sum_{k=1}^{\infty} \frac{1}{k} \cos(kx) \quad , \quad \sum_{k=1}^{\infty} \frac{1}{k!} \cos(kx) \quad , \quad \sum_{k=1}^{\infty} \frac{1}{1+k^2} \cos(kx)$$

I. AW&H, page 24[1]: 1, 4 (Be sure to read the footnote below.)

Then determine the intervals on which $\sum_{k=0}^{\infty} \frac{1}{x^k}$ is uniformly convergent.

Then do #3 on page 24.

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[1]Problem 1.2.1 asks for "the range of uniform convergence". This is poorly stated and misleading. What should be asked for are "the intervals of uniform convergence", and the given answer "$0 < s \leq x < \infty$" should be interpreted as "all intervals of the form $[s, \infty)$ with $0 < s < \infty$".

J. We've seen that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{for} \quad -1 < x < 1 \quad ,$$

and that this series converges uniformly on every $[a, b]$ where $-1 < a < b < 1$. Using this, differentiation, and integration, find formulas for the sums of the following series:

$$\sum_{k=0}^{\infty} kx^{k-1} \quad , \quad \sum_{k=1}^{\infty} kx^k \quad , \quad \sum_{k=1}^{\infty} \frac{1}{k} x^{k+1} \quad , \quad \sum_{k=1}^{\infty} \frac{1}{k} x^k \qquad .$$

Also, find power series representations for the following functions

$$\frac{1}{1+x} \quad , \quad \frac{1}{(1+x)^2} \quad , \quad \frac{1}{1+x^2} \quad , \quad \ln|1-x| \quad , \quad \arctan(x) \qquad .$$

Hints:

$$\ln|1-x| = -\int_0^x \frac{1}{1-t}\,dt \qquad \text{and} \qquad \arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt$$

K. For each of the following, determine the radius of convergence, the interval of convergence, and an interval over which the series converges uniformly (if one exists):

$$\sum_{k=0}^{\infty} kx^k \quad , \quad \sum_{k=1}^{\infty} \frac{1}{k} x^k \quad , \quad \sum_{k=1}^{\infty} \frac{1}{k^2} x^k \quad , \quad \sum_{k=0}^{\infty} \frac{1}{k!} x^k \quad ,$$

$$\sum_{k=0}^{\infty} k!\, x^k \quad , \quad \sum_{k=0}^{\infty} 2^k\, x^k \quad , \quad \sum_{k=0}^{\infty} \frac{(-3)^k}{4^k} x^k$$

L. Read/skim the subsection AW&H on Taylor series (page 25 and following) and the section on the basic binomial series (pages 33 and 34, don't bother with page 35 except for the definition of !! in line (1.75) and (1.74)). Then do the following from AW&H:

1. problem #8 on page 32

2. problems (7), 10, (11), 13 from the set starting on page 36

3. problem #3 on page 37