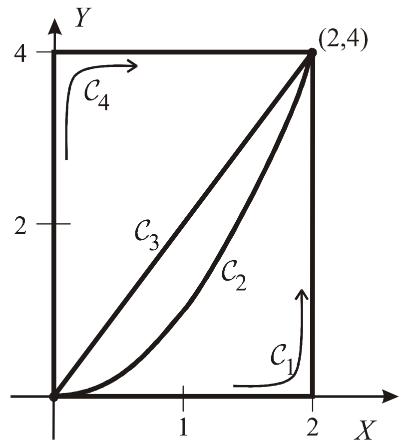
Homework Handout IX

A. For the following problems, we will use Cartesian coordinates to describe points in the plane, and will let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ be the four curves from $(0,0)$ to $(2,4)$ indicated in the figure. Note that:

$a)$ $\mathcal{C}_1$ and $\mathcal{C}_4$ consist of portions of the $X-$ and Y–axes,

$b)$ $\mathcal{C}_3$ is a straight line, and

$c)$ each point (x,y) in $\mathcal{C}_2$ satisfies $y = x^2$.



In addition, assume ϕ and $\mathbf{F}$ are, respectively, a scalar field and a vector field with coordinate formulas

$$\phi(x,y) = x^2 y \qquad \text{and} \qquad \mathbf{F}(x,y) = 2xy\,\mathbf{i} + \left(x - y^2\right)\mathbf{j} \quad .$$

1. Find parametrizations for each of these four curves (the parametrizations for $\mathcal{C}_1$ and $\mathcal{C}_4$ can each be in "two pieces").

2. Determine " $d\boldsymbol{r}$ " for each of the parametrizations just found, and then compute

$$\int_{\mathcal{C}_k} \mathbf{F} \cdot d\boldsymbol{r} \qquad \text{for } k = 1,\ 2,\ 3,\ \text{and } 4 \quad .$$

3. Determine " ds " for each of the parametrizations just found, and then *set up* (computing if practical)

$$\int_{\mathcal{C}_k} \phi \, ds \qquad \text{for } k = 1,\ 2,\ 3,\ \text{and } 4 \quad .$$

4. Using your results from the above, compute

$$\int_{\Gamma} \mathbf{F} \cdot d\boldsymbol{r}$$

where

a. $\Gamma = \mathcal{C}_2 - \mathcal{C}_3$ (I.e, take $\mathcal{C}_2$ from $(0,0)$ to $(2,4)$, then go backwards on $\mathcal{C}_3$ to $(0,0)$.)

b. $\Gamma = \mathcal{C}_1 - \mathcal{C}_4$

B. For these problems, we are using cylindrical coordinates $\{(\rho, \phi, z)\}$ in three-dimensional Euclidean space.

1. Suppose we have a curve $\mathcal{C}$ in the plane with a cylindrical coordinate parametrization,

$$\boldsymbol{r}(t) \sim (\rho(t), \phi(t), z(t)) \quad .$$

Determine the formulas for $d\boldsymbol{r}$ and ds in terms of $\frac{d\rho}{dt}$, $\frac{d\phi}{dt}$, $\frac{dz}{dt}$, $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{k}$.

2. Assume that, in fact, our curve $\mathcal{C}$ is given by

$$\boldsymbol{r}(t) \sim (\rho(t), \phi(t), z(t)) = (3, 2\pi t, t) \qquad \text{for} \quad 0 < t < 5 \quad .$$

a. Sketch the curve.

b. Compute $\int_{\mathcal{C}} \phi \, ds$ where $\phi(\boldsymbol{x}) =$ the square of the distance from the origin to $\boldsymbol{x}$.

c. Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\boldsymbol{r}$ where $\mathbf{F}$ is the vector field with cylindrical coordinate formula

$$\mathbf{F}(r, \phi, z) = \cos(\phi)\mathbf{e}_\rho + z\,\mathbf{e}_\phi + \rho\,\mathbf{k} \qquad .$$

3. Compute $\int_{\mathcal{C}} \mathbf{e}_\rho \times d\boldsymbol{r}$ and $\int_{\mathcal{C}} \mathbf{e}_\phi \times d\boldsymbol{r}$ when $\mathcal{C}$ is the semicircle paramtrized by

$$\boldsymbol{r}(t) \sim (\rho, \phi, z) = (3, \phi, 5) \qquad \text{for} \quad 0 < \phi < \pi \qquad .$$

C. Determine the formulas for $d\boldsymbol{r}$ and ds in terms of the standard spherical coordinate system $\{(r, \theta, \phi)\}$.

D. Determine the formulas for $d\boldsymbol{r}$ and ds in terms of the parabolic coordinate system $\{(u, v)\}$ from problem ***L*** in *Homework Handout VII.*

E. Let $\mathbf{F}$ be the conservative vector field given by $\mathbf{F} = -\boldsymbol{\nabla}\psi$ where ψ is the scalar field whose Cartesian coordinate formula is given by

$$\psi(x, y, z) = \frac{9}{x^2 + 2y^2} e^{-(x^2+z^2)} \qquad .$$

1. Find the Cartesian formula for $\mathbf{F}$.

2. Find $\int_{\mathcal{C}} \mathbf{F} \cdot d\boldsymbol{r}$ assuming $\mathcal{C}$ is the helix from problem ***B2***, above. Compute this integral the easy way — using the fact that $\mathbf{F}$ is conservative, NOT using the Cartesian formula for $\mathbf{F}$.

3. Find $\int_{\mathcal{C}} \mathbf{F} \cdot d\boldsymbol{r}$ when $\mathcal{C}$ is any closed curve (i.e., any “loop”.)

F. Test whether each (2-dimensional) field given below is conservative and then, if it is conservative, find a potential ψ for it:

1. $\mathbf{F}(x,y) \;=\; 2xy\,\mathbf{i} \;+\; (x^2+6y^2)\mathbf{j}$

2. $\mathbf{G}(x,y) \;=\; 8x^2\,\mathbf{i} \;-\; xy^2\,\mathbf{j}$

G. Exercise 3.9.3 on page 180 of AW&H.

H. In a previous homework set, you showed $\boldsymbol{\nabla}\times\mathbf{r}=\mathbf{0}$. Keeping this and what we know about conservative vector fields in mind, do exercise 3.7.4 on page 164 of AW&H.

I. Exercise 3.9.1 on page 180 of AW&H (Be intelligent. Observe that the vector field is radially symmetric, and use results from problems in *Homework Handout VIII* on radially symmetric problems.)

J. Let us assume that the Universe is an N-dimensional Euclidean space with the Earth at the center (i.e., that is where we place the origin of any Cartesian coordinate system). The earth's gravational force $\mathbf{G}$ is "obviously" radially symmetric and conservative. For various reasons, it is also known to be solenoidal (i.e., divergence free). Using these facts, along with facts derived previously about such forces, derive the most general formula possible for $\mathbf{G}$ assuming

1. $N=3$.

2. $N=2$.

(Note: You will have to solve a second-order differential equation for the potential ϕ . You should be able to solve it by first letting $y=\phi'$ and solving the corresponding first-order differential equation for y . Then find ϕ . Just in case you need it, there is a brief review of differential equations as an appendix in our online notes.)

K. (Optional: Read about vector potentials on pages 172 to 180 of AW&H, and then do exercises 3.9.4, (3.9.5), (3.9.6) starting on page 180).

L. Exercise 3.10.16 on page 199 of AW&H.