Homework Handout VIII

A. In a previous problem, you sketched the curves

$$\boldsymbol{x}(t) \sim (\rho(t), \phi(t)) = (t, 2\pi t) \quad \text{with} \quad 0 \leq t$$

$$\boldsymbol{x}(t) \sim (\rho(t), \phi(t)) = \left(t^2, \frac{\pi}{3}\right) \quad \text{with} \quad -\infty \leq t \leq \infty$$

$$\boldsymbol{x}(t) \sim (\rho(t), \phi(t)) = (4, t) \quad \text{with} \quad 0 \leq t \leq 4\pi$$

Now:

1. Find $\frac{d\boldsymbol{x}}{dt}$ in terms of the polar coordinates and the associated tangent vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$. Also, on your sketch of each curve, sketch $\frac{d\boldsymbol{x}}{dt}$ at different points on the curve.

2. Find $\frac{ds}{dt}$, the rate at which the arclength along the curve varies as t varies $\left(\text{use your formula for } \frac{d\boldsymbol{x}}{dt}\right)$. Then write out the integral that gives the length of the curve traced out at t goes from 0 to 1 (if the integral is simple enough, evaluate it!).

3. Find $\frac{d^2\boldsymbol{x}}{dt^2}$ in terms of the polar coordinates and the associated tangent vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$. (Doing this requires that you've found the acceleration formula for polar coordinates, which is exercise 9.3 in the online notes.)

B. Let $\{(u, v)\}$ be the first coordinate system in the *Three Coordinate Systems for the Plane* handout, and let $\boldsymbol{x}(t) \sim (u(t), v(t))$ be the position of George the Gerbil at time t.

1. Find the general formulas for George's velocity, speed and acceleration,

$$\frac{d\boldsymbol{x}}{dt} \quad , \quad \frac{ds}{dt} \quad \text{and} \quad \frac{d^2\boldsymbol{x}}{dt^2} \quad .$$

2. Assume $(u(t), v(t)) = (2t, 3t)$ and do the following:

a. Sketch the resulting curve on the plane.

b. Compute George's velocity, speed and acceleration at each time t.

c. Set up and compute (if practical) the integral giving the distance traveled by George between $t = 0$ and $t = 3$.

3. Assume $(u(t), v(t)) = (t^2, t)$ and do the following:

a. Sketch the resulting curve on the plane.

b. Compute George's velocity, speed and acceleration at each time t.

c. Set up and compute (if practical) the integral giving the distance traveled by George between $t = 0$ and $t = 3$.

C. Let $\{(u, v)\}$ be the parabolic coordinate system for the plane from exercise ***L*** of Homework *Handout VII*. Using your results from that exercise, find the Christoffel symbols for this coordinate system and the partial derivatives

$$\frac{\partial\, \overrightarrow{\varepsilon_u}}{\partial u} \quad , \quad \frac{\partial\, \overrightarrow{\varepsilon_u}}{\partial v} \quad , \quad \frac{\partial\, \overrightarrow{\varepsilon_v}}{\partial u} \quad \text{and} \quad \frac{\partial\, \overrightarrow{\varepsilon_v}}{\partial v}$$

in terms of this coordinate system and $\{\overrightarrow{\varepsilon_u}, \overrightarrow{\varepsilon_v}\}$.

D. Redo exercise ***B***, above, but using the parabolic coordinate system $\{(u, v)\}$ for the plane from exercise ***L*** of Homework *Handout VII*. Make use of the results from the last problem, above.

E. Let $\{(u, v)\}$ be the coordinate system for the plane from exercise ***N*** of Homework *Handout VII* with $a = 1$. Using your results from that exercise, find the Christoffel symbols for this coordinate system and the partial derivatives

$$\frac{\partial\, \overrightarrow{\varepsilon_u}}{\partial u} \quad , \quad \frac{\partial\, \overrightarrow{\varepsilon_u}}{\partial v} \quad , \quad \frac{\partial\, \overrightarrow{\varepsilon_v}}{\partial u} \quad \text{and} \quad \frac{\partial\, \overrightarrow{\varepsilon_v}}{\partial v}$$

in terms of this coordinate system and $\{\overrightarrow{\varepsilon_u}, \overrightarrow{\varepsilon_v}\}$.

F. Redo exercise ***B***, above, but using the coordinate system $\{(u, v)\}$ for the plane from exercise ***N*** of Homework *Handout VII*. Make use of the results from the last problem, above.

G. Let $\{(u, v, \theta)\}$ be the polar parabolic coordinate system for the plane from exercise ***M*** of Homework *Handout VII*, and let $\mathbf{x}(t) \sim (u(t), v(t), \theta(t))$ be the position of George the Gerbil at time t.

1. Find the general formulas for George's velocity and speed and acceleration,

$$\frac{d\mathbf{x}}{dt} \quad , \quad \frac{ds}{dt} \quad \text{and} \quad \frac{d^2\mathbf{x}}{dt^2} \quad .$$

2. Find the general formulas for George's velocity, speed and acceleration,

3. Assume $(u(t), v(t), \theta(t)) = (2\,,\, 3 + \sin(t)\,,\, t)$ and do the following:

a. Sketch the resulting curve on the plane as t goes from 0 to 2π.

b. Compute George's velocity and speed and acceleration at each time t.

c. Set up and compute (if practical) the integral giving the distance traveled by George between $t = 0$ and $t = 2\pi$.

d. Compute George's acceleration at each time t. Compute whatever Christoffel symbols are needed for this.

H 1. The polar coordinate formulas for two scalar fields Ψ_1 and Ψ_2 are given below. For each, find the corresponding Cartesian coordinate formulas:

a. $\psi_1(\rho, \phi) = \rho^2 \sin(\phi)$

b. $\psi_2(\rho, \phi) = \phi \sin(\rho)$

2. The Cartesian coordinate formulas for two scalar fields Ψ_3 and Ψ_4 are given below. For each, find the corresponding polar coordinate formulas:

a. $\psi_3(x, y) = x^2 + y$

b. $\psi_4(x, y) = \dfrac{1}{x^2 + y^2}$

3. Is there a moral to this exercise?

I. (In the plane) For the following, use the two Cartesian coordinate systems and corresponding orthonormal bases

$$\{ (x, y) \} \text{ with } \{ \mathbf{i}, \mathbf{j} \} \qquad \text{and} \qquad \{ (s, t) \} \text{ with } \{ \mathbf{e}_s, \mathbf{e}_t \}$$

where

$$\mathbf{e}_s = \frac{1}{\sqrt{5}}[2\mathbf{i} + \mathbf{j}] \quad \text{and} \quad \mathbf{e}_t = \frac{1}{\sqrt{5}}[\mathbf{i} - 2\mathbf{j}]$$

and with the origin $\boldsymbol{O}$ being the same point for each:

1 a. Find $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{e}_s$ and $\mathbf{e}_t$.

b. Sketch the two coordinate systems, showing their relation to each other.

2. Let Ψ and $\mathbf{V}$ be the scalar field and the vector field with $\{(x, y)\}$ formulas

$$\Psi(\boldsymbol{x}) = \psi(x, y) = x^2 - 2y$$

and

$$\mathbf{V}(\boldsymbol{x}) = \big(x^2 - 2y\big)\mathbf{i} + (y - x)\,\mathbf{j}$$

Find the formulas for Ψ and $\mathbf{V}$ in terms of the $\{(s, t)\}$ with $\{ \mathbf{e}_s, \mathbf{e}_t \}$ system. Use what we learned about "change of basis" matrices $\mathbf{M}_{\mathcal{AB}}$.

J. Assume we have two <u>*orthogonal*</u> coordinate systems (with associated scaling factors and bases) for some N-dimensional space $\mathcal{S}$,

$$\{(x^1,\ldots,x^N)\} \quad , \quad \{h_1,\ldots,h_n\} \quad , \quad \mathcal{B} = \{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$$

and

$$\{(x^{1\prime},\ldots,x^{N\prime})\} \quad , \quad \{h_1',\ldots,h_n'\} \quad , \quad \mathcal{B}' = \{\mathbf{e}_1',\ldots,\mathbf{e}_n'\} \quad .$$

Let $\mathbf{V}$ be a vector field on $\mathcal{S}$ with coordinate/component formulas

$$\mathbf{V}(\boldsymbol{x}) = \sum_{k=1}^{N} V^k(x^1,x^2,\ldots,x^N)\,\mathbf{e}_k \qquad \text{where} \quad \boldsymbol{x} \sim (x^1,x^2,\ldots,x^N)$$

and

$$\mathbf{V}(\boldsymbol{x}) = \sum_{k=1}^{N} V^{k\prime}(x^{1\prime},x^{2\prime},\ldots,x^{N\prime})\,\mathbf{e}_k' \qquad \text{where} \quad \boldsymbol{x} \sim (x^{1\prime},x^{2\prime},\ldots,x^{N\prime}) \quad .$$

1. Using what we know about "change of basis" matrices, and the results from problem ***K*** in *Homework Handout VII*, show that the components of this vector field are related by either

$$V^{i\prime} = \sum_{j=1}^{N} \frac{h_j}{h_i'}\frac{\partial x^j}{\partial x^{i\prime}} V^j \qquad \text{or} \qquad V^{i\prime} = \sum_{j=1}^{N} \frac{h_i'}{h_j}\frac{\partial x^{i\prime}}{\partial x^j} V^j \quad .$$

Convince yourself that these are equivalent formulas for $V^{i\prime}$ (that comes from the result obtained in problem ***K*** of *Homework Handout VII*). Hence, in practice, you can use whichever one is most convenient.

2. What is the corresponding matrix (matrices) $\mathbf{U}$ for computing

$$|\,\mathbf{V}\,\rangle_{\mathcal{B}'} = \mathbf{U}|\,\mathbf{V}\,\rangle_{\mathcal{B}} \quad ?$$

3. To what does the above simplify to if both coordinate systems are Cartesian? (Confirm this observation by redoing the previous exercise using the partial derivative formulas just derived.)

K 1. Using the above, find the general formulas for converting the Cartesian coordinate formula for a vector field on the plane to the corresponding polar coordinate formula, and for converting the polar coordinate formula for a vector field to the corresponding Cartesian coordinate formula.

2. Convert the following from Cartesian to polar, or from polar to Cartesian, as appropriate:

a. $\mathbf{V}(\boldsymbol{x}) = xy\,\mathbf{i} + \dfrac{y}{x}\,\mathbf{j}$

b. $\mathbf{V}(\boldsymbol{x}) = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

c. $\mathbf{V}(\boldsymbol{x}) = x^2\,\mathbf{i} - y^2\,\mathbf{j}$

d. $\mathbf{V}(\boldsymbol{x}) = \sin(\phi)\,\mathbf{e}_\rho + \cos(\phi)\,\mathbf{e}_\phi$

e. $\mathbf{V}(\boldsymbol{x}) = \rho^2\,\mathbf{e}_\rho + 4\,\mathbf{e}_\phi$

f. $\mathbf{V}(\boldsymbol{x}) = 0\,\mathbf{e}_\rho + \rho\,\mathbf{e}_\phi$

L. Find $\mathbf{grad}(\phi)$ where $\phi(\boldsymbol{x}) = 2x^2 + \sin(yz)$.

M. Find $\operatorname{div}(\mathbf{F})$ and $\mathbf{curl}(\mathbf{F})$ where $\mathbf{F}(\boldsymbol{x}) = \dfrac{x^2 y}{z}\,\mathbf{i} + \sin\bigl(2x^2 + 2y^2\bigr)\,\mathbf{j} + \cos\bigl(x^2 + z^2\bigr)\,\mathbf{k}$

N. Let $\mathbf{F}(x,y,z) = (a_1 x + b_1 y + c_1 z)\,\mathbf{i} + (a_2 x + b_2 y + c_2 z)\,\mathbf{j} + (a_3 x + b_3 y + c_3 z)\,\mathbf{k}$.
Find values for constants a_1, b_1, ... and c_3 (not all zero!) so that

1. $\mathbf{F}$ is solenoidal. ***2.*** $\mathbf{F}$ is irrotational.

O. (*Central Forces, Part 1*) For any Cartesian coordinate system $\{(x^1, x^2, \ldots, x^N)\}$ and position $\boldsymbol{x}$, let $\mathbf{r}(\boldsymbol{x})$ be the position vector corresponding to $\boldsymbol{x}$,

$$\mathbf{r} = \mathbf{r}(\boldsymbol{x}) = \overrightarrow{\boldsymbol{Ox}} = \sum_{k=1}^{N} x^k \mathbf{e}_k \quad ,$$

and let

$$r = r(\boldsymbol{x}) = \left\|\mathbf{r}(\boldsymbol{x})\right\| \qquad \text{and} \qquad \widehat{\mathbf{r}} = \widehat{\mathbf{r}}(\boldsymbol{x}) = \frac{\mathbf{r}(\boldsymbol{x})}{r(\boldsymbol{x})} \quad .$$

($\widehat{\mathbf{r}}$ is standard notation for the unit vector in the "radial direction", not to be confused with just plain $\mathbf{r}$. Alternatively, you can use $\mathbf{e}_r$ instead of $\widehat{\mathbf{r}}$.) Verify the following by computation using the Cartesian formulas:

1. $\mathbf{grad}(r) = \boldsymbol{\nabla} r(\boldsymbol{x}) = \widehat{\mathbf{r}}$

2. $\operatorname{div}(\mathbf{r}) = \boldsymbol{\nabla} \cdot \mathbf{r}(\boldsymbol{x}) = N$

3. $\mathbf{curl}(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{r}(\boldsymbol{x}) = \mathbf{0}$ (assuming $N = 3$)

P. (*Central Forces, Part 2*) Assume our space is N-dimensional and Euclidean; let $\mathbf{r}$, r and $\widehat{\mathbf{r}}$ be as in problem ***O***, and let f and g be any two suitably differentiable real-valued functions on $(0,\infty)$.

Verify all the following using the results from problem ***O***, the relations between $\mathbf{r}$, r and $\widehat{\mathbf{r}}$, and the appropriate product and chain rule from the subsection at the end of section 9.3 of the online notes. At no point should you use the Cartesian formulas for anything.

1. $\mathbf{grad}(g(r)) = g'(r)\,\widehat{\mathbf{r}}$

2. $\operatorname{div}(\widehat{\mathbf{r}}) = \boldsymbol{\nabla} \cdot \widehat{\mathbf{r}}(\boldsymbol{x}) = (N-1)\,r^{-1}$

3. $\operatorname{div}(g(r)\,\widehat{\mathbf{r}}) = g'(r) + (N-1)\,\dfrac{g(r)}{r}$

4. $\operatorname{div}(f(r)\,\mathbf{r}) = rf'(r) + Nf(r)$

5. $\mathbf{curl}(g(r)\,\widehat{\mathbf{r}}) = \mathbf{0}$

Q. Starting on page 152 of AW&H, do exercises 3.5.1, 3.5.2, and 3.5.3 (and figure out what the last problem is, geometrically).

R. Let $\boldsymbol{x} \sim (x, y)$ be a point on the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \quad ,$$

and find the formula for the unit vector $\mathbf{n}$ at $\boldsymbol{x}$ which is perpendicular to the ellipse and points away from the inside of the ellipse.

S. Starting on page 152 of AW&H, do exercises 3.5.8a or b ($\mathbf{A}$ and $\mathbf{B}$ are vector-valued functions of t) and 3.5.9 (but be cautious of naively treating this as a triple vector product)..

T. Starting on page 157 of AW&H, do exercises 3.6.1, look at 3.6.5&6, and do problems 3.6.7 and 3.6.8.

U. Using the standard polar coordinate formulas, compute

1. the gradient of each of the following scalar fields:

g. $\psi_1(\rho, \phi) = \rho^2 \sin(\phi)$ ***h.*** $\psi_2(\rho, \phi) = \phi \sin(\rho)$

2. the divergence of each of the following vector fields:

a. $\mathbf{F}(\boldsymbol{x}) = \sin(\phi)\,\mathbf{e}_\rho + \cos(\phi)\,\mathbf{e}_\phi$ ***b.*** $\mathbf{F}(\boldsymbol{x}) = \rho^2\,\mathbf{e}_r + 4\,\mathbf{e}_\phi$

c. $\mathbf{F}(\boldsymbol{x}) = \mathbf{e}_\rho$ ***d.*** $\mathbf{F}(\boldsymbol{x}) = \mathbf{e}_\phi$

3. the Laplacian of each of the following scalar fields:

a. $\psi_1(\rho, \phi) = \rho^2 \sin(\phi)$ ***b.*** $\psi_2(\rho, \phi) = \phi \sin(\rho)$

c. $\psi_3(\rho, \phi) = \rho^p$ where p is any real constant.

V. Using the cylindrical coordinate formulas, compute

1. the divergence of each of the following vector fields:

a. $\mathbf{F}(\boldsymbol{x}) = \sin(\phi)\,\mathbf{e}_\rho + \cos(\phi)\,\mathbf{e}_\phi + \mathbf{e}_z$ ***b.*** $\mathbf{F}(\boldsymbol{x}) = z^2\,\mathbf{e}_\rho$

c. $\mathbf{F}(\boldsymbol{x}) = \mathbf{e}_\rho$ ***d.*** $\mathbf{F}(\boldsymbol{x}) = \mathbf{e}_\phi$

2. the curl of each of the following vector fields:

a. $\mathbf{F}(\boldsymbol{x}) = \sin(\phi)\,\mathbf{e}_\rho + \cos(\phi)\,\mathbf{e}_\phi + \mathbf{e}_z$ ***b.*** $\mathbf{F}(\boldsymbol{x}) = z^2\,\mathbf{e}_\rho$

c. $\mathbf{F}(\boldsymbol{x}) = \mathbf{e}_\rho$ ***d.*** $\mathbf{F}(\boldsymbol{x}) = \mathbf{e}_\phi$

W. Find the formulas for the gradient and divergence in

1. the parabolic coordinate system from problem ***L*** of *Homework Handout VII*.

2. the elliptic coordinate system from problem ***N*** of *Homework Handout VII*.

X. Find the formulas for the gradient, divergence and curl in

1. the polar parabolic coordinate system from problem ***M*** of *Homework Handout VII*.

2. the oblate spheroidal coordinate system from problem ***O*** of *Homework Handout VII*.

Y. (*Central Forces, Part 3*) Assume our space is N-dimensional and Euclidean; let $r = r(\boldsymbol{x})$ be as in problems ***O*** and ***P***, and let g be any suitably differentiable real-valued function on $(0, \infty)$.

1. Compute $\nabla^2 g(r)$

2. Verify that $\psi(r) = \ln r$ is a solution to Laplace's equation in two-dimensional Euclidean space.

3. Compute $\nabla^2(r^\gamma)$ in N-dimensional Euclidean space (where γ is any real constant). Then, for $N = 3, 4, 5, \dots$; find a corresponding nonzero solution to Laplace's equation.

Z. Skim §3.6 of AW&H (Example 3.6.2 is important to anyone studying E&M or optics, but we won't be refering to it in this course). Starting on page 157, do 3.6.9, 3.6.10, 3.6.11 and 3.6.14

AA. Find the formulas for the Laplacian in

1. the parabolic coordinate system from problem ***L*** of *Homework Handout VII*.

2. the elliptic coordinate system from problem ***N*** of *Homework Handout VII*.

3. the polar parabolic coordinate system from problem ***M*** of *Homework Handout VII*.

4. the oblate spheroidal coordinate system from problem ***O*** of *Homework Handout VII*.