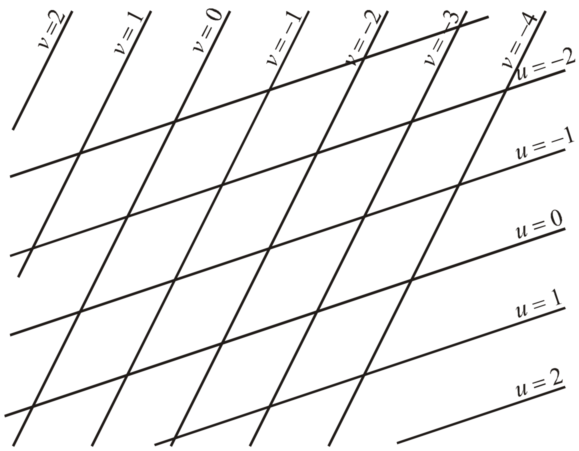
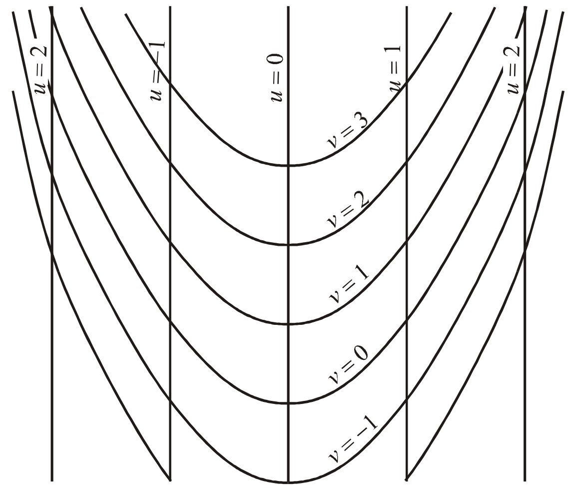
Homework Handout VII

A. Two $\{(u,v)\}$ coordinate systems for the plane are sketched below. On each, plot the points $(0,0)$, $(1,2)$ and $(-1,-2)$. Also, sketch the curve (well as you can) $u = v$.

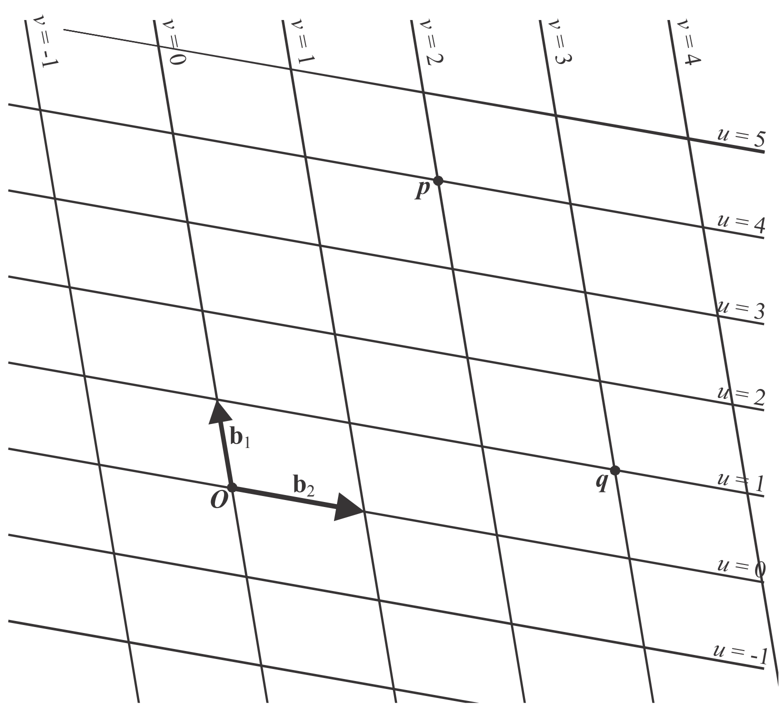




B. Let $\{\,(u,v)\,\}$ be a basis-based coordinate system for the plane based on a basis $\{\,\mathbf{b}_1\,,\,\mathbf{b}_2\,\}$ as indicated below. Assume the smallest angle between $\mathbf{b}_1$ and $\mathbf{b}_2$ is $\dfrac{2\pi}{3}$ and that

$$\|\mathbf{b}_1\| = 2 \qquad \text{and} \qquad \|\mathbf{b}_2\| = 3 \quad .$$

1. What are the (u,v) coordinates of the points $\boldsymbol{p}$ and $\boldsymbol{q}$?
2. Plot the point $\boldsymbol{x} \sim (3,-1)$.
3. Find dist($\boldsymbol{p}$,$\boldsymbol{q}$), the distance between points $\boldsymbol{p}$ and $\boldsymbol{q}$.
4. Sketch the straight line
$$u = 2v - 1 \quad .$$
5. Sketch the curve $u = v^2$.



C. Sketch the curve traced out by each $\boldsymbol{x}(t)$ given below.
(Assume, as appropriate, that $\{(x,y)\}$ and $\{(\rho,\phi)\}$ are, respectively, the standard Cartesian and polar coordinate systems for two-dimensional Euclidean space, and that $\{(x,y,z)\}$ and $\{(r,\theta,\phi)\}$ are, respectively, the standard Cartesian and spherical coordinate systems for three-dimensional Euclidean space. See the figures on pages 188 and 182 of AWH.)

1. $\mathbf{x}(t) \sim (x(t), y(t)) = (t, t^2)$ with $-1 \leq t \leq 2$
2. $\mathbf{x}(t) \sim (x(t), y(t), z(t)) = (t, \sin(t), \cos(t))$ with $0 \leq t$
3. $\mathbf{x}(t) \sim (\rho(t), \phi(t)) = (t, 2\pi t)$ with $0 \leq t$
4. $\mathbf{x}(t) \sim (\rho(t), \phi(t)) = \left(t^2, \frac{\pi}{3}\right)$ with $-\infty \leq t \leq \infty$
5. $\mathbf{x}(t) \sim (\rho(t), \phi(t)) = (4, t)$ with $0 \leq t \leq 4\pi$
6. $\mathbf{x}(t) \sim (r(t), \theta(t), \phi(t)) = \left(4, \frac{\pi}{3}, t\right)$ with $0 \leq t \leq 2\pi$

D. Find parameterizations for the following curves. Assume Cartesian coordinates. If an orientation is indicated, be sure your parametrization is appropriate for that orientation.

1. The straight line from $\boldsymbol{a}$ to $\boldsymbol{b}$ where $\boldsymbol{a} \sim (0,0)$ and $\boldsymbol{b} \sim (2,3)$.
2. The straight line from $\boldsymbol{a}$ to $\boldsymbol{b}$ where $\boldsymbol{a} \sim (2,3)$ and $\boldsymbol{b} \sim (0,0)$.
3. The straight line from $\boldsymbol{a}$ to $\boldsymbol{b}$ where $\boldsymbol{a} \sim (2,3)$ and $\boldsymbol{b} \sim (6,10)$.
4. The straight line from $\boldsymbol{a}$ to $\boldsymbol{b}$ where $\boldsymbol{a} \sim (2,3,8)$ and $\boldsymbol{b} \sim (6,10,0)$.
5. The parabola given by $x = y^2$.
6. The circle of radius 3 about the origin.
7. The circle of radius 3 about the point $\boldsymbol{p} \sim (3,4)$.

E. For each of the following two curves with Cartesion parameterizations,

$$\mathbf{x}(t) \sim (x(t), y(t)) = (t, t^2) \quad \text{with} \quad -1 < t < 2$$

$$\mathbf{x}(t) \sim (x(t), y(t), z(t)) = (t, \sin(t), \cos(t)) \quad \text{with} \quad 0 \leq t \qquad ,$$

1. Sketch the curve (again).
2. Find $\frac{d\boldsymbol{x}}{dt}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and, if appropriate, $\mathbf{k}$. Also, on your sketch of each curve, sketch $\frac{d\boldsymbol{x}}{dt}$ at different points on the curve.
3. Find $\frac{ds}{dt}$, the rate at which the arclength along the curve varies as t varies $\left(\text{use your formula for } \frac{d\boldsymbol{x}}{dt}\right)$. Then write out the integral that gives the length of the parametrized curve (if the integral is simple enough, evaluate it!).

F. In a previous problem, you sketched the curves

$$\boldsymbol{x}(t) \sim (\rho(t), \phi(t)) = (t, 2\pi t) \quad \text{with} \quad 0 \leq t$$

$$\boldsymbol{x}(t) \sim (\rho(t), \phi(t)) = \left(t^2, \frac{\pi}{3}\right) \quad \text{with} \quad -\infty \leq t \leq \infty$$

$$\boldsymbol{x}(t) \sim (\rho(t), \phi(t)) = (4, t) \quad \text{with} \quad 0 \leq t \leq 4\pi$$

Go back to your sketches and add corresponding unit tangent vectors $\mathbf{T}$ at various points on each curve.

G. In this problem, we are using Cartesian coordinates for a three-dimensional Euclidean space, and assuming that we have an object whose position at time t is

$$\boldsymbol{x}(t) \sim (R\cos(\omega t), R\sin(\omega t), 0)$$

where R and ω are constants. Note that this describes circular motion with radius R and angular frequency ω about the origin $\boldsymbol{O}$. Also let $\mathbf{r}$ be the corresponding positions vector,

$$\mathbf{r}(t) = \overrightarrow{\boldsymbol{Ox}}(t) \quad .$$

Evaluate the following in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

1. Evaluate $\mathbf{v} = \frac{d\boldsymbol{x}}{dt}$ and $\mathbf{a} = \frac{d^2\boldsymbol{x}}{dt^2}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

2. Evaluate $\mathbf{r} \times \mathbf{v}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

3. Verify that $\mathbf{a} = -\omega^2\,\mathbf{r}$**.**

H. Three different coordinate systems for either the Euclidean plane or the upper half plane are described on the sheet *Three Coordinate Systems for the Plane*. For convenience, each is denoted as a "$\{(u, v)\}$ system" instead of a "$\{(x^{1\prime}, x^{2\prime})\}$ system". For each:

Pick a couple of values for (u, v) coordinates and then, for each choice:

1. Plot the point with those (u, v)-coordinates (for the last one, you'll first have to sketch the coordinate curves yourself).

2. Sketch $\overrightarrow{\boldsymbol{\varepsilon}_u}$ and $\overrightarrow{\boldsymbol{\varepsilon}_v}$ at that location (remember: $\overrightarrow{\boldsymbol{\varepsilon}_u}$ points in the direction of *increasing* u, not constant u). (Use the Cartesian system for the "true lengths.")

3. Sketch $\mathbf{e}_u$ and $\mathbf{e}_v$ at that location.

4. Try to determine (educated guess) whether the scaling factors h_u and h_v are greater than, less than, or equal to one at this point. (Use the Cartesian system for the "true lengths.")

I. Again, consider the three different coordinate systems for the Euclidean plane or the upper half plane described on the sheet *Three Coordinate Systems for the Plane*. For each:

1. Recall the points you plotted in a previous exercise, and determine the corresponding (x, y) coordinates.

2. Find the set of change of coordinate formulas relating the $\{(u, v)\}$ system to the standard Cartesian $\{(x, y)\}$ system.

3. Find all the following:

a. $\overrightarrow{\varepsilon_u}$ and $\overrightarrow{\varepsilon_v}$ (expressed in terms of the $\{(u, v)\}$ system, but with $\mathbf{i}$ and $\mathbf{j}$.)

b. the scaling factors h_u and h_v , along with the corresponding $\mathbf{e}_u$ and $\mathbf{e}_v$. (Express $\mathbf{e}_u$ and $\mathbf{e}_v$ in terms of the $\{(u, v)\}$ system, but with $\mathbf{i}$ and $\mathbf{j}$.)

c. the components of the metric g_{uu} , g_{vv} , g_{uv} and g_{uv} .

4. Decide whether the $\{(u, v)\}$ system is orthogonal.

J. Suppose we have <u>two</u> Cartesian coordinate systems with corresponding orthonormal basis for some Euclidean space,

$$\left\{ \left(x^1 , x^2 , \ldots , x^N\right) \right\} \quad \text{with} \quad \mathcal{B} = \{ \mathbf{e}_1 , \mathbf{e}_2 , \ldots , \mathbf{e}_N \} \ \text{(and origin } \mathcal{O} \text{)}$$

and

$$\left\{ \left(x^{1\prime} , x^{2\prime} , \ldots , x^{N\prime}\right) \right\} \quad \text{with} \quad \mathcal{B}' = \{ \mathbf{e}_1' , \mathbf{e}_2' , \ldots , \mathbf{e}_N' \} \ \text{(and origin } \mathcal{O} \text{)} \quad .$$

1. Verify the following equations:

a. $$\mathbf{e}_i' = \sum_{k=1}^{N} \frac{\partial x^k}{\partial x^{i\prime}} \mathbf{e}_k$$

b. $$\mathbf{e}_j = \sum_{k=1}^{N} \frac{\partial x^{k\prime}}{\partial x^j} \mathbf{e}_k'$$

c. $$\langle \mathbf{e}_i' \,|\, \mathbf{e}_j \rangle = \frac{\partial x^j}{\partial x^{i\prime}} = \frac{\partial x^{i\prime}}{\partial x^j}$$

d. $$\mathbf{e}_i' = \sum_{k=1}^{N} \frac{\partial x^{i\prime}}{\partial x^k} \mathbf{e}_k$$

e. $$\mathbf{e}_j = \sum_{k=1}^{N} \frac{\partial x^j}{\partial x^{k\prime}} \mathbf{e}_k'$$

2. Find (in terms of partial derivatives) the unitary matrix $\mathbf{U}$ such that $|\,\mathbf{v}\,\rangle_{\mathcal{B}'} = \mathbf{U}|\,\mathbf{v}\,\rangle_{\mathcal{B}}$.

K. Assume we have two <u>*orthogonal*</u> coordinate systems (with associated scaling factors and bases),

$$\left\{\left(x^1, \ldots, x^n\right)\right\} \quad , \quad \{h_1, \ldots, h_n\} \quad , \quad \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$$

and

$$\left\{\left(x^{1\prime}, \ldots, x^{n\prime}\right)\right\} \quad , \quad \{h_1', \ldots, h_n'\} \quad , \quad \{\mathbf{e}_1', \ldots, \mathbf{e}_n'\} \quad .$$

Using the "orthogonality", show that

$$\frac{h_j}{h_k'} \frac{\partial x^j}{\partial x^{k\prime}} = \mathbf{e}_k' \cdot \mathbf{e}_j = \frac{h_k'}{h_j} \frac{\partial x^{k\prime}}{\partial x^j} \quad .$$

L. A parabolic coordinate system $\{(u,v)\}$ on the "right-half" of the Euclidean plane is related to a Cartesian system $\{(x,z)\}$ by

$$x = uv \qquad \text{and} \qquad z = \frac{v^2 - u^2}{2} \qquad \text{for} \quad u \geq 0 \quad \text{and} \quad v \geq 0 \quad .$$

1. By solving for z in terms of, first, x and u, and then in terms of x and v, verify that the constant u coordinate curves are given by

$$z = \frac{1}{2}\left(\frac{x^2}{u^2} - u^2\right) \qquad \text{for} \quad u > 0$$

and that the constant v coordinate curves are given by

$$z = \frac{1}{2}\left(v^2 - \frac{x^2}{v^2}\right) \qquad \text{for} \quad v > 0 \quad .$$

Describe these curves geometrically (i.e., straight lines, circles, ellipses, ...) (Hint: It's parabolic system). What are the $u = 0$ and the $v = 0$ coordinate curves?

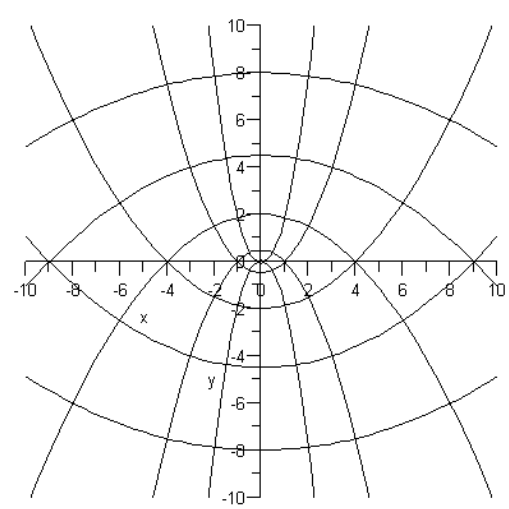
2. The five coordinate curves corresponding to

$$u = .5,\ 1,\ 2,\ 3 \text{ and } 4 \ ,$$

along with the five coordinate curves corresponding to

$$v = .5,\ 1,\ 2,\ 3 \text{ and } 4$$

have been sketched (using Maple) below. Identity each curve with the corresponding value of either u or v. Also, plot the point $(u,v) = (1,3)$. (Strictly speaking, the curves should only be in the right-half plane.)



3. Find all the following for this coordinate system (and sketch the vectors at different points):

a. $\overrightarrow{\varepsilon_u}$ and $\overrightarrow{\varepsilon_v}$ (expressed in terms of the $\{(u, v)\}$ system, but with $\mathbf{i}$ and $\mathbf{k}$.)

b. the scaling factors h_u and h_v , along with the corresponding $\mathbf{e}_u$ and $\mathbf{e}_v$. (Express $\mathbf{e}_u$ and $\mathbf{e}_v$ in terms of the $\{(u, v)\}$ system, but with $\mathbf{i}$ and $\mathbf{k}$.)

c. the components of the metric g_{uu} , g_{vv} , g_{uv} and g_{uv} .

4. Decide whether this parabolic system is orthogonal.

M. Let us convert the two-dimensional system in the last problem to a three-dimensional system by rotating it about the Z-axis. Letting θ be the standard polar angle in the XY-plane, we have the coordinate system $\{(u, v, \theta)\}$ which is related to the standard Cartesian system $\{(x, y, z)\}$ by

$$x = uv\cos(\theta) \qquad \text{and} \qquad y = uv\sin(\theta) \qquad \text{and} \qquad z = \frac{v^2 - u^2}{2} \quad .$$

As before $u \geq 0$ and $v \geq 0$.

1. What do the $u =$ constant surfaces look like? What do the $v =$ constant surfaces look like? What do the $\theta =$ constant surfaces look like?

2. Find all the following for this coordinate system (and attempt to sketch the vectors at different points):

a. $\overrightarrow{\varepsilon_u}$, $\overrightarrow{\varepsilon_v}$ and $\overrightarrow{\varepsilon_\theta}$ (expressed in terms of the $\{(u, v, \theta)\}$ system, with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.)

b. the scaling factors h_u , h_v and h_θ, along with the corresponding $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_\theta$. (Express $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_\theta$ in terms of the $\{(u, v, \theta)\}$ system, with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.)

c. the components of the metric g_{uu} , g_{vv} , $g_{\theta\theta}$, g_{uv} , $\dots$.

3. Decide whether this "polar parabolic" system is orthogonal.

N. Let a be any fixed positive number, and consider the two-dimensional coordinate system $\{(u, v)\}$ on the Euclidean plane related to a Cartesian system $\{(x, y)\}$ by

$$x = a\cosh(u)\cos(v) \qquad \text{and} \qquad y = a\sinh(u)\sin(v)$$

for $u > 0$ and $0 \leq v < 2\pi$.

1. Verify that these coordinates satisfy

$$\frac{x^2}{a^2\cosh^2(u)} + \frac{y^2}{a^2\sinh^2(u)} = 1 \qquad \text{and} \qquad \frac{x^2}{a^2\cos^2(v)} - \frac{y^2}{a^2\sin^2(v)} = 1 \quad ,$$

and observe that the $u =$ constant curves are ellipses, and the $v =$ constant curves are hyperbolas. (This is said to be an elliptic system because of the constant-u curves.)

2. Find all the following for this coordinate system (and sketch the vectors at different points):

a. $\overrightarrow{\varepsilon_u}$ and $\overrightarrow{\varepsilon_v}$ (expressed in terms of the $\{(u,v)\}$ system, but with $\mathbf{i}$ and $\mathbf{j}$.)

b. the scaling factors h_u and h_v , along with the corresponding $\mathbf{e}_u$ and $\mathbf{e}_v$. (Express $\mathbf{e}_u$ and $\mathbf{e}_v$ in terms of the $\{(u,v)\}$ system, but with $\mathbf{i}$ and $\mathbf{j}$.)

c. the components of the metric g_{uu} , g_{vv} , g_{uv} and g_{uv} .

3. Decide whether this parabolic system is orthogonal.

O. We can obtain the "oblate spheroidal" coordinate system for three-dimensional Euclidean space by replacing the y in the last problem with z , and then "rotating" the coordinate system described in the last problem about the Z-axis. The change of coordinate formulas are

$$x = a\cosh(u)\cos(v)\cos(\theta) \qquad , \qquad y = a\cosh(u)\cos(v)\sin(\theta)$$

and

$$z = a\sinh(u)\sin(v) \quad ,$$

with $u > 0$ and $0 \leq v < 2\pi$.

1. What do the $u =$ constant surfaces look like? What do the $v =$ constant surfaces look like? What do the $\theta =$ constant surfaces look like?

2. Find all the following for this coordinate system (and attempt to sketch the vectors at different points):

a. $\overrightarrow{\varepsilon_u}$, $\overrightarrow{\varepsilon_v}$ and $\overrightarrow{\varepsilon_\theta}$ (expressed in terms of the $\{(u,v,\theta)\}$ system, with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.)

b. the scaling factors h_u , h_v and h_θ, along with the corresponding $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_\theta$. (Express $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_\theta$ in terms of the $\{(u,v,\theta)\}$ system, with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.)

c. the components of the metric g_{uu} , g_{vv} , $g_{\theta\theta}$, g_{uv} , $\ldots$.

3. Decide whether this system is orthogonal.