Homework Handout VI

A. The following all involve linear operators on a two-dimensional traditional vector space $\mathcal{V}$ with standard basis $\mathcal{B} = \{ \mathbf{i}, \mathbf{j} \}$. Try to do all of these problems without refering to the matrix for the operator.

1. What are all the eigenvalues and eigenvectors for the "magnification by 2" operator given by $\mathcal{M}_2(\mathbf{v}) = 2\mathbf{v}$.

2. Consider the vector projection onto $\mathbf{a}$, $\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v})$, where $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$. What is one eigenvector corresponding to eigenvalue $\lambda = 1$, and what is one eigenvector corresponding to eigenvalue $\lambda = 0$? (Hint: See example 7.1 in the notes.)

3. Let $\mathcal{R}_\theta$ be the rotation of every vector counterclockwise by a fixed angle θ. Does $\mathcal{R}_\theta$ have any eigen-pairs if $0 < \theta < 2\pi$? Why or why not?

B. The following all involve linear operators on a three-dimensional traditional vector space $\mathcal{V}$ with standard basis $\mathcal{B} = \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \}$. Try to do all of these problems without refering to the matrix for the operator.

1. What are the eigenvalues for the vector projection onto $\mathbf{a}$, $\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v})$, where $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$? Also, describe all the corresponding eigenvectors.

2. What are the eigenvalues for the projection onto the plane spanned by the $\mathbf{i}$ and $\mathbf{j}$ vectors, $\overrightarrow{\text{pr}}_{\{\mathbf{i},\mathbf{j}\}}(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) = v_1\mathbf{i} + v_2\mathbf{j}$? Also, describe all the corresponding eigenvectors.

3. Let $\mathcal{R}_{\mathbf{k},\theta}$ be the rotation of every vector about $\mathbf{k}$ by a fixed angle θ (as in problem ***B4*** in *Homework Set V*), and assume $0 < \theta < 2\pi$. Find one eigenvalue and corresponding eigenvector. (Compare with the above two-dimensional "clockwise rotation" problem.)

4. Let $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and find an eigenvalue and corresponding eigenvector for the cross product operator $\mathcal{K}_{\mathbf{a}}(\mathbf{v}) = \mathbf{a} \times \mathbf{v}$.

C. For each of the following matrices, find the eigenvalues and corresponding eigenvectors. For each eigenvalue, express the corresponding eigenvectors as linear combinations of an orthonormal set of eigenvectors (remember: this orthonormal set will often consist of just one unit vector!).

1. $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 3i \\ -3i & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{bmatrix}$

3. The 3×3 matrices in problems 1, 2, 3, 8, 10, and 12 starting on page 308 of AWH..

D. Find the eigenvalues and corresponding eigenvectors for the following two matrices, and then answer the question: *Must every* 2×2 *matrix have a linearly independent set of two eigenvectors?*.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

E. Let $\mathcal{V}$ be a two-dimensional traditional vector space $\mathcal{V}$ with standard basis $\mathcal{B}$, and let $\mathbf{R}$ be the matrix for $\mathcal{R}_{\pi/2}$, the rotation of every vector in $\mathcal{V}$ counterclockwise by $\frac{\pi}{2}$ (i.e., 90°). Verify that $\mathbf{R}$ has eigenvalues $\lambda_1 = -i$ and $\lambda_2 = i$, and corresponding eigenvectors

$$|\,\mathbf{v}_1\,\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \text{and} \qquad |\,\mathbf{v}_2\,\rangle = \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad .$$

Why does this not contradict the fact that the correct answer to **A3**, above, was that the rotation operator on $\mathcal{V}$ has no eigenvalues or eigenvectors?

F. Diagonalize each of the following Hermitian matrices. That is, for each matrix $\mathbf{H}$ find the unitary matrix $\mathbf{U}$ and diagonal matrix $\mathbf{D}$ such that $\mathbf{U}\mathbf{H}\mathbf{U}^\dagger = \mathbf{D}$. (Feel free to use the fact that I'm reusing matrices from a previous problem.)

1. $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 3i \\ -3i & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

2. The 3×3 matrices in problems 1, 2, 3, 8, 10, and 12 starting on page 308 of AWH..

G. Find an Hermitian matrix $\mathbf{H}$ with eigenvalue/eigenvector pairs

$$(\lambda_1\,,\,\mathbf{v}_1) = (2\,,\,2\,\mathbf{i} + 3\,\mathbf{j}) \qquad \text{and} \qquad (\lambda_2\,,\,\mathbf{v}_2) = (5\,,\,3\,\mathbf{i} - 2\,\mathbf{j})$$

(You should normalize the eigenvectors, first.)

H. You choose three different nonzero real values λ_1, λ_2, and λ_3, along with an orthonormal set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ (each of which has at least two nonzero components), and construct an Hermitian matrix $\mathbf{H}$ with eigenvalue/eigenvector pairs

$$(\lambda_1\,,\,\mathbf{v}_1) \quad , \quad (\lambda_2\,,\,\mathbf{v}_2) \quad \text{and} \quad (\lambda_3\,,\,\mathbf{v}_3) \quad .$$

I. Assume $\mathbf{A}$ is an anti-Hermitian matrix (i.e., $\mathbf{A}^\dagger = -\mathbf{A}$).

1. Let $\mathbf{B} = i\mathbf{A}$, and show that $\mathbf{B}$ is Hermitian.

2. Using the fact that $i\mathbf{A}$ is Hermitian:

i. Verify that the eigenvalues of $\mathbf{A}$ are imaginary.

ii. What can you say about "diagonalizing $\mathbf{A}$"?