Homework Handout V

A. Let $\mathcal{V}$ be a two-dimensional traditional vector space with standard basis $\mathcal{A} = \{\mathbf{i}, \mathbf{j}\}$. Using this basis for $\mathcal{V}$, do the following (unless otherwise indicated, view $\mathcal{V}$ as both the "input space" and the "output space").[1]

1. Find the matrix for the "magnification by 2" operator given by $\mathcal{M}_2(\mathbf{v}) = 2\mathbf{v}$.
2. Find the matrix for the projection onto the $\mathbf{i}$ vector, $\vec{\text{pr}}_{\mathbf{i}}(v_1\mathbf{i} + v_2\mathbf{j}) = v_1\mathbf{i}$.
3. Find the matrix for the vector projection onto $\mathbf{a}$, $\vec{\text{pr}}_{\mathbf{a}}(\mathbf{v})$, where $\mathbf{a} = 2\mathbf{i} + 1\mathbf{j}$. Then use the matrix to find $\vec{\text{pr}}_{\mathbf{a}}(\mathbf{v})$ when $\mathbf{v} = 5\mathbf{i} - \mathbf{4j}$.
4. Find the matrix for the linear operator $\mathcal{L}$ such that $\mathcal{L}(\mathbf{i}) = 2\mathbf{i} - 4\mathbf{j}$ and $\mathcal{L}(\mathbf{j}) = 3\mathbf{i} - \mathbf{5j}$. Then use the matrix to find $\mathcal{L}(\mathbf{v})$ when $\mathbf{v} = 5\mathbf{i} - \mathbf{4j}$
5. Find the matrix for $\mathcal{R}_\theta$, the rotation of every vector counterclockwise by a fixed angle θ. Then use the matrix to find $\mathcal{R}_\theta(\mathbf{v})$ when $\theta = \frac{\pi}{6}$ and $\mathbf{v} = 5\mathbf{i} - 4\mathbf{j}$
6. Find the matrix for the rotation of every vector clockwise by a fixed angle θ.
7. Find the matrix for the dot product with $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathcal{D}_{\mathbf{a}}(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}$. (In this case, the "target space" is the space of all real numbers $\mathbb{R}$. Use $\{1\}$ as the basis for $\mathbb{R}$.)

B. Let $\mathcal{V}$ be a three-dimensional traditional vector space with standard basis $\mathcal{A} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Using this basis for $\mathcal{V}$, do the following (unless otherwise indicated, view $\mathcal{V}$ as both the "input space" and the "output space").

1. Find the matrix for the "magnification by 2" operator given by $\mathcal{M}_2(\mathbf{v}) = 2\mathbf{v}$.
2. Find the matrix for the vector projection onto $\mathbf{a}$, $\vec{\text{pr}}_{\mathbf{a}}(\mathbf{v})$, where $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then use the matrix to find $\vec{\text{pr}}_{\mathbf{a}}(\mathbf{v})$ when $\mathbf{v} = 5\mathbf{i} - \mathbf{7j} + 9\mathbf{k}$.
3. Find the matrix for the projection onto the plane spanned by the $\mathbf{i}$ and $\mathbf{j}$ vectors, $\vec{\text{pr}}_{\{\mathbf{i},\mathbf{j}\}}(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) = v_1\mathbf{i} + v_2\mathbf{j}$.
4. Find the matrix for $\mathcal{R}_{\mathbf{k},\theta}$, the rotation of every vector about $\mathbf{k}$ by a fixed angle θ. (This means that vectors parallel to $\mathbf{k}$ do not move, while the vectors in the plane spanned by $\{\mathbf{i}, \mathbf{j}\}$ are rotated — as in a previous exercise by the angle θ in the direction the fingers of your right hand would curl if your thumb points in the direction of $\mathbf{k}$.) (Compare with the above two-dimensional "clockwise rotation" problem.)

   Then use the matrix to find $\mathcal{R}_{\mathbf{k},\theta}(\mathbf{v})$ when $\theta = \frac{\pi}{6}$ and $\mathbf{v} = 5\mathbf{i} - \mathbf{4j} + 3\mathbf{k}$.

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[1]Be intelligent. Where posssible, find the matrix for the operation using the "boxed formula" or its slightly simplified version for the case where $\mathcal{V} = \mathcal{W}$ (in the notes **Elementary Linear Transform Theory**).

5. Find the matrix for $\mathcal{R}_{\mathbf{i},\theta}$, the rotation of every vector about $\mathbf{i}$ by a fixed angle θ . Then use the matrix to find $\mathcal{R}_{\mathbf{i},\theta}(\mathbf{v})$ when $\theta = \frac{\pi}{4}$ and $\mathbf{v} = 5\mathbf{i} - \mathbf{4j} + 3\mathbf{k}$.

6. (optional) Find the matrix for $\mathcal{R}_{\mathbf{j},\theta}$, the rotation of every vector about $\mathbf{j}$ by a fixed angle θ .

7. Find the matrix for the dot product with $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\,\mathbf{k}$, $\mathcal{D}_{\mathbf{a}}(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}$. (In this case, the "target space" is the space of all real numbers $\mathbb{R}$. Use $\{\,1\,\}$ as the basis for $\mathbb{R}$.)

8. Find the matrix for the cross product with $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\,\mathbf{k}$, $\mathcal{K}_{\mathbf{a}}(\mathbf{v}) = \mathbf{a} \times \mathbf{v}$. Then use the matrix to find $\mathcal{K}_{\mathbf{a}}(\mathbf{v})$ when $\mathbf{v} = 5\mathbf{i} - \mathbf{7j} + 9\mathbf{k}$.

C. Let $\mathcal{P}^3$ be the vector space of all third-degree polynomials. Remember, this is the set of every function p of the form

$$p(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3$$

where the α_k's are complex constants. For the following, use the basis

$$\mathcal{B} = \{\,1\,,\,z\,,\,z^2\,,\,z^3\,\}$$

and the inner product

$$\langle\, p \,|\, q \,\rangle = \int_0^1 p^*(x)\,q(x)\,dx \quad .$$

1. Let $\mathcal{D}$ be the basic differential operator $\mathcal{D}(p(z)) = p'(z)$. Verify that this is a linear operator on $\mathcal{P}^3$ (with $\mathcal{P}^3$ also being the target space), and find the matrix $\mathbf{D}$ for this operator .

2. Let $\mathcal{J}$ be the integral operator

$$\mathcal{J}(p(z)) = \int_0^1 p(x)\,\sqrt{x}\;dx \quad .$$

Verify that this is a linear operator on $\mathcal{P}^3$ (with $\mathbb{C}$ being the target space), and find the matrix $\mathbf{J}$ for this operator.

3. (optional) Let $\mathcal{J}$ be the integral operator just defined. Find a single polynomial $r(z)$ in $\mathcal{P}^3$ such that

$$\mathcal{J}(p(z)) = \langle\, r \,|\, p \,\rangle \quad \text{for every } p \in \mathcal{P}^3 \quad .$$

D. The theory we developed does not just apply to finite-dimensional vector spaces. We can apply it to linear operators on an infinite-dimensional vector space just as well, provided this space has a basis. Of course, the matrices may have infinitely many rows or columns. Keeping this in mind, Let $\mathcal{P}$ be the space of all polynomials. This space has basiss

$$\mathcal{B} = \{ 1 , z , z^2 , z^3 , \dots \} \quad .$$

Using this basis,

1. Find the matrix $\mathbf{D}$ for the basic differential operator $\mathcal{D}(p(z)) = p'(z)$.

2. Verify that

$$\mathcal{J}(p(z)) = \int_0^1 p(x)\sqrt{x}\ dx$$

is a linear operator on $\mathcal{P}$ (with $\mathbb{C}$ being the target space), and find the matrix $\mathbf{J}$ for this operator.

E. Let $\mathcal{V}$ be the vector space of all functions on the interval $(0,1)$ that can be written as linear combinations of the complex exponentials in the set

$$\mathcal{B} = \{ \mathbf{b}_n = e^{i2\pi nx} \ : \ n \text{ is any integer} \} \quad .$$

Note that $\mathcal{B}$ is a basis for this space. For this space, use the "energy norm" inner productt

$$\langle f \,|\, g \rangle = \int_0^1 f^*(x)\, g(x)\, dx \quad .$$

1. If you have not already done so, verify that $\mathcal{B}$ is orthogonal with respect to this inner product.

2. Find the entries in the $\infty \times \infty$ matrix for the basic second-order differential operator $\Delta(f) = f''$.

F. Let $\mathcal{V}$ be a two-dimensional traditional vector space with standard basis $\mathcal{A} = \{ \mathbf{i}, \mathbf{j} \}$, and let $\mathcal{B} = \{ \mathbf{b}_1, \mathbf{b}_2 \}$ be the orthonormal basis given by

$$\mathbf{b}_1 = \frac{1}{\sqrt{5}}[2\mathbf{i} + \mathbf{j}] \quad \text{and} \quad \mathbf{b}_2 = \frac{1}{\sqrt{5}}[\mathbf{i} - 2\mathbf{j}] \quad .$$

Go ahead and compute $\mathbf{M}_{\mathcal{AB}}$ and $\mathbf{M}_{\mathcal{BA}}$ for use in the finding the matrix with respect to $\mathcal{B}$ for each of the following linear transforms:

1. The "magnification by 2" operator given by $\mathcal{M}_2(\mathbf{v}) = 2\mathbf{v}$. (You found the corresponding matrix with respect to $\mathcal{A}$ in problem ***A1***.)

2. The operator $\mathcal{L}$ for which $\mathbf{L}_{\mathcal{A}} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

3. The vector projection onto $\mathbf{a}$, $\vec{\text{pr}}_{\mathbf{a}}(\mathbf{v})$, where $\mathbf{a} = 2\mathbf{i} + 1\mathbf{j}$. (You found the corresponding matrix with respect to $\mathcal{A}$ in problem ***A3***.) (You should get a particularly simple matrix. Why?)

4. The linear operator $\mathcal{L}$ given in problem ***A4***.

5. The "counterclockwise rotation by angle θ" operator $\mathcal{R}_\theta$ given in problem ***A4***.

G. Let $\mathcal{V}$ be a three-dimensional traditional vector space with standard basis $\mathcal{A} = \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \}$, and let $\mathcal{B} = \{ \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \}$ be given by

$$\mathbf{b}_1 = \frac{1}{\sqrt{5}}[\mathbf{i} + 2\mathbf{j}] \quad , \quad \mathbf{b}_2 = \frac{1}{\sqrt{6}}[2\mathbf{i} - \mathbf{j} + \mathbf{k}] \quad .$$

and

$$\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2 = \frac{1}{\sqrt{30}}[2\mathbf{i} - \mathbf{j} - 5\mathbf{k}] \quad .$$

In problem ***F*** in Homework Handout IV, you verified that $\mathcal{B}$ is orthonormal, and you computed (I hope) $\mathbf{M}_{\mathcal{AB}}$ and $\mathbf{M}_{\mathcal{BA}}$. Using this, find the matrix with respect to $\mathcal{B}$ for each of the following linear transforms:

1. The vector projection onto $\mathbf{a}$, $\vec{\text{pr}}_{\mathbf{a}}(\mathbf{v})$, where $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. (You found the corresponding matrix with respect to $\mathcal{A}$ in problem ***B2***.) (You should get a particularly simple matrix. Why?)

2. The cross product with $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathcal{K}_{\mathbf{a}}(\mathbf{v}) = \mathbf{a} \times \mathbf{v}$. (You found the corresponding matrix with respect to $\mathcal{A}$ in problem ***B8***.)

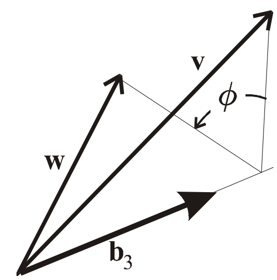
H. Let $\mathcal{V}$, $\mathcal{A}$ and $\mathcal{B}$ be the vector space and orthonormal basis from the previous problem. In the following problems, finding the matrix of the given operator with respect to $\mathcal{B}$ should be 'easy'. Use this fact and the change of basis formulas to also find the matrix with respect to $\mathcal{A}$ for the given operator.

1. The projection onto the plane spanned by the $\mathbf{b}_1$ and $\mathbf{b}_2$ vectors, $\overrightarrow{\text{pr}}_{\{\mathbf{b}_1,\mathbf{b}_2\}}(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + v_3\mathbf{b}_3) = v_1\mathbf{b}_1 + v_2\mathbf{b}_2$. Find the matrix of this operator first with respect to $\mathcal{B}$ and then with respect to $\mathcal{A}$.

2. The cross product with $\mathbf{b}_3$, $\mathcal{K}_{\mathbf{b}_3}(\mathbf{v}) = \mathbf{b}_3 \times \mathbf{v}$. Find the matrix of this operator first with respect to $\mathcal{B}$ and then with respect to $\mathcal{A}$.

I. Here we are considering traditional vectors in 3-dimensional space with standard basis

$$\mathcal{A} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \quad .$$



Let ϕ be some angle, let $\mathbf{b}_3$ be some unit vector, and let $\mathcal{R}$ be the linear transformation which is the rotation through angle ϕ about $\mathbf{b}_3$.

(See the figure, in which $\mathbf{w} = \mathcal{R}(\mathbf{v})$.)

Using $\quad \phi = \frac{\pi}{6}$ and $\mathbf{b}_3 = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$:

1. Find an orthonormal basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for which $\mathbf{R}_{\mathcal{B}}$, the matrix for $\mathcal{R}$ with respect to $\mathcal{B}$, is relatively simple and easily computed (hint: I gave you $\mathbf{b}_3$). Express each $\mathbf{b}_k$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and compute that matrix $\mathbf{R}_{\mathcal{B}}$.

2. Finding and using the appropriate unitary matrices, compute the matrix $\mathbf{R}_{\mathcal{A}}$ for $\mathcal{R}$ with respect to the standard basis $\mathcal{A} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

3. Find $\mathbf{w} = \mathcal{R}(\mathbf{v})$ where $\mathbf{v} = 1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

J. Redo the last problem, but with $\phi = \frac{\pi}{4}$ and $\mathbf{b}_3 = \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$,

K. (optional) “Linear functionals” and the “Riesz theorem”:

1. Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces, and let $\mathcal{L}[\mathcal{V},\mathcal{W}]$ denote the collection of all linear operators from $\mathcal{V}$ into $\mathcal{W}$. Verify that $\mathcal{L}[\mathcal{V},\mathcal{W}]$ is, itself, a vector space (with the “vectors” being the linear operators).[2]

2. A *linear functional* on a vector space $\mathcal{V}$ is simply a linear operator from $\mathcal{V}$ into $\mathbb{C}$. In the terms given just above, a linear functional on $\mathcal{V}$ is one of the linear operators in $\mathcal{L}[\mathcal{V},\mathbb{C}]$. The space of all linear functionals on $\mathcal{V}$ (i.e., the space $\mathcal{L}[\mathcal{V},\mathbb{C}]$) is often called the *dual space of* $\mathcal{V}$.

a. Let $\mathbf{g}$ be any fixed vector in $\mathcal{V}$ and define the operator $\Gamma_{\mathbf{g}}$ by $\Gamma_{\mathbf{g}}(\mathbf{v}) = \langle\, \mathbf{g} \,|\, \mathbf{v} \,\rangle$. Show that $\Gamma_{\mathbf{g}}$ is a linear functional on $\mathcal{V}$.

Now assume $\mathcal{V}$ is finite dimensional with orthonormal basis $\mathcal{B} = \{\, \mathbf{e}_1,\ \mathbf{e}_2\,,\ \dots\,,\ \mathbf{e}_N \,\}$. Let Γ be any linear functional on $\mathcal{V}$ and do the following:

b. Determine the size of the matrix $\mathbf{G}$ for Γ, and give the formula for finding its entries.

c. Show that there is a vector $\mathbf{g}$ in $\mathcal{V}$ such that

$$\Gamma(\mathbf{v}) = \langle\, \mathbf{g} \,|\, \mathbf{v} \,\rangle \quad \text{for each } \mathbf{v} \text{ in } \mathcal{V} \quad .$$

(The Riesz theorem for finite-dimensional vector spaces is that, for each linear functional Γ on a finite-dimensional vector space $\mathcal{V}$ with inner product, there is a corresponding vector $\mathbf{g}$ in $\mathcal{V}$ such that the above equation holds. You've just proven it! This theorem can be generalized to some (but not all) cases where $\mathcal{V}$ is infinite dimensional. The cases where it can be generalized can be rather significant.)

L. (optional – assumes you've at least read the previous problem) Consider $\mathcal{P}$, the space of all polynomials with the energy norm inner product given in problem ***D***, along with the integral operator $\mathcal{J}$ defined by

$$\mathcal{J}(p(z)) = \int_0^1 p(x)\,\sqrt{x}\ dx \quad .$$

d. Verify that $\mathcal{J}$ is a linear functional on $\mathcal{P}$.

e. Verify that there is no polynomial r such that

$$\mathcal{J}(p(z)) = \langle\, r \,|\, p \,\rangle \quad \text{for every } p \in \mathcal{P}^3 \quad .$$

(Hence, the Riesz theorem does not hold for $\mathcal{P}$.)

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[2]Linear operators on this space are called *tensors*. They can be represented by “matrices of matrices”.