Homework Handout IV

A. Let $$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2i}{\sqrt{30}} & \frac{-i}{\sqrt{30}} & \frac{5i}{\sqrt{30}} \end{bmatrix} \quad .$$

1. Compute $\mathbf{U}^\mathsf{T}$ and $\mathbf{U}^\dagger$.
2. Verify that $\mathbf{U}$ is unitary.
3. Let $\mathcal{B} = \{ \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 \}$ be any orthonormal set of vectors (so $\langle \mathbf{e}_j | \mathbf{e}_k \rangle = \delta_{jk}$), and let $\{ \mathbf{b}_1 , \mathbf{b}_2 , \mathbf{b}_3 \}$ be given by
$$[\, \mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3 \,] = [\, \mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3 \,] \mathbf{U} \quad .$$
   - ***a.*** Write out the complete set of formulas for the $\mathbf{b}_j$'s in terms of the $\mathbf{e}_k$'s.
   - ***b.*** Compute each $\langle \mathbf{b}_j \,|\, \mathbf{b}_k \rangle$ and verify that $\{ \mathbf{b}_1 , \mathbf{b}_2 , \mathbf{b}_3 \}$ is an orthonormal set (as claimed in the lecture).
   - ***c.*** Find the $\mathbf{e}_k$'s in terms of the $\mathbf{b}_j$'s. (Remember to make use of the fact that $\mathbf{U}$ is unitary!)

B. For each of the following matrices, find values for a and b so that the resulting matrix is orthogonal. (Multiple answers are possible.)

$$\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} \quad , \quad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ a & b \end{bmatrix} \quad , \quad \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ a & b \end{bmatrix}$$

C. Find a second and third row for

$$\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

so that the resulting matri$x$ is orthogonal, and then verify that your matrix satisfies $|\det(\mathbf{A})| = 1$ and $\mathbf{A}^\mathsf{T} = \mathbf{A}^\dagger$.

D. Find complex numbers a and b so that

$$\mathbf{U} = \begin{bmatrix} \frac{3+4i}{\sqrt{54}} & \frac{2+5i}{\sqrt{54}} \\ a & b \end{bmatrix}$$

is unitary.

E. Here we are considering traditional vectors in 3-dimensional space. Let

$$\mathcal{A} = \{ \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 \} = \{ \mathbf{i} , \mathbf{j} , \mathbf{k} \}$$

be the standard basis, and let

$$\mathcal{B} = \{ \mathbf{b}_1 , \mathbf{b}_2 , \mathbf{b}_3 \}$$

be given by

$$\mathbf{b}_1 = \mathbf{i} + 2\mathbf{j} \quad , \quad \mathbf{b}_2 = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b}_3 = 4\mathbf{j} + 3\mathbf{k} \quad .$$

1. Compute the matrices $\mathbf{M}_{\mathcal{A}\mathcal{A}}$, $\mathbf{M}_{\mathcal{A}\mathcal{B}}$, $\mathbf{M}_{\mathcal{B}\mathcal{A}}$ and $\mathbf{M}_{\mathcal{B}\mathcal{B}}$.
2. Find the matrix $\mathbf{A}$ such that $|\,\mathbf{v}\,\rangle_{\mathcal{A}} = \mathbf{A}|\,\mathbf{v}\,\rangle_{\mathcal{B}}$ for every vector $\mathbf{v}$ in space, and use it to find $|\,\mathbf{v}\,\rangle_{\mathcal{A}}$ when $\mathbf{v} = 3\,\mathbf{b}_1 + 4\,\mathbf{b}_2 + 6\,\mathbf{b}_3$.
3. How would you find the matrix $\mathbf{B}$ such that $|\,\mathbf{v}\,\rangle_{\mathcal{B}} = \mathbf{B}|\,\mathbf{v}\,\rangle_{\mathcal{A}}$ for every vector $\mathbf{v}$ in space.
4. (optional) Find the matrix $\mathbf{B}$ such that $|\,\mathbf{v}\,\rangle_{\mathcal{B}} = \mathbf{B}|\,\mathbf{v}\,\rangle_{\mathcal{A}}$ for every vector $\mathbf{v}$ in space, and use it to find $|\,\mathbf{v}\,\rangle_{\mathcal{B}}$ when $\mathbf{v} = 3\,\mathbf{i} + 4\,\mathbf{j} + 6\,\mathbf{k}$.

F. Here we are considering traditional vectors in 3-dimensional space. Let

$$\mathcal{A} = \{ \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 \} = \{ \mathbf{i} , \mathbf{j} , \mathbf{k} \}$$

be the standard basis, and let

$$\mathcal{B} = \{ \mathbf{b}_1 , \mathbf{b}_2 , \mathbf{b}_3 \}$$

be given by

$$\mathbf{b}_1 = \frac{1}{\sqrt{5}}[\mathbf{i} + 2\mathbf{j}] \quad , \quad \mathbf{b}_2 = \frac{1}{\sqrt{6}}[2\mathbf{i} - \mathbf{j} + \mathbf{k}] \quad \text{and} \quad \mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$$

1. Compute $\mathbf{b}_3$ and verify that $\mathcal{B}$ is orthonormal.
2. Find the matrix $\mathbf{U}$ such that $|\,\mathbf{v}\,\rangle_{\mathcal{B}} = \mathbf{U}|\,\mathbf{v}\,\rangle_{\mathcal{A}}$ for every vector $\mathbf{v}$ in space.
3. Find $|\,\mathbf{u}\,\rangle_{\mathcal{A}}$ and $|\,\mathbf{u}\,\rangle_{\mathcal{B}}$ assuming $\mathbf{u} = 3\,\mathbf{i} + 4\,\mathbf{j} + 6\,\mathbf{k}$.
4. Find $|\,\mathbf{w}\,\rangle_{\mathcal{A}}$ and $|\,\mathbf{w}\,\rangle_{\mathcal{B}}$ assuming $\mathbf{w} = 3\,\mathbf{b}_1 + 4\,\mathbf{b}_2 + 6\,\mathbf{b}_3$.
5. Find $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in terms of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$.
6. Is there a vector $\mathbf{a}$ such that

$$|\,\mathbf{a}\,\rangle_{\mathcal{A}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad |\,\mathbf{a}\,\rangle_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad ?$$

Why or why not?

G. (Still using traditional vectors in the plane.) In an earlier exercise you found values a and b so that

$$\mathbf{A} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ a & b \end{bmatrix}$$

is orthogonal. Using this $\mathbf{A}$ and the orthonormal bases

$$\mathcal{A} = \{\,\mathbf{i}\,,\,\mathbf{j}\,\} \qquad \text{and} \qquad \mathcal{B} = \{\,\mathbf{b}_1\,,\,\mathbf{b}_2\,\}$$

where $|\,\mathbf{v}\,\rangle_{\mathcal{B}} = \mathbf{A}|\,\mathbf{v}\,\rangle_{\mathcal{A}}$ for every vector $\mathbf{v}$ in the plane, sketch the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ in the plane showing their relation to the standard basis.

H. In an earlier exercise you found complex values a and b so that

$$\mathbf{U} = \begin{bmatrix} \frac{3+4i}{\sqrt{54}} & \frac{2+5i}{\sqrt{54}} \\ a & b \end{bmatrix}$$

is unitary. Using this $\mathbf{U}$ and the standard basis $\mathcal{A} = \{\,\mathbf{i}\,,\,\mathbf{j}\,\}$, find the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ so that $|\,\mathbf{v}\,\rangle_{\mathcal{B}} = \mathbf{U}|\,\mathbf{v}\,\rangle_{\mathcal{A}}$ for every vector $\mathbf{v}$ in the plane where $\mathcal{B} = \{\,\mathbf{b}_1\,,\,\mathbf{b}_2\,\}$.

I. Here we are considering the space of "traditional vectors in the plane" but with complex scalars. Let $\mathcal{A} = \{\,\mathbf{i}\,,\,\mathbf{j}\,\}$ be the standard basis and let

$$\mathcal{B} = \{\,\mathbf{b}_1\,,\,\mathbf{b}_2\,\}$$

where

$$\mathbf{b}_1 = \frac{1}{\sqrt{5}}(\,2i\mathbf{i} + 1\,\mathbf{j}) \qquad \text{and} \qquad \mathbf{b}_2 = \frac{1}{\sqrt{5}}(-i\,\mathbf{i} + 2\,\mathbf{j})$$

1. Confirm that $\mathcal{B}$ is orthonormal.

2. Find the matrix $\mathbf{U}$ such that $|\,\mathbf{v}\,\rangle_{\mathcal{B}} = \mathbf{U}|\,\mathbf{v}\,\rangle_{\mathcal{A}}$ for every vector $\mathbf{v}$.

3. Let $\mathbf{v} = 8\mathbf{i} + 2\,\mathbf{j}$, and find $|\,\mathbf{v}\,\rangle_{\mathcal{B}}$.

J. Here we are considering the "vector space" of all functions that can be written

$$f(x) = a_{-2}e^{-i4\pi x} + a_{-1}e^{-i2\pi x} + a_0 1 + a_1 e^{i2\pi x} + a_2 e^{i4\pi x}$$

where the a_k's can be any complex values. The inner product, as before, is

$$\langle f \mid g \rangle = \int_0^1 f^*(x)\, g(x)\, dx \quad .$$

From previous exercises, you know

$$\mathcal{A} = \left\{ 1 , e^{i2\pi x} , e^{-i2\pi x} , e^{i4\pi x} , e^{-i4\pi x} \right\}$$

and

$$\mathcal{B} = \left\{ 1 , \sqrt{2}\cos(2\pi x) , \sqrt{2}\sin(2\pi x) , \sqrt{2}\cos(4\pi x) , \sqrt{2}\sin(4\pi x) \right\}$$

are two orthonormal bases for this space. Now:

1. Using "$e^{i\theta} = \cos(\theta) + i\sin(\theta)$", find the matrix $\mathbf{A}$ such that

$$\left[1,\ e^{i2\pi x},\ e^{-i2\pi x},\ e^{i4\pi x},\ e^{-i4\pi x}\right]$$
$$= \left[1,\ \sqrt{2}\cos(2\pi x),\ \sqrt{2}\sin(2\pi x),\ \sqrt{2}\cos(4\pi x),\ \sqrt{2}\sin(4\pi x)\right]\mathbf{A}$$

2. Find the matrices $\mathbf{B}$ and $\mathbf{C}$ such that

$$|\, f \,\rangle_{\mathcal{B}} = \mathbf{B}|\, f \,\rangle_{\mathcal{A}} \qquad \text{and} \qquad |\, f \,\rangle_{\mathcal{A}} = \mathbf{C}|\, f \,\rangle_{\mathcal{B}} \quad .$$

3. Suppose

$$|\, f \,\rangle_{\mathcal{A}} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \quad .$$

i. What is the formula for $f(x)$ in terms of exponentials?

ii. What is the formula for $f(x)$ in terms of sine and cosines?

4. Suppose

$$|\, g \,\rangle_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \quad .$$

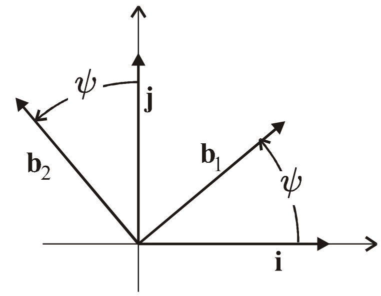
i. What is the formula for $g(x)$ in terms of exponentials?

ii. What is the formula for $g(x)$ in terms of sine and cosines?

K. ("Rotated" bases for the plane) Here we are considering the traditional vectors in the plane with two orthonormal bases:

$$\mathcal{A} = \{\mathbf{i}, \mathbf{j}\} \qquad \text{and} \qquad \mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$$

where $\mathcal{A}$ is the standard basis and $\mathcal{B}$ is generated by rotating $\mathcal{A}$ counterclockwise through some angle ψ (see figure).



1. Find $\mathbf{b}_1$ and $\mathbf{b}_2$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

2. Based on your answer to part ***a***, above, and a "Big Theorem" from class, what is the matrix $\mathbf{U}$ such that, for every vector $\mathbf{v}$ in the plane,

$$|\mathbf{v}\rangle_{\mathcal{B}} = \mathbf{U}|\mathbf{v}\rangle_{\mathcal{A}} \quad ?$$

3. Find the components of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ with respect to both $\mathcal{A}$ and $\mathcal{B}$ (using $\mathbf{U}$ where possible).

4. Find the components of $\mathbf{v} = 3\mathbf{b}_1 + 4\mathbf{b}_2$ with respect to both $\mathcal{A}$ and $\mathcal{B}$ (using $\mathbf{U}$ where possible).

5. Find $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{b}_1$ and $\mathbf{b}_2$.

L. Here we are considering traditional vectors in 3-dimensional space with standard basis

$$S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \quad .$$

1. Find a right-handed orthonormal basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ such that $\mathbf{b}_3$ points in the same direction as $\mathbf{A} = \mathbf{i} + \mathbf{j}$.

2. Using the $\mathcal{B}$ you found in ***1***,

i. Find the unitary matrices $\mathbf{U}$ and $\mathbf{W}$ such that $|\mathbf{v}\rangle_{\mathcal{B}} = \mathbf{U}|\mathbf{v}\rangle_{S}$ and $|\mathbf{v}\rangle_{S} = \mathbf{W}|\mathbf{v}\rangle_{\mathcal{B}}$ for every vector $\mathbf{v}$.

ii. Find $|\mathbf{u}\rangle_S$ and $|\mathbf{u}\rangle_{\mathcal{B}}$ assuming $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.

iii. Find $|\mathbf{w}\rangle_S$ and $|\mathbf{w}\rangle_{\mathcal{B}}$ assuming $\mathbf{w} = 3\mathbf{b}_1 + 4\mathbf{b}_2 + 6\mathbf{b}_3$.

iv. Find $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in terms of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$.

3&4. Repeat the above using $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

M. Here we are considering traditional vectors in 4-dimensional space of traditional vectors with orthnormal basis

$$\mathcal{A} = \{ \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 , \mathbf{e}_4 \}$$

Using the component formulas developed in section 5.6 of the online notes:

1. Find the area of the parallelogram (in the plane spanned by $\mathcal{S}_2 = \{ \mathbf{e}_1 , \mathbf{e}_2 \}$) generated by

i. the vectors $\mathbf{v}_1 = 3\mathbf{e}_1 + 2\mathbf{e}_2$ and $\mathbf{v}_2 = 5\mathbf{e}_1 - 4\mathbf{e}_2$

ii. the vectors with componenents $| \mathbf{w}_1 \rangle_{\mathcal{S}_2} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$ and $| \mathbf{w}_2 \rangle_{\mathcal{S}_2} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$.

2. Find the volume of the parallelepiped (in the space spanned by $\mathcal{S}_3 = \{ \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 \}$) generated by the vectors

i. the vectors $\mathbf{v}_1 = 2\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_3$, $\mathbf{v}_2 = 5\mathbf{e}_1 - 4\mathbf{e}_2 + 3\mathbf{e}_3$

and $\mathbf{v}_1 = \mathbf{e}_2 - \mathbf{e}_3$

ii. the vectors with componenents $| \mathbf{w}_1 \rangle_{\mathcal{S}_3} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $| \mathbf{w}_2 \rangle_{\mathcal{S}_3} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$

and $| \mathbf{w}_3 \rangle_{\mathcal{S}_3} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

3. Find the four-dimensional volume of the hyper-parallelepipedgenerated by the vectors with components

$$| \mathbf{w}_1 \rangle_{\mathcal{A}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \quad , \quad | \mathbf{w}_2 \rangle_{\mathcal{A}} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \end{bmatrix} \quad , \quad | \mathbf{w}_3 \rangle_{\mathcal{A}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix} \quad \text{and} \quad | \mathbf{w}_4 \rangle_{\mathcal{A}} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

N. Here we are considering traditional vectors in 3-dimensional space. Let

$$\mathcal{A} = \{ \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 \} = \{ \mathbf{i} , \mathbf{j} , \mathbf{k} \}$$

be the standard basis, and let

$$\mathcal{B} = \{ \mathbf{b}_1 , \mathbf{b}_2 , \mathbf{b}_3 \}$$

be given by

$$\mathbf{b}_1 = \mathbf{i} + 2\mathbf{j}$$

$$\mathbf{b}_2 = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

$$\mathbf{b}_3 = 4\mathbf{j} + 3\mathbf{k}$$

This is the same two bases used in exercise ***E*** . Remember that you computed the matrices $\mathbf{M}_{\mathcal{AB}}$, $\mathbf{M}_{\mathcal{AB}}$, $\ldots$ in that exercise. Using these matrices (as appropriate) and the appropriate "special case" volume formula from section 5.6 of the online notes, find the volume of the parallelogram generated by the vectors

$$\mathbf{v}_1 = \alpha\,\mathbf{b}_1 \quad , \quad \mathbf{v}_2 = \beta\,\mathbf{b}_2 \quad \text{and} \quad \mathbf{v}_3 = \gamma\,\mathbf{b}_3$$

when

1. $\alpha = 1$, $\beta = 1$ and $\gamma = 1$.

2. $\alpha = 5$, $\beta = 5$ and $\gamma = 5$.

3. $\alpha = 2$, $\beta = 3$ and $\gamma = 4$