Homework Handout II

A. For the following, $\mathcal{V}$ is a three-dimensional space of traditional vectors with standard basis

$$\mathcal{S} = \{ \mathbf{i} , \mathbf{j} , \mathbf{k} \} \quad .$$

(If you prefer, use $\{ \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 \}$ or $\{ \mathbf{x} , \mathbf{y} , \mathbf{z} \}$.)
Also, let

$$\mathcal{B} = \{ \mathbf{b}_1 , \mathbf{b}_2 , \mathbf{b}_3 \}$$

where

$$\mathbf{b}_1 = \mathbf{i} + 2\mathbf{j} \quad , \quad \mathbf{b}_2 = 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{b}_3 = 2\mathbf{i} - 3\mathbf{j} \quad .$$

1. Solve for $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in terms of $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$.
2. Is $\mathcal{B}$ a basis for $\mathcal{V}$? Give a reason for your answer.
3. Let $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. What is $\mathbf{v}$ in terms of $\mathcal{B}$?
4. Find the following (with $\mathbf{v}$ as above):

$$|\mathbf{i}\rangle_{\mathcal{S}} \quad , \quad |\mathbf{j}\rangle_{\mathcal{S}} \quad , \quad |\mathbf{k}\rangle_{\mathcal{S}} \quad , \quad |\mathbf{b}_1\rangle_{\mathcal{S}} \quad , \quad |\mathbf{b}_2\rangle_{\mathcal{S}} \quad , \quad |\mathbf{b}_3\rangle_{\mathcal{S}} \quad , \quad |\mathbf{v}\rangle_{\mathcal{S}} \quad ,$$

$$|\mathbf{i}\rangle_{\mathcal{B}} \quad , \quad |\mathbf{j}\rangle_{\mathcal{B}} \quad , \quad |\mathbf{k}\rangle_{\mathcal{B}} \quad , \quad |\mathbf{b}_1\rangle_{\mathcal{B}} \quad , \quad |\mathbf{b}_2\rangle_{\mathcal{B}} \quad , \quad |\mathbf{b}_3\rangle_{\mathcal{B}} \quad \text{and} \quad |\mathbf{v}\rangle_{\mathcal{B}}$$

5. Compute $\langle \mathbf{b}_i \,|\, \mathbf{b}_j \rangle$ (i.e., $\mathbf{b}_i \cdot \mathbf{b}_j$) for all possible i's and j's.
6. Let $\mathbf{v} = v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2 + v_3 \mathbf{b}_3$ and $\mathbf{w} = w_1 \mathbf{b}_1 + w_2 \mathbf{b}_2 + w_3 \mathbf{b}_3$.

   Find the corresponding component formulas for $\langle \mathbf{v} \,|\, \mathbf{w} \rangle$ and $\|\mathbf{v}\|$.

   $\left(\text{Note: } \langle \mathbf{v} \,|\, \mathbf{w} \rangle \neq v_1 w_1 + v_2 w_2 + v_3 w_3 \quad \text{and} \quad \|\mathbf{v}\| \neq \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} \; ! \right)$

7. (optional) Suppose $\mathbf{c}$ is any vector in $\mathcal{V}$ and let the components of $\mathbf{c}$ with respect to our two bases be denoted by

$$|\mathbf{c}\rangle_{\mathcal{S}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad \text{and} \quad |\mathbf{c}\rangle_{\mathcal{B}} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \quad .$$

   Find the formulas for computing the α_k's from the β_k's, and for computing the β_k's from the α_k's.

B. Consider (but don't yet bother solving, yet) the differential equation

$$y'' + y = 0 \qquad .$$

1. Suppose y_1 and y_2 are two solutions to this differential equation. Verify that any linear combination of these two solutions is also a solution.
2. Let $\mathcal{S}$ be the set of all solutions to this differential equation. Is $\mathcal{S}$ a vector space? Explain.
3. What is the general solution to this differential equation? What does it tell you about a possible basis for $\mathcal{S}$ and the dimension of $\mathcal{S}$?

C. Let $\mathcal{S}$ be the set of all solutions to some given homogeneous linear differential equation

$$ay'' + by' + cy = 0$$

where a , b , and c are known functions. Show that $\mathcal{S}$ is a vector space. (If you recall enough from your old differential equations class, you can even state the dimension of $\mathcal{S}$.)

D. Compute $\langle \mathbf{v} \,|\, \mathbf{w} \rangle$, $\langle \mathbf{w} \,|\, \mathbf{v} \rangle$ and $\|\mathbf{v}\|$ when the vector space is $\mathbb{C}^2$, $\mathbf{v} = (3i \,,\, 2 + 3i)$ and $\mathbf{w} = (4 \,,\, 5 + 2i)$.

E. Compute the "energy norm" inner product of two functions f and g on the interval $[0, 1]$,

$$\langle f \,|\, g \rangle = \int_0^1 f^*(x)\, g(x)\, dx \ ,$$

for the following choices of f and g (simplify your answers as much as practical):

1. $f(x) = 3 + (2 + 3i)x$ and $g(x) = 5x - 2ix^2$
2. $f(x) = 3 + (2 + 3i)e^{i2\pi x}$ and $g(x) = e^{i\pi x}$
3. $f(x) = e^{i2\pi x}$ and $g(x) = 2 + x$

F. Let $\mathbf{a}$ and $\mathbf{v}$ be two (nonzero) vectors from a vector space $\mathcal{V}$ with an inner product $\langle \cdot \,|\, \cdot \rangle$. Define the "generalized projection of vector $\mathbf{v}$ onto vector $\mathbf{a}$" by

$$\vec{\text{pr}}_{\mathbf{a}}(\mathbf{v}) = \frac{\langle \mathbf{a} \,|\, \mathbf{v} \rangle}{\|\mathbf{a}\|^2}\, \mathbf{a} \qquad .$$

and define the corresponding "generalized projection of $\mathbf{v}$ orthogonal to $\mathbf{a}$" by

$$\vec{\text{or}}_{\mathbf{a}}(\mathbf{v}) = \mathbf{v} - \vec{\text{pr}}_{\mathbf{a}}(\mathbf{v}) \qquad .$$

Note that we automatically have that $\overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v})$ is "parallel" to $\mathbf{a}$, and that

$$\mathbf{v} = \overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v}) + \overrightarrow{\text{or}}_{\mathbf{a}}(\mathbf{v}) \qquad .$$

Now confirm that the set $\{ \overrightarrow{\text{pr}}_{\mathbf{a}}(\mathbf{v}) \, , \overrightarrow{\text{or}}_{\mathbf{a}}(\mathbf{v}) \}$ is orthogonal.

G. Let $\mathcal{V}$ be the linear space of all functions of the form

$$f(x) = \alpha_{-2} e^{-i4\pi x} + \alpha_{-1} e^{-i2\pi x} + \alpha_0 + \alpha_1 e^{i2\pi x} + \alpha_2 e^{i4\pi x}$$

where α_{-2} , α_{-1} , α_0 , α_1 and α_2 are constants.

1. Using the inner product

$$\langle f \,|\, g \rangle = \int_0^1 f^*(x)\, g(x)\, dx \ ,$$

verify that both

$$\mathcal{B}_E = \left\{ e^{-i4\pi x} \, , \, e^{-i2\pi x} \, , \, 1 \, , \, e^{i2\pi x} \, , \, e^{i4\pi x} \right\}$$

and

$$\mathcal{B}_T = \{ 1 \, , \, \cos(2\pi x) \, , \, \sin(2\pi x) \, , \, \cos(4\pi x), \, \sin(4\pi x) \}$$

are orthogonal bases for V.

2. What is $|\, e^{i2\pi x} \rangle_{\mathcal{B}_T}$? $|\, \sin(4\pi x) \rangle_{\mathcal{B}_E}$? (That is, find the components of each function with respect to the indicated basis.)

3. Construct the orthonormal basis corresponding to $\mathcal{B}_E$ and the orthonormal basis corresponding to $\mathcal{B}_T$.

H. Let $\mathcal{V}$ be a three-dimensional space of traditional vectors with a "standard" basis

$$\mathcal{S} = \{ \mathbf{i} \, , \mathbf{j} \, , \mathbf{k} \} \qquad .$$

(If you prefer, use $\{ \mathbf{e}_1 \, , \mathbf{e}_2 \, , \mathbf{e}_3 \}$ or $\{ \mathbf{x} \, , \mathbf{y} \, , \mathbf{z} \}$.)
Using the Gram-Schmidt procedure, construct an orthonormal basis for $\mathcal{V}$ from

$$\mathcal{B} = \{ \mathbf{b}_1 \, , \mathbf{b}_2 \, , \mathbf{b}_3 \}$$

where

$$\mathbf{b}_1 = \mathbf{i} + 2\mathbf{j} \qquad , \qquad \mathbf{b}_2 = 3\mathbf{j} - \mathbf{k} \qquad \text{and} \qquad \mathbf{b}_3 = 2\mathbf{i} - 3\mathbf{j} \quad .$$

I. You should have already convinced yourself that the space $\mathcal{P}$ of all polynomials has the basis

$$\{1\,,\,x,\,x^2,\,x^3,\,x^4,\,x^5,\,\ldots\,\} \qquad .$$

However, this basis is not orthonormal or even orthogonal with respect to the inner product

$$\langle\, f \,|\, g\,\rangle \;=\; \int_0^1 f^*(x)\, g(x)\, dx \;\; .$$

Let

$$\Phi \;=\; \{\, \phi_0(x) \;,\; \phi_1(x)\,,\; \phi_2(x)\,,\; \phi_3(x)\,,\; \phi_4(x)\,,\; \phi_5(x)\,,\; \ldots\,\}$$

be the corresponding orthonormal basis generated from the above basis by the Gram-Schmidt procedure.

1. Find the formulas for $\phi_0(x)\,,\,\phi_1(x)$ and $\phi_2(x)$.

2. Find the components of $f(x) = 2 + 3x^2$ with respect to basis Φ .