Homework Handout X

Unless otherwise stated, **n** always denotes the "outward pointing unit normal vector field" on the indicated curve or surface.

Also, don't forget that $\iint_{\mathcal{S}} dA$ is just the area of $\mathcal{S}$, and $\iiint_{\mathcal{R}} dV$ is just the volulme of $\mathcal{R}$.

A. Compute $\displaystyle\iint_{\mathcal{S}} \psi \, dA$ where $\mathcal{S}$ is a sphere of radius 3 and ψ is the scalar field whose coordinate formula is $\psi(x, y, z) = x^2$ (using a Cartesian coordinate system whose origin is at the center of the sphere). Use the "latitude and longitude" coordinate system $\{(\theta, \phi)\}$ on $\mathcal{S}$ arising from the standard spherical coordinate system $\{(r, \theta, \phi)\}$.

B. The following concern the parabolic coordinate systems from problems ***L*** and ***M*** of *Homework Handout VII*. Remember, these are orthogonal coordinate systems.

1. Find the element of area dA for the Euclidean plane in terms of the parabolic $\{(u, v)\}$ coordinate system from problem ***L*** of *Homework Handout VII*. Then, using this coordinate system, evaluate

$$\iint_{\mathcal{R}} \psi \, dA$$

where $\mathcal{R}$ is the region bounded by the coordinate curves

$$u = 0 \quad , \quad u = 3 \quad , \quad v = 0 \quad \text{and} \quad v = 2$$

and ψ is the scalar field with $\{(u, v)\}$ coordinate formula $\psi(u, v) = 4u + \sqrt{v}$.

2. Let $\{(u, v, \theta)\}$ be the polar parabolic coordinate system for three-dimensional Euclidean space from problem ***M*** of *Homework Handout VII*, and assume $\mathcal{S}$ is the coordinate surface $v = 3$. Convince yourself that $\{(u, \theta)\}$ is a coordinate system for this surface, and find the corresponding formula for dA.

C. Repeat the previous problem using the coordinate systems from problems ***N*** and ***O*** of *Homework Handout VII*.

D. Several surface integrals are given below. Each surface is in three-dimensional Euclidean space and is the graph of some function of two variables. For each:

i. Find the corresponding formula for dA on this surface, and

ii. Set up the double integral for computation in terms of the appropriate two coordinates. Don't forget the limits on the integrals. If the resulting integral is simple enough, compute it.

1. $\iint_{\mathcal{S}} dA$ where $\mathcal{S}$ is the graph of $z = 1 + 2x + 3y$ with $0 < x < 4$ and $0 < y < 6$.

2. $\iint_{\mathcal{S}} (z^2 - x^2)\, dA$ where $\mathcal{S}$ is the graph of $z = 1 + 2x + 3y$ with $0 < x < 4$ and $0 < y < 6$.

3. $\iint_{\mathcal{S}} xz\ dA$ where $\mathcal{S}$ is the graph of $z = 4 - x^2$ with $0 < x < 3$ and $0 < y < 2$.

4. $\iint_{\mathcal{S}} 14\, dA$ where $\mathcal{S}$ is the graph of $z = x^2 + y^2$ with $0 < x < 2$ and $0 < y < 2$.

5. $\iint_{\mathcal{S}} dA$ where $\mathcal{S}$ is the graph of $z = 4 - \sqrt{x^2 + y^2}$ with $x^2 + y^2 < 4$. (But use polar coordinates, not Cartesian!)

6. $\iint_{\mathcal{S}} dA$ where $\mathcal{S}$ is the portion in the first octant of the plane passing throught the points having (x, y, z) coordinates

$$(3, 0, 0) \quad , \quad (0, 1, 0) \quad \text{and} \quad (0, 0, 2) \quad .$$

E. Using spherical coordinates, compute $\iiint_{\mathcal{B}} \psi\, dV$ where $\mathcal{B}$ is a (solid) ball of radius 3 and ψ is the scalar field whose coordinate formula is $\psi(x, y, z) = x^2$ (using a Cartesian coordinate system whose origin is at the center of the sphere).

F. Find the element of volume, dV, in terms of

1. The polar parabolic coordinate system from problem ***M*** of *Homework Handout VII.*
2. The oblate spheroidal coordinate system from problem ***O*** of *Homework Handout VII.*

G. Note that, if $\mathcal{S}$ is a surface (in three-dimensional Euclidean space) given as the graph of $z = f(u, v)$ for some coordinate system on "the XY-plane", then $\mathcal{S}$ is also the surface of all (u, v, z) satisfying $\psi(u, v, z) = 0$ where ψ is the scalar field with coordinate formula

$$\psi(u, v, z) \ = \ f(u, v) \ - \ z \quad .$$

Recall that $\boldsymbol{\nabla}\psi$ is orthogonal to surfaces on which ψ is constant.[1] Using these facts, find explicit formulas in terms of the appropriate coordinate systems for both a normal vector field and a unit normal vector field on each surface described below (stolen from a previous problem). Also, attempt to sketch the surfaces along with the unit normal vectors at various points.

1. $\mathcal{S}$ is the graph of $z = 1 + 2x + 3y$.
2. $\mathcal{S}$ is the graph of $z = 4 - x^2$.

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[1]If you don't recall this, go back to the discussion of the geometric significance of the gradient on pages 9-14 to 9-16 in our online notes.

3. $\mathcal{S}$ is the graph of $z = x^2 + y^2$ (use the Cartesian coordinate system).

4. $\mathcal{S}$ is the graph of $z = x^2 + y^2$ (use the cylindrical coordinate system).

5. $z = 4 - \sqrt{x^2 + y^2}$ with $x^2 + y^2 < 4$. (use the cylindrical coordinate system).

H. Let $\mathcal{H}$ be the upper half of the sphere of radius 3 centered at the origin of a standard Cartesian coordinate system, and let $\mathbf{n}$ be the unit normal vector field on this hemisphere $\mathcal{H}$ that points in the "upward direction" (i.e, at each point not on the edge of $\mathcal{H}$, the $\mathbf{k}$ component of $\mathbf{n}$ is positive).

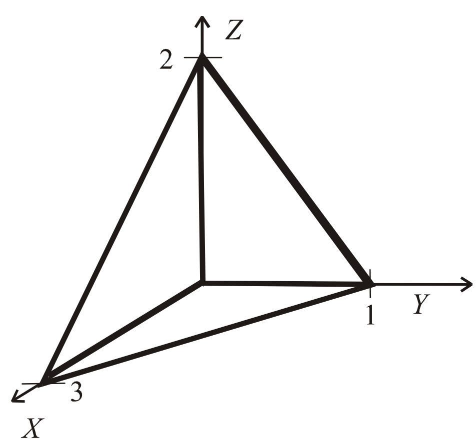
1. What is the formula for $\mathbf{n}$ using each of the standard coordinate systems: Cartesian, cylindrical and spherical?

2. Set up for evaluation (evaluate if possible) each of the following integrals, using the coordinate system you feel would lead to the easiest to compute integrals:

a. $\iint_{\mathcal{H}} \mathbf{F} \cdot \mathbf{n}\, dA$ where $\mathbf{F}(r, \theta, \phi) = r\, \mathbf{e}_r$

b. $\iint_{\mathcal{H}} \mathbf{F} \cdot \mathbf{n}\, dA$ where $\mathbf{F}(x, y, z) = \mathbf{k}$

I. Let $\mathcal{R}$ be the solid region in the first octant bounded by the plane sketched here, and $\mathcal{S}$ be the surface consisting of the four triangular regions bounding $\mathcal{R}$, and let $\mathbf{n}$ be the outward pointing unit normal vector field on $\mathcal{S}$.



In addition, let $\mathcal{S}_{\text{top}}$ and $\mathcal{S}_{\text{bottom}}$ be the triangular surfaces on the top and on the bottom of $\mathcal{R}$, respectively (so $\mathcal{S}_{\text{top}}$ is the largest triangular region sketched, and $\mathcal{S}_{\text{bottom}}$ is in the XY-plane).
Compute each of the following:

1. $\iiint_{\mathcal{R}} (2x + z)\, dV$

2. $\iint_{\mathcal{S}_{\text{top}}} (x\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{n}\, dA$

3. $\iint_{\mathcal{S}_{\text{bottom}}} (x\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{n}\, dA$

4. $\iint_{\mathcal{S}} (x\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{n}\, dA$

J. Assume $\mathcal{C}$ is the triangle in the XY-plane having corners at $(0,0)$, $(4,0)$ and $(0,3)$, and let $\mathcal{R}$ be the region enclosed by this triangle. Compute the following after using either the "fundamental theorem of multidimensional calculus" or using Gauss's theorem to convert the given integral to an easier-to-compute area integral.

1. $\int_{\mathcal{C}} (4x - y^2) n^1 \, ds \quad \left(n^1 = \mathbf{i} \cdot \mathbf{n}\right)$

2. $\int_{\mathcal{C}} (4x - y) \, n^2 \, ds \quad \left(n^2 = \mathbf{j} \cdot \mathbf{n}\right)$

3. $\int_{\mathcal{C}} (4x - y) \mathbf{n} \, ds$

4. $\int_{\mathcal{C}} (\, 3x \, \mathbf{i} + 4y \, \mathbf{j} \,) \cdot \mathbf{n} \, ds$

5. $\int_{\mathcal{C}} (\, 3y \, \mathbf{i} + 4x \, \mathbf{j} \,) \cdot \mathbf{n} \, ds$

6. $\int_{\mathcal{C}} \left(\, 2y \, \mathbf{i} + 3y^2 \, \mathbf{j} \,\right) \cdot \mathbf{n} \, ds$

K. Evaluate $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dA$ where, using a Cartesian coordinate system,

$$\mathbf{F}(x,y,z) = (2x - 3y)\,\mathbf{i} + (4y - z^2)\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k} \quad ,$$

and

1. $\mathcal{S}$ is the surface consisting of the sides of the rectangular box with corners at $(0,0,0)$ and $(2,3,4)$, and with sides parallel to the coordinate planes

2. $\mathcal{S}$ is the sphere of radius 2 about the origin.

L. Let $\mathcal{H}$ be the upper half of the sphere of radius 3 centered at the origin of a Cartesian coordinate system, and let $\mathcal{D}$ be the disk (in the XY-plane) of radius 3 at the base of $\mathcal{H}$.

1. Let $\mathcal{R}$ be the region enclosed by $\mathcal{H}$ and $\mathcal{D}$, and verify that, for any (differentiable) vector field $\mathbf{F}$,

$$\iint_{\mathcal{H}} \mathbf{F} \cdot \mathbf{n} \, dA = \iiint_{\mathcal{R}} \boldsymbol{\nabla} \cdot \mathbf{F} \, dV + \iint_{\mathcal{D}} \mathbf{F} \cdot \mathbf{k} \, dA \quad .$$

2. Now evaluate $\iint_{\mathcal{H}} \mathbf{k} \cdot \mathbf{n} \, dA$.

M. Exercises 3.8.1, 3.8.2, 3.8.3 starting on page 62 of AW&H.

N. In the following, assume $\mathcal{R}$ is a bounded solid and $\mathcal{S}$ is its boundary.[2]

1. Let ϕ and ψ be any two sufficiently differentiable scalar fields on $\mathcal{R}$, and verify the following equations:

a. $\boldsymbol{\nabla} \cdot (\phi \, \boldsymbol{\nabla}\psi) = (\boldsymbol{\nabla}\phi) \cdot (\boldsymbol{\nabla}\psi) + \phi \, \nabla^2 \psi$

[2]However, these formulas, which will be used next semester when we study partial differential equations, are valid whenever $\mathcal{R}$ is any bounded region in "N-dimensional Euclidean space" and $\mathcal{S}$ is its boundary.

b. $\displaystyle\iiint_{\mathcal{R}} \left[(\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\psi) + \phi\,\nabla^2\psi\right] dV = \iint (\phi\,\boldsymbol{\nabla}\psi)\cdot\mathbf{n}\,dA$

2. Show that, for any two sufficiently differentiable scalar fields u and v,

$$\iiint_{\mathcal{R}} \left[u\,\nabla^2 v - v\,\nabla^2 u\right] dV = \iint_{\mathcal{S}} [u\,\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u]\cdot\mathbf{n}\,dA \quad .$$

(Hint: Use the results from the previous part with "$(\phi,\psi) = (u,v)$" and $(\phi,\psi) = (v,u)$".

O. Using Green's theorem, find $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \left(x^2 - y\right)\mathbf{i} + (4y - x)\,\mathbf{j}$ and $\mathcal{C}$ is the rectangle in the XY-plane with corners $(0,0)$ and $(3,2)$, oriented "counterclockwise".

P. Exercise 3.8.5 on page 169 of AW&H [Consider using Green's theorem, also], (3.8.10 THEN 3.8.9) (note: $d\boldsymbol{\lambda} = d\boldsymbol{r}$)

Q. For this, our space is a sphere of radius R. This is a two-dimensional non-Euclidean space. A good coordinate system for this space is the "latitude/longitude angles" (θ,ϕ) from the spherical coordinate system with "$\theta = 0$" corresponding to the North Pole, and "$\theta = \pi$" corresponding to the South Pole. Recall that $\{(\theta,\phi)\}$ is an orthogonal system.

1. Let ψ and $\mathbf{F}$ be, respectively, a scalar field and a vector field on this sphere, and find the $\{(\theta,\phi)\}$–coordinate/component formulas for $\boldsymbol{\nabla}\psi$, $\boldsymbol{\nabla}\cdot\mathbf{F}$ and $\boldsymbol{\nabla}^2\psi$.

2. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(\theta,\phi) = \theta\,\mathbf{e}_\theta \quad ,$$

and let ϵ be some small positive value (it can be $\frac{\pi}{100}$ if you wish). Set

$$\mathcal{C} = \mathcal{C}_\epsilon + \mathcal{C}_{\pi/2}$$

where $\mathcal{C}_\epsilon$ and $\mathcal{C}_{\pi/2}$ are the circles on the sphere with $\theta = \epsilon$ and $\theta = \dfrac{\pi}{2}$ (the equator), respectively, and let $\mathcal{R}$ be all of the sphere between these two circles (so $\mathcal{R}$ is all of the northern hemisphere except for a small disk about the North Pole).

Compute both

$$\iint_{\mathcal{R}} \boldsymbol{\nabla}\cdot\mathbf{F}\,dA \qquad \text{and} \qquad \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,ds$$

and verify that

$$\iint_{\mathcal{R}} \boldsymbol{\nabla}\cdot\mathbf{F}\,dA = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{n}\,ds \quad .$$